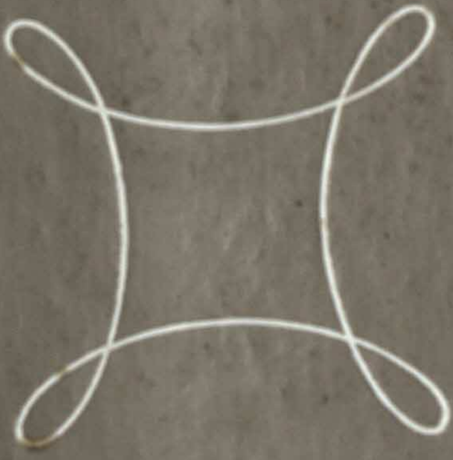
MATHEMATICAL HANDBOOK

HIGHER MATHEMATICS

M. VYGODSKY



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М. Я. ВЫГОДСКИЙ

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M. Vygodsky

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This handbook is a continuation of the *Handbook of Elementary Mathematics* by the same author and includes material usually studied in mathematics courses of higher educational institutions.

The designation of this handbook is twofold.

Firstly, it is a reference work in which the reader can find definitions (what is a vector product?) and factual information, such as how to find the surface of a solid of revolution or how to expand a function in a trigonometric series, and so on. Definitions, theorems, rules and formulas (accompanied by examples and practical hints) are readily found by reference to the comprehensive index or table of contents.

Secondly, the handbook is intended for systematic reading. It does not take the place of a textbook and so full proofs are only given in exceptional cases. However, it can well serve as material for a first acquaintance with the subject. For this purpose, detailed explanations are given of basic concepts, such as that of a scalar product (Sec. 104), limit (Secs. 203-206), the differential (Secs. 228-235), or infinite series (Secs. 270, 366-370). All rules are abundantly illustrated with examples, which form an integral part of the handbook (see Secs. 50-62, 134, 149, 264-266, 369, 422, 498, and others). Explanations indicate how to proceed when a rule ceases to be valid; they also point out errors to be avoided (see Secs. 290, 339, 340, 379, and others).

The theorems and rules are also accompanied by a wide range of explanatory material. In some cases, emphasis is placed on bringing out the *content* of a theorem to facilitate a grasp of the proof. At other times, special examples are illustrated and the reasoning is such as to provide a complete proof of the theorem if applied to the general case (see Secs. 148, 149, 369, 374). Occasionally, the explanation simply refers the reader to the sections on which the proof is based. Material given in small print may be omitted in a *first reading*; however, this does not mean it is not important.

Considerable attention has been paid to the historical background of mathematical entities, their origin and development. This very often helps the user to place the subject matter in its proper perspective. Of particular interest in this respect are Secs. 270, 366 together with Secs. 271, 383, 399, and 400, which, it is hoped, will give the reader a clearer understanding of Taylor's series than is usually obtainable in a formal exposition. Also, biographical information from the lives of mathematicians has been included where deemed advisable.

PLANE ANALYTIC GEOMETRY

1. The Subject of Analytic Geometry

The school *(elementary)* course of geometry treats of the properties of rectilinear figures and the circle. Most important are constructions; calculations play a subordinate role in the theory, although their practical significance is great. Ordinarily, the choice of a construction requires ingenuity. That is the chief difficulty when solving problems by the methods of elementary geometry.

Analytic geometry grew out of the need for establishing uniform techniques for solving geometrical problems, the aim being to apply them to the study of curves, which are of particular importance in practical problems.

This aim was achieved in the coordinate method (see Secs. 2 to 4). In this method, calculations are fundamental, while constructions play a subordinate role. As a result, solving problems by the method of analytic geometry requires much less inventiveness.

The origins of the coordinate method go back to the works of the ancient Greek mathematicians, in particular *Apollonius* (3-2 century B. C.). The coordinate method was systematically elaborated in the first half of the 17th century in the works of Fermat ¹⁾ and Descartes.²⁾ However, they considered only plane curves. It was Euler ³⁾ who first applied the coordinate method in a systematic study of space curves and surfaces.

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¹⁾ Pierre Fermat (1601-1655), celebrated French mathematician, one of the forerunners of Newton and Leibniz in developing the differential calculus; made a great contribution to the theory of numbers. Most of Fermat's works (including those on analytic geometry) were not published during the author's lifetime.

²⁾ Rene Descartes (1596-1650), celebrated French philosopher and mathematician. The year 1637, which saw the publication of his *Geometrie*, an appendix to his philosophical treatise, is taken to be the date of birth of analytic geometry.

³⁾ Leonhard Euler (1707-1783), born in Switzerland, wrote over 800 scientific papers and made important discoveries in all of the physico-mathematical sciences.

2. Coordinates

The coordinates of a point are quantities which determine the position of the point (in space, in a plane or on a curved surface, on a straight or curved line). If, for instance, a point M lies somewhere on a straight line $X'X$ (Fig. 1), then its position may be defined by a single number in the following manner: choose on $X'X$ some initial point O and measure the segment OM in, say, centimetres. The result will be a number x, either positive or negative, depending on the direction of OM (to the right or to the left if the straight line is horizontal). The number x is the coordinate of the point M.

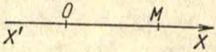

Fig. 1

The value of the coordinate x depends on the choice of the initial point O, on the choice of the positive direction on the straight line and also on the scale unit.

3. Rectangular Coordinate System

The position of a point in a plane is determined by two coordinates. The simplest method is the following.

Two mutually perpendicular straight lines $X'X$ and $Y'Y$ (Fig. 2) are drawn. These are termed *coordinate axes*. One (usually drawn horizontally) is the *axis of abscissas*, or the x-axis (in our case, $X'X$), and the other is the *axis of ordinates*, or the y-axis ($Y'Y$). The point O, the point of intersection of the two axes, is called the *origin of coordinates* or simply the *origin*. A unit of length (scale unit) is chosen. It may be arbitrary but is the same for both axes.

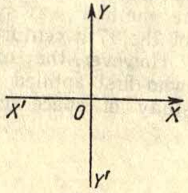

Fig. 2

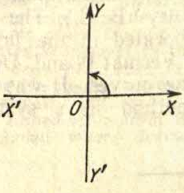

Fig. 3

On each axis a positive direction is chosen (indicated by an arrow). In Fig. 2, the ray OX is the positive direction of the x-axis and the ray OY is the positive direction of the y-axis.

It is customary to choose the positive directions (Fig. 3) so that a counterclockwise rotation of the ray OX through 90° will bring it to coincidence with the positive ray OY.

The coordinate axes $X'X$, $Y'Y$ (with established positive directions and an appropriate scale unit) form a *rectangular coordinate system*.

4. Rectangular Coordinates

The position of a point M in a plane in the rectangular coordinate system (Sec. 3) is determined as follows. Draw MP parallel to $Y'Y$ to intersection with the x-axis at the point P (Fig. 4) and MQ parallel to $X'X$ to its intersection with the y-axis at the point Q. The numbers x and y which measure the segments OP and OQ by means of the chosen scale unit (sometimes by means of the segments themselves) are called the *rectangular coordinates* (or, simply, *coordinates*) of the point M. These numbers are positive or negative depending on the directions of the segments OP and OQ. The number x is the *abscissa* of the point M and the number y is its *ordinate*.

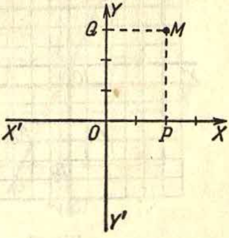

Fig. 4

In Fig. 4, the point M has abscissa $x=2$ and ordinate $y=3$ (the scale unit is 0.4 cm.) This information is usually written briefly as M (2, 3). Generally, the notation $M(a, b)$ means that the point M has abscissa $x=a$ and ordinate $y=b$.

Examples. The points indicated in Fig. 5 are designated as follows: $A_1(+2, +4)$, $A_2(-2, +4)$, $A_3(+2, -4)$, $A_4(-2, -4)$, $B_1(+5, 0)$, $B_2(0, -6)$, $O(0, 0)$.

Note. The coordinates of a given point M will be different in a different rectangular coordinate system.

5. Quadrants

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The four quadrants formed by the coordinate axes are numbered as shown in Fig. 6. The table below shows the signs of the coordinates of points in the different quadrants.

Coordinates \ Quadrant	I	II	III	IV
Abscissa	+	−	−	+
Ordinate	+	+	−	−

The point A_1 in Fig. 5 lies in the first quadrant, A_2 in the second, A_4 in the third, and the point A_3 lies in the fourth quadrant.

If a point lies on the axis of abscissas (for instance, B_1 in Fig. 5), then its ordinate y is zero. If a point lies on the axis of ordinates (point B_2, for example, in Fig. 5), then its abscissa is zero.

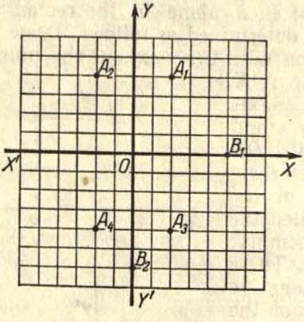

Fig. 5

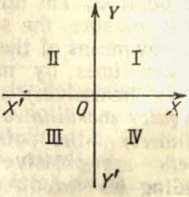

Fig. 6

6. Oblique Coordinate System

There are also other systems of coordinates besides the rectangular system. The oblique system (which most resembles the rectangular coordinate system) is constructed as follows (Fig. 7): draw two nonperpendicular straight lines $X'X$ and $Y'Y$ *(coordinate axes)* and proceed as in the construction of the rectangular coordinate system (Sec. 3). The coordinates $x=OP$ (abscissa) and $y=PM$ (ordinate) are defined as in Sec. 4.

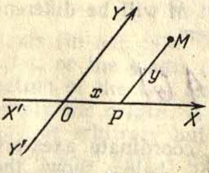

Fig. 7

The rectangular and oblique systems of coordinates come under the generic heading of the *cartesian coordinate system*.

Among coordinate systems other than the cartesian type, frequent use is made of the *polar system of coordinates* (see Sec. 73).

7. The Equation of a Line

Consider the equation $x+y=3$, which relates an abscissa x and an ordinate y. This equation is satisfied by the set of pairs of values x, y, for example, $x=1$, $y=2$, $x=2$ and $y=1$, $x=3$ and $y=0$, $x=4$ and $y=-1$, and so on. Each pair of coordinates (in the given coordinate system) is associated with a single point (Sec. 4). Fig. 8*a* depicts points $A_1(1, 2)$, $A_2(2, 1)$, $A_3(3, 0)$, $A_4(4, -1)$, all of which lie on a single straight line UV. Any other point whose coordinates satisfy the equation $x+y=3$ will also lie on the same line. Conversely, for any point lying on the straight line UV, the coordinates x, y satisfy the equation $x+y=3$.

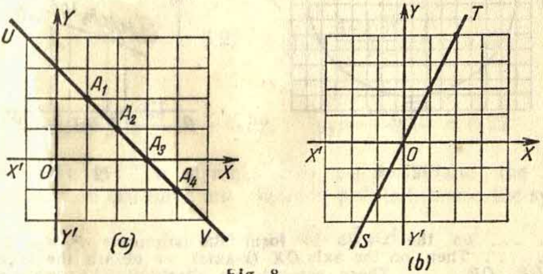

Fig. 8

Accordingly, one says that the *equation* $x+y=3$ *is the equation of the straight line UV*, or the *equation* $x+y=3$ *represents (defines) the straight line UV*. Similarly, we can say that the equation of the straight line ST (Fig. 8*b*) is $y=2x$, the equation $x^2+y^2=49$ defines a circle (Fig. 9), the radius of which contains 7 scale units and the centre of which lies at the origin of coordinates (see Sec. 38).

Generally, the *equation which relates the coordinates x and y is called the equation of the line (curve) L* provided the following two conditions hold: (1) the *coordinates x, y of any point M of the line L satisfy the equation*, (2) the *coordinates x, y of any point not lying on the line L do not satisfy the equation.*

The coordinates of an arbitrary point M on the line L are called *running (moving, or current) coordinates* since the line L can be formed by moving the point M.

In Fig. 10, let M_1, M_2, M_3, ... be consecutive positions of a point M on a line L. Drop a series of perpendiculars M_1P_1, M_2P_2, M_3P_3, ... on the x-axis to form the segments P_1M_1, P_2M_2, P_3M_3, ... Then, on the axis OX (x-axis) we obtain the segments OP_1, OP_2, OP_3, ... These segments are abscissas. The word comes from the Latin *abscindere*, meaning "to cut off". The term "ordinate" comes from the Latin *ordinatim ducta*, meaning "conducted in an orderly manner".

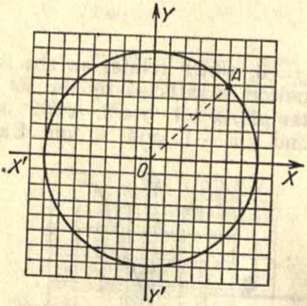

Fig. 9

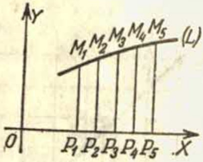

Fig. 10

By representing each point in the plane by its coordinates, and each line by an equation that relates the running coordinates, we reduce geometrical problems to analytical (computational) problems. Hence, the name *"analytic geometry"*.

8. The Mutual Positions of a Line and a Point

In order to state whether a point M lies on a certain line L, it is sufficient to know the coordinates of M and the equation of the line L. If the coordinates of M satisfy the equation of L, then M lies on L; otherwise it does not lie on L.

Example. Does the point A (5, 5) lie on the circle $x^2+y^2=49$ (Sec. 7)?

Solution. Put the values $x=5$ and $y=5$ into the equation $x^2+y^2=49$. The equation is not satisfied and so the point A does not lie on the circle.

9. The Mutual Positions of Two Lines

In order to state whether two lines have common points and if they do, how many, one has to know the equations of the lines. If the equations are simultaneous, then there are common points, otherwise there are no common points. The number of common points is equal to the number of solutions of the system of equations.

Example 1. The straight line $x+y=3$ (Sec. 7) and the circle $x^2+y^2=49$ have two points in common because the system

$$x+y=3, \quad x^2+y^2=49$$

has two solutions:

$$x_1=\frac{3+\sqrt{89}}{2}\approx 6.22, \quad y_1=\frac{3-\sqrt{89}}{2}=-3.22$$

and

$$x_2=\frac{3-\sqrt{89}}{2}\approx -3.22, \quad y_2=\frac{3+\sqrt{89}}{2}\approx 6.22$$

Example 2. The straight line $x+y=3$ and the circle $x^2+y^2=4$ do not have any common points because the system

$$x+y=3, \quad x^2+y^2=4$$

has no (real) solutions.

10. The Distance Between Two Points

The distance d between the points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ is given by the formula

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \qquad (1)$$

Example. The distance between the points $M(-2.3, 4.0)$ and $N(8.5, 0.7)$ is

$$d=\sqrt{(8.5+2.3)^2+(0.7-4)^2}=\sqrt{10.8^2+3.3^2}\approx 11.3$$

(scale units).

Note 1. The order of the points M and N is immaterial; N may be taken first and M second.

Note 2. The distance d is taken positive and so the square root in formula (1) has only one sign (positive).

11. Dividing a Line-Segment in a Given Ratio

In Fig. 11 take the points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$. It is required to find the coordinates x and y of the point K which divides the segment A_1A_2 in the ratio

$$A_1K:KA_2=m_1:m_2$$

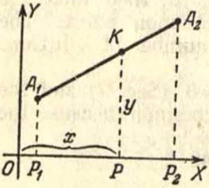

Fig. 11

The solution is given by the formulas

$$\left.\begin{aligned} x&=\frac{m_2x_1+m_1x_2}{m_1+m_2}, \\ y&=\frac{m_2y_1+m_1y_2}{m_1+m_2} \end{aligned}\right\} \qquad (1)$$

If the ratio $m_1:m_2$ is denoted by the letter λ, then (1) assumes the nonsymmetrical form

$$x=\frac{x_1+\lambda x_2}{1+\lambda}, \quad y=\frac{y_1+\lambda y_2}{1+\lambda} \qquad (2)$$

Example 1. Given the point $B(6, -4)$ and the point O coincident with the origin. Find the point K which divides BO in the ratio 2:3.

Solution. In formula (1) substitute

$$m_1=2, \quad m_2=3, \quad x_1=6, \quad y_1=-4, \quad x_2=0, \quad y_2=0$$

This yields

$$x=\frac{18}{5}=3.6, \quad y=-\frac{12}{5}=-2.4$$

which are the coordinates of the desired point K.

Note 1. The expression "the point K divides the segment A_1A_2 in the ratio $m_1:m_2$" means that the ratio $m_1:m_2$ is equal to the ratio of the segments $A_1K:KA_2$ taken in *this* order and not in the reverse order. In Example 1, the point $K\ (3.6\ -2.4)$ divides the segment BO in the ratio 2:3 and the segment OB in the ratio 3:2.

Note 2. Let the point K divide the segment A_1A_2 *externally*; that is, let the point lie on a continuation of the segment A_1A_2. Then formulas (1) and (2) hold true if we affix a minus sign to the quantity $m_1:m_2=\lambda$.

Example 2. Given the points $A_1(1, 2)$ and $A_2(3, 3)$. Find the point, on the continuation of the segment A_1A_2, that is twice as far from A_1 as from A_2.

Solution. We have $\lambda=m_1:m_2=-2$ (so that we can put $m_1=-2$, $m_2=1$, or $m_1=2$, $m_2=-1$). By formula (1) we find

$$x=\frac{1\cdot 1+(-2)\cdot 3}{-2+1}=5, \quad y=\frac{1\cdot 2+(-2)\cdot 3}{-2+1}=4$$

11a. Midpoint of a Line-Segment

The coordinates of the midpoint of a line-segment A_1A_2 are equal to the half-sums of the corresponding coordinates of its end-points:

$$x=\frac{x_1+x_2}{2}, \quad y=\frac{y_1+y_2}{2}$$

These formulas are obtained from (1) and (2), Sec. 11, by putting $m_1=m_2=1$ or $\lambda=1$.

12. Second-Order Determinant[1]

The notation $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ denotes the very same thing as $ad-bc$.

Examples.

$$\begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix}=2\cdot 5-3\cdot 7=-11,$$

$$\begin{vmatrix} 3 & -4 \\ 6 & 2 \end{vmatrix}=3\cdot 2-6\cdot(-4)=30$$

The expression $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called a *determinant of the second order.*

13. The Area of a Triangle

Let the points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $A_3(x_3, y_3)$ be the vertices of a triangle. Then the area of the triangle is given by the formula

$$S=\pm\frac{1}{2}\begin{vmatrix} x_1-x_3 & y_1-y_3 \\ x_2-x_3 & y_2-y_3 \end{vmatrix} \tag{1}$$

On the right side we have a second-order determinant (Sec. 12). We assume the area of a triangle to be positive and take the positive sign in front of the determinant if the value of the determinant is positive; we take the minus sign if it is negative.

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[1] Determinants are explained in detail in Secs. 182 to 185.

Example. Find the area of a triangle with vertices $A(1, 3)$, $B(2, -5)$ and $C(-8, 4)$.

Solution. Taking A as the first vertex, B as the second and C as the third, we find

$$\begin{vmatrix} x_1-x_3 & y_1-y_3 \\ x_2-x_3 & y_2-y_3 \end{vmatrix} = \begin{vmatrix} 1+8 & 3-4 \\ 2+8 & -5-4 \end{vmatrix} = \begin{vmatrix} 9 & -1 \\ 10 & -9 \end{vmatrix} = -81+10=-71$$

In formula (1) we take the minus sign and get

$$S=-\frac{1}{2}\cdot(-71)=35.5$$

However, if we take A for the first vertex, C for the second and B for the third, then

$$\begin{vmatrix} x_1-x_3 & y_1-y_3 \\ x_2-x_3 & y_2-y_3 \end{vmatrix} = \begin{vmatrix} 1-2 & 3+5 \\ -8-2 & 4+5 \end{vmatrix} = \begin{vmatrix} -1 & 8 \\ -10 & 9 \end{vmatrix} = 71$$

In formula (1) we have to take the plus sign, which again yields $S=35.5$.

Note. If the vertex A_3 coincides with the origin of coordinates, then the area of the triangle is given by the formula

$$S=\pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \qquad (2)$$

This is a special case of formula (1) for $x_3=y_3=0$.

14. The Straight Line. An Equation Solved for the Ordinate (Slope-Intercept Form)

Any straight line not parallel to the axis of ordinates may be represented by an equation of the form

$$y=ax+b \qquad (1)$$

Here, a is the tangent of the angle α (Fig. 12) formed by a straight line and the positive direction of the axis of abscissas [1] ($a=\tan\alpha=\tan\angle XLS$), and the number b is equal

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[1] The initial side of the angle α is the ray OX. On the straight line SS' we can take any one of the rays LS, LS'. The angle XLS is considered positive if a rotation which brings to coincidence the rays LX and LS is performed in the same direction as the rotation through 90° that brings to coincidence the axis OX and the axis OY (that is, counterclockwise in the customary arrangement).

in magnitude to the length of the segment OK intercepted by the straight line on the axis of ordinates; the number b is positive or negative depending on the direction of the segment OK. If the straight line passes through the origin, $b=0$.

The quantity a is called the *slope* and the quantity b, the *initial ordinate*.

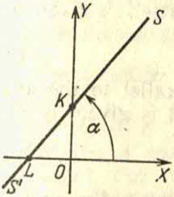

Fig. 12

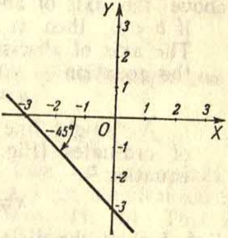

Fig. 13

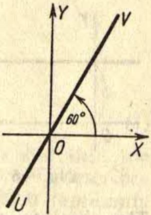

Fig. 14

Example 1. Write the equation of a straight line (Fig. 13) which forms an angle $\alpha=-45^\circ$ with the x-axis and intercepts an initial ordinate $b=-3$.

Solution. The slope $a=\tan(-45^\circ)=-1$. The desired equation is $y=-x-3$.

Example 2. What line does the equation $3x=\sqrt{3}\,y$ represent?

Solution. Solving for y we find $y=\sqrt{3}\,x$. From the slope $a=\sqrt{3}$ we find the angle α: since $\tan\alpha=\sqrt{3}$, it follows that $\alpha=60^\circ$ (or $\alpha=240^\circ$). The initial ordinate $b=0$, and so this equation represents the straight line UV (Fig. 14) which passes through the origin and forms with the x-axis an angle of 60° (or 240°).

Note 1. Unlike the other types of equations of a straight line (see Secs. 30 and 33), Eq. (1) is *solved for the ordinate and is termed the slope-intercept form of the equation of a straight line.*

Note 2. A straight line parallel to the axis of ordinates cannot be represented by an equation solved for the ordinate. Compare Sec. 15.

15. A Straight Line Parallel to an Axis

A straight line parallel to the axis of abscissas (Fig. 15) is given by the equation[1]

$$y=b \tag{1}$$

where b is equal, in absolute value, to the distance from the axis of abscissas to the straight line. If $b>0$, then the straight line lies above the axis of abscissas (see Fig. 15); if $b<0$, then it is below the axis. The axis of abscissas itself is given by the equation

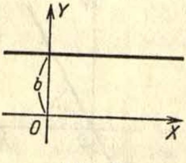

Fig. 15

$$y=0 \tag{1a}$$

A straight line parallel to the axis of ordinates (Fig. 16) is given by the equation[2]

$$x=f \tag{2}$$

The absolute value of f gives the distance from the axis of ordinates to the straight line. If $f>0$, the straight line lies to the right of the axis of ordinates (see Fig. 16); if $f<0$, it lies to the left of the axis. The axis of ordinates itself is given by the equation

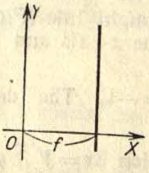

Fig. 16

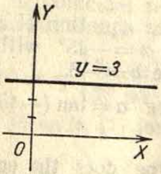

Fig. 17

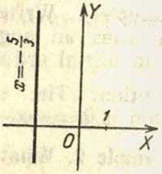

Fig. 18

$$x=0 \tag{2a}$$

Example 1. Write the equation of the straight line that intercepts the initial ordinate $b=3$ and is parallel to the x-axis (Fig. 17).

Answer. $y=3$.

---

[1] Eq. (1) is a special case of the equation $y=ax+b$ solved for the ordinate (Sec. 14). The slope $a=0$.

[2] Eq. (2) is a special case of $x=a'y+b'$ solved for the abscissa. The slope $a'=0$.

Example 2. What kind of line is given by the equation $3x+5=0$?

Solution. Solving the equation for x, we get $x=-\frac{5}{3}$. The equation represents a straight line which is parallel to the y-axis and lies to the left of it at a distance of $\frac{5}{3}$ (Fig. 18). The quantity $f=-\frac{5}{3}$ may be called the *initial abscissa*.

16. The General Equation of the Straight Line

The equation

$$Ax+By+C=0 \tag{1}$$

(where A, B, C can take on any values, provided that the coefficients A and B are not simultaneously zero[1]) describes a straight line (cf. Secs. 14, 15). This equation represents any straight line, and so it is called the *general equation of the straight line*.

If $A=0$, i.e. Eq. (1) does not contain x, then it represents a straight line parallel[2] to the x-axis (Sec. 15).

If $B=0$, i.e. Eq. (1) does not contain y, then it describes a straight line parallel[2] to the y-axis.

When B is not equal to zero, Eq. (1) may be solved for the ordinate y; then it is reduced to the form

$$y=ax+b\left(\text{where } a=-\frac{A}{B},\ b=-\frac{C}{B}\right) \tag{2}$$

Thus, the equation $2x-4y+5=0$ ($A=2$, $B=-4$, $C=5$) reduces to the form

$$y=0.5x+1.25$$

$\left(a=-\frac{2}{-4}=0.5,\ b=\frac{-5}{-4}=1.25\right)$ solved for the ordinate (initial ordinate $b=1.25$, slope $a=0.5$, so that $\alpha\approx 26°34'$; see Sec. 14).

Similarly, for $A\neq 0$ Eq. (1) may be solved for x.

If $C=0$, i.e. Eq. (1) does not contain the absolute term, it describes a straight line passing through the origin (Sec. 8).

---

[1] For $A=B=0$ we have either the identity $0=0$ (if $C=0$) or something senseless like $5=0$ (for $C\neq 0$).

[2] The x-axis is included in the group of straight lines parallel to the x-axis. The same goes for lines parallel to the y-axis (the y-axis itself is included).

17. Constructing a Straight Line on the Basis of Its Equation

To construct a straight line, it suffices to fix two of its points. For example, one can take the points of intersection with the axes (if the straight line is not parallel to any axis and does not pass through the origin); when the line is parallel to one of the axes or passes through the origin, we have only one point of intersection). For greater precision, it is advisable to find one or two check points.

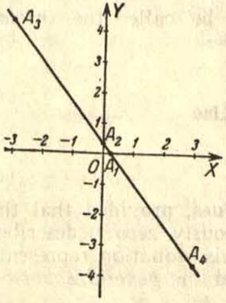

Fig. 19

Example. Construct the straight line $4x+3y=1$. Putting $y=0$, we find (Fig. 19) the point of intersection of the straight line with the axis of abscissas: $A_1\left(\frac{1}{4},\ 0\right)$. Putting $x=0$, we get the point of intersection with the axis of ordinates: $A_2\left(0,\ \frac{1}{3}\right)$. These points are too close to one another and so let us specify another two values of the abscissa, say, $x=-3$ and $x=+3$, which yield the points $A_3\left(-3,\ \frac{13}{3}\right)$, $A_4\left(3,\ -\frac{11}{3}\right)$. Draw the straight line $A_4A_1A_2A_3$.

18. The Parallelism Condition of Straight Lines

The condition that two straight lines given by the equations

$$y=a_1x+b_1, \tag{1}$$

$$y=a_2x+b_2 \tag{2}$$

be parallel is the equality of the slopes

$$a_1=a_2 \tag{3}$$

The straight lines (1) and (2) are parallel if the slopes are not equal.[1]

Example 1. The straight lines $y=3x-5$ and $y=3x+4$ are parallel since their slopes are equal ($a_1=a_2=3$).

---

[1] Here, and henceforward, two coincident straight lines are considered parallel.

Example 2. The straight lines $y=3x-5$ and $y=6x-8$ are not parallel since their slopes are not equal ($a_1=3$, $a_2=6$).

Example 3. The straight lines $2y=3x-5$ and $4y=6x-8$ are parallel since their slopes are equal $\left(a_1=\frac{3}{2}, a_2=\frac{6}{4}=\frac{3}{2}\right)$.

Note 1. If the equation of one of two straight lines does not contain an ordinate (i.e. the straight line is parallel to the y-axis), then it is parallel to the other straight line, provided that the equation of the latter does not contain y either. For example, the straight lines $2x+3=0$ and $x=5$ are parallel, but the straight lines $x-3=0$ and $x-y=0$ are not parallel.

Note 2. If two straight lines are given by the equations

$$\left.\begin{aligned}A_1x+B_1y+C_1=0,\\ A_2x+B_2y+C_2=0\end{aligned}\right\} \tag{4}$$

then the condition of parallelism is

$$A_1B_2-A_2B_1=0 \tag{5}$$

or, in the notation of Sec. 12,

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}=0$$

Example 4. The straight lines

$$2x-7y+12=0$$

and

$$x-3.5y+10=0$$

are parallel since

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}=\begin{vmatrix} 2 & -7 \\ 1 & -3.5 \end{vmatrix}=2\cdot(-3.5)-1\cdot(-7)=0$$

Example 5. The straight lines

$$2x-7y+12=0$$

and

$$3x+2y-6=0$$

are not parallel since

$$\begin{vmatrix} 2 & -7 \\ 3 & 2 \end{vmatrix}=25\neq 0$$

Note 3. Equality (5) may be written as

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}\,. \tag{6}$$

which states that the *condition for the straight lines (4) being parallel is the proportionality of the coefficients of the running coordinates.*[1] Compare Examples 4 and 5. If the absolute terms are proportional as well, i. e. if

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2} \tag{7}$$

then the straight lines (4) are not only parallel but are also coincident. Thus, the equations

$$3x+2y-6=0$$

and

$$6x+4y-12=0$$

describe one and the same straight line.

19. The Intersection of Straight Lines

To find the point of intersection of the straight lines

$$A_1x+B_1y+C_1=0 \tag{1}$$

and

$$A_2x+B_2y+C_2=0 \tag{2}$$

it is necessary to solve the system of equations (1) and (2). As a rule, this system yields a unique solution and we obtain the desired point (Sec. 9). The only possible exception is the equality of the ratios $\frac{A_1}{A_2}$ and $\frac{B_1}{B_2}$, i. e. when the straight lines are parallel (see Sec. 18, Notes 2 and 3).

Note. If the given straight lines are parallel and do not coincide, then the system (1)-(2) has no solution; if they coincide, there is an infinity of solutions.

Example 1. Find the points of intersection of the straight lines $y=2x-3$ and $y=-3x+2$. Solving the system of equations, we find $x=1$, $y=-1$. The straight lines intersect at the point $(1, -1)$.

Example 2. The straight lines

$$2x-7y+12=0, \quad x-3.5y+10=0$$

are parallel and do not coincide since the ratios 2:1 and (−7):(−3.5) are equal, but they are not equal to the ratio

---

[1] It may turn out that one of the quantities A_2 or B_2 (but not both together, see Sec. 16) is equal to zero. Then the proportion (6) may be understood in the meaning that the corresponding numerator is also zero. The proportion (7) has the same meaning for $C_2=0$

12:10 (cf. Example 4, Sec. 18). The given system of equations has no solution.

Example 3. The straight lines $3x+2y-6=0$ and $6x+4y-12=0$ coincide since the ratios 3:6, 2:4 and (−6):(−12) are equal. The second equation is obtained from the first by multiplying by 2. This system has an infinity of solutions.

20. The Perpendicularity Condition of Two Straight Lines

The condition that two straight lines given by the equations

$$y=a_1x+b_1, \tag{1}$$
$$y=a_2x+b_2 \tag{2}$$

be perpendicular is the relation

$$a_1a_2=-1 \tag{3}$$

which states that two straight lines are perpendicular if the product of their slopes is equal to −1, and they are not perpendicular if the product is not equal to −1.

Example 1. The straight lines $y=3x$ and $y=-\frac{1}{3}x$ are perpendicular since $a_1a_2=3\cdot\left(-\frac{1}{3}\right)=-1$.

Example 2. The straight lines $y=3x$ and $y=\frac{1}{3}x$ are not perpendicular since $a_1a_2=3\cdot\frac{1}{3}=1$.

Note 1. If the equation of one of the two straight lines does not contain an ordinate (i. e. the straight line is parallel to the y-axis), then it is perpendicular to the other straight line provided that the equation of the latter does not contain an abscissa (then the second straight line is parallel to the axis of abscissas), otherwise the straight lines are not perpendicular. For example, the straight lines $x=5$ and $3y+2=0$ are perpendicular and the straight lines $x=5$, and $y=2x$ are not perpendicular.

Note 2. If two straight lines are given by the equations

$$A_1x+B_1y+C_1=0, \quad A_2x+B_2y+C_2=0 \tag{4}$$

then the condition for their being perpendicular is

$$A_1A_2+B_1B_2=0 \tag{5}$$

Example 3. The straight lines $2x+5y=8$ and $5x-2y=3$ are perpendicular; indeed, $A_1=2$, $A_2=5$, $B_1=5$, $B_2=-2$, and so $A_1A_2+B_1B_2=10-10=0$.

Example 4. The straight lines $\frac{1}{2}x-\frac{1}{3}y=0$ and $2x-3y=0$ are not perpendicular since $A_1A_2+B_1B_2=2$.

21. The Angle Between Two Straight Lines

Let two nonperpendicular straight lines L_1, L_2 (taken in a specific order) be given by the equations

$$y=a_1x+b_1, \tag{1}$$

$$y=a_2x+b_2. \tag{2}$$

Then the formula [1]

$$\tan\Theta=\frac{a_2-a_1}{1+a_1a_2} \tag{3}$$

yields the angle through which the first straight line must be rotated in order to make it parallel to the second line.

Example 1. Find the angle between the straight lines $y=2x-3$ and $y=-3x+2$ (Fig. 20.)

Here, $a_1=2$, $a_2=-3$. By formula (3), we find

$$\tan\Theta=\frac{-3-2}{1+2\cdot(-3)}=1$$

whence $\Theta=+45°$. This means that when the straight line $y=2x-3$ (AB in Fig. 20) is turned through the angle $+45°$ about the point of intersection $M(1, -1)$ of the given straight lines (Example 1, Sec. 19), it will coincide with the straight line $y=-3x+2$ (CD in Fig. 20). It is also possible to take $\Theta=180°+45°=225°$, $\Theta=-180°+45°=-135°$, and so on. (These angles are denoted by Θ_1, Θ_2 in Fig. 20).

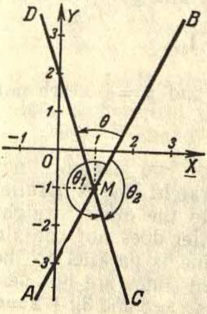

Fig. 20

Example 2. Find the angle between the straight lines $y=-3x+2$ and $y=2x-3$. Here, the lines are the same as in Example 1, but the straight line CD (see Fig. 20) is the first one and AB is the second. Formula (3) yields $\tan\Theta=-1$,

---

[1] On its applicability when the straight lines L_1, L_2 are perpendicular, see Note 1 below.

i. e. $\Theta=-45°$ (or $\Theta=135°$ or $\Theta=-225°$, etc.). This is the angle through which the straight line CD must be rotated to bring it into coincidence with AB.

Example 3. Find the straight line that passes through the origin and intersects the straight line $y=2x-3$ at an angle of 45°.

Solution. The sought-for straight line is given by the equation $y=ax$ (Sec. 14). The slope a may be found from (3) by taking the slope of the given straight line in place of a_1 (i. e. by putting $a_1=2$); in place of a_2 we take the slope a of the desired straight line, and in place of Θ, an angle of $+45°$ or $-45°$. We then get

$$\frac{a-2}{1+2a}=\pm 1$$

The problem has two solutions: $y=-3x$ (the straight line AB in Fig. 21) and $y=\frac{1}{3}x$ (the straight line CD).

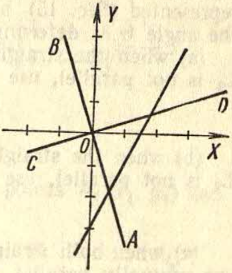

Fig. 21

Note 1. If the straight lines (1) and (2) are perpendicular ($\Theta=\pm 90°$), then the expression $1+a_1a_2$ in the denominator of (3) vanishes (Sec. 20) and the quotient $\frac{a_2-a_1}{1+a_1a_2}$ ceases to exist.[1] At the same time, $\tan\Theta$ ceases to exist (becomes infinite). Taken literally, formula (3) is meaningless; in this case it has a conventional meaning, namely that each time the denominator of (3) vanishes the angle Θ is to be considered $\pm 90°$ (both a rotation through $+90°$ and one through $-90°$ brings either of the perpendicular straight lines to coincidence with the other).

Example 4. Find the angle between the straight lines $y=2x-3$ and $y=-\frac{1}{2}x+7$ $\left(a_1=2,\ a_2=-\frac{1}{2}\right)$. If we first ask whether these straight lines are perpendicular, the answer is yes by the characteristic (3) of Sec. 20 so that we obtain $\Theta=\pm 90°$ even without formula (3). Formula (3) yields the same result. We get

---

[1] The numerator a_2-a_1 is not zero since the slopes a_1, a_2 (Sec. 18) are equal only in the case of parallel straight lines.

$$\tan\Theta=\frac{-\frac{1}{2}-2}{1+\left(-\frac{1}{2}\right)\cdot 2}=\frac{-2\frac{1}{2}}{0}$$

In accordance with Note 1, this equality is to be understood in the meaning that $\Theta=\pm 90°$.

Note 2. If even one of the straight lines L_1, L_2 (or both) is parallel to the y-axis, then formula (3) cannot be applied because then one of the straight lines (or both) cannot be represented (Sec. 15) by an equation of the form (1). Then the angle Θ is determined in the following manner:

(a) when the straight line L_2 is parallel to the y-axis and L_1 is not parallel, use the formula

$$\tan\Theta=\frac{1}{a_1}$$

(b) when the straight line L_1 is parallel to the y-axis and L_2 is not parallel, use the formula

$$\tan\Theta=-\frac{1}{a_1}$$

(c) when both straight lines are parallel to the y-axis, they are mutually parallel, so that $\tan\Theta=0$.

Note 3. The angle between the straight lines given by the equations

$$A_1x+B_1y+C_1=0 \tag{4}$$

and

$$A_2x+B_2y+C_2=0 \tag{5}$$

may be found from the formula

$$\tan\Theta=\frac{A_1B_2-A_2B_1}{A_1A_2+B_1B_2} \tag{6}$$

When $A_1A_2+B_1B_2=0$, formula (6) is given a conventional meaning (see Note 1) and $\Theta=\pm 90°$. Compare Sec. 20, formula (5).

22. The Condition for Three Points Lying on One Straight Line

The three points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $A_3(x_3, y_3)$ lie on one straight line if and only if[1]

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}=0 \tag{1}$$

---

[1] The left side of (1) is written in the form of a determinant (see Sec. 12).

This formula also states (Sec. 13) that the area of the "triangle" $A_2A_3A_1$ is zero.

Example 1. The points $A_1(-2, 5)$, $A_2(4, 3)$, $A_3(16, -1)$ lie on one straight line since

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix} = \begin{vmatrix} 4+2 & 3-5 \\ 16+2 & -1-5 \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 18 & -6 \end{vmatrix} =$$
$$=6\cdot(-6)-(-2)\cdot 18=0$$

Example 2. The points $A_1(-2, 6)$, $A_2(2, 5)$, $A_3(5, 3)$ do not lie on one straight line since

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix} = \begin{vmatrix} 2+2 & 5-6 \\ 5+2 & 3-6 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 7 & -3 \end{vmatrix} = -5$$

23. The Equation of a Straight Line Through Two Points (Two-Point Form)

A straight line passing through two points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ is given by the equation[1]

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x-x_1 & y-y_1 \end{vmatrix} = 0 \qquad (1)$$

It states that the given points A_1, A_2 and the variable point $A(x, y)$ lie on one straight line (Sec. 22).

Eq. (1) may be represented (see note below) in the form

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \qquad (2)$$

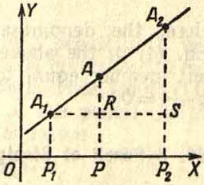

Fig. 22

This equation expresses the proportionality of the sides of the right triangles A_1RA and A_1SA_2 depicted in Fig. 22, where

$$x_1=OP_1, \quad x_2=OP_2, \quad x=OP,$$
$$x-x_1=A_1R, \quad x_2-x_1=A_1S;$$
$$y_1=P_1A_1, \quad y_2=P_2A_2, \quad y=PA,$$
$$y-y_1=RA, \quad y_2-y_1=SA_2$$

Example 1. Form the equation of the straight line passing through the points (1, 5) and (3, 9).

---

[1] The left side of (1) is written in the form of a determinant (see Sec. 12).

Solution. Formula (1) gives

$$\begin{vmatrix} 3-1 & 9-5 \\ x-1 & y-5 \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} 2 & 4 \\ x-1 & y-5 \end{vmatrix} = 0$$

that is, $2(y-5)-4(x-1)=0$ or $2x-y+3=0$.

Formula (2) yields $\frac{x-1}{2}=\frac{y-5}{4}$. Whence we again get $2x-y+3=0$.

Note. When $x_2=x_1$ (or $y_2=y_1$), one of the denominators of (2) is zero; then Eq. (2) should be taken to mean that the corresponding numerator is zero. See Example 2 below (also the footnote on page 34).

Example 2. Form the equation of a straight line that passes through the points $A_1(4, -2)$ and $A_2(4, 5)$. Eq. (1) yields

$$\begin{vmatrix} 0 & 7 \\ x-4 & y+2 \end{vmatrix} = 0 \tag{3}$$

i. e. $0(y+2)-7(x-4)=0$, or $x-4=0$.

Eq. (2) is written as

$$\frac{x-4}{0}=\frac{y+2}{7} \tag{4}$$

Here, the denominator of the left member is zero. Taking Eq. (4) in the above meaning, we put the numerator of the left member equal to zero and we obtain the same result: $x-4=0$.

24. A Pencil of Straight Lines

The collection of lines passing through one point $A_1(x_1, y_1)$ (Fig. 23) is termed a *(central) pencil of lines* through a point.

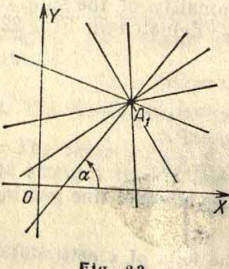

Fig. 23

The point A_1 is called the *vertex of the pencil*. Each one of the lines of the pencil (with the exception of that which is parallel to the axis of ordinates; see Note 1) may be represented by the equation

$$y-y_1=k(x-x_1) \tag{1}$$

Here, k is the slope of the line under consideration ($k=\tan\alpha$). Eq. (1) is called the *equation of the pencil*. The quantity k (the *parameter* of the pencil) characterizes the direction of the line; it varies from one line of the pencil to the next.

The value of the parameter k may be found if some other condition is given which (together with the condition that the line belong to the pencil) defines the position of the line; see Example 2.

Example 1. Form the equation of a pencil with vertex at the point $A_1(-4, -8)$.

Solution. By (1) we have

$$y+8=k(x+4)$$

Example 2. Find the equation of a straight line that passes through the point $A_1(1, 4)$ and is perpendicular to the straight line $3x-2y=12$.

Solution. The desired line belongs to a pencil with vertex (1, 4). The equation of the pencil is $y-4=k(x-1)$. To find the value of the parameter k, note that the desired line is perpendicular to the straight line $3x-2y=12$; the slope of the latter is $\frac{3}{2}$. We have (Sec. 20) $\frac{3}{2}k=-1$, i. e. $k=-\frac{2}{3}$. The desired line is given by the equation $y-4=-\frac{2}{3}(x-1)$ or $y=-\frac{2}{3}x+4\frac{2}{3}$.

Note 1. A straight line belonging to a pencil with vertex at $A_1(x_1, y_1)$ and parallel to the y-axis is given by the equation $x-x_1=0$. This equation is not obtainable from (1), no matter what the value of k. *All* lines of the pencil (*without exception*) may be represented by the equation

$$l(y-y_1)=m(x-x_1) \tag{2}$$

where l and m are arbitrary numbers (not equal to zero simultaneously). When $l \neq 0$, we can divide Eq. (2) by l. Then, denoting $\frac{m}{l}$ in terms of k, we get (1). But if we put $l=0$, then Eq. (2) takes the form $x-x_1=0$.

Note 2. The equation of a pencil containing two intersecting straight lines L_1, L_2 given by the equations

$$A_1x+B_1y+C_1=0, \quad A_2x+B_2y+C_2=0$$

is of the form

$$m_1(A_1x+B_1y+C_1)+m_2(A_2x+B_2y+C_2)=0 \tag{3}$$

Here, m_1, m_2 are arbitrary numbers (not simultaneously zero). In particular, for $m_1=0$ we get the line L_2, for $m_2=0$ we have the line L_1. In place of (3) we can write the equation

$$A_1x+B_1y+C_1+\lambda(A_2x+B_2y+C_2)=0 \tag{4}$$

in which all possible values are given to only one letter λ, but it is not possible to obtain the equation of the line L_2 from (4).

Eq. (1) is a special case of Eq. (4) when the straight lines L_1 and L_2 are given by the equations $y=y_1$, $x=x_1$ (they are then parallel to the axes of coordinates).

Example 3. Form the equation of a straight line which passes through the point of intersection of the lines $2x-3y-1=0$, $3x-y-2=0$ and is perpendicular to the straight line $y=x$.

Solution. The desired line (which definitely does not coincide with the line $3x-y-2=0$) belongs to the pencil

$$2x-3y-1+\lambda(3x-y-2)=0 \tag{5}$$

The slope of the line (5) is $k=\frac{3\lambda+2}{\lambda+3}$. Since the desired line is perpendicular to the line $y=x$, it follows (Sec. 20) that $k=-1$. Hence, $\frac{3\lambda+2}{\lambda+3}=-1$, i. e. $\lambda=-\frac{5}{4}$. Substituting $\lambda=-\frac{5}{4}$ into (5), we get (after simplifications)

$$7x+7y-6=0$$

Note 3. If the lines L_1, L_2 are parallel (but noncoincident), Eq. (3) represents, for all possible values of m_1, m_2, all straight lines parallel to the two given lines. A set of mutually parallel straight lines is termed a *pencil of parallel lines* (parallel pencil). Thus, Eq. (3) represents either a central pencil or a parallel pencil.

25. The Equation of a Straight Line Through a Given Point and Parallel to a Given Straight Line (Point-Slope Form)

1. A straight line passing through a point $M_1(x_1, y_1)$ parallel to a straight line $y=ax+b$ is given by the equation

$$y-y_1=a(x-x_1) \tag{1}$$

Cf. Sec. 24.

Example 1. Form the equation of a straight line which passes through the point $(-2, 5)$ and is parallel to the straight line

$$5x-7y-4=0$$

Solution. The given line may be represented by the equation $y=\frac{5}{7}x-\frac{4}{7}$ $\left(\text{here } a=\frac{5}{7}\right)$. The equation of the line is $y-5=\frac{5}{7}[x-(-2)]$ or $7(y-5)=5(x+2)$ or $5x-7y+45=0$.

2. A straight line which passes through a point $M_1(x_1, y_1)$ and is parallel to the straight line $Ax+By+C=0$ is given by the equation

$$A(x-x_1)+B(y-y_1)=0 \tag{2}$$

Example 2. Solving Example 1 ($A=5$, $B=-7$) by formula (2), we find $5(x+2)-7(y-5)=0$.

Example 3. Form the equation of a straight line which passes through the point $(-2, 5)$ and is parallel to the straight line $7x+10=0$.

Solution. Here $A=7$, $B=0$. Formula (2) yields $7(x+2)=0$, or $x+2=0$. Formula (1) is not applicable since the given equation cannot be solved for y (the given straight line is parallel to the y-axis, cf. Sec. 15).

26. The Equation of a Straight Line Through a Given Point and Perpendicular to a Given Straight Line

1. A straight line which passes through a point $M_1(x_1, y_1)$ and is perpendicular to a straight line $y=ax+b$ is given by the equation

$$y-y_1=-\frac{1}{a}(x-x_1) \tag{1}$$

Cf. Sec. 24, Example 2.

Example 1. Form the equation of a straight line which passes through the point $(2, -1)$ and is perpendicular to the straight line

$$4x-9y=3$$

Solution. The given line may be represented by the equation $y=\frac{4}{9}x-\frac{1}{3}$ $\left(a=\frac{4}{9}\right)$. The equation of the desired line is $y+1=-\frac{9}{4}(x-2)$ or $9x+4y-14=0$.

2. A straight line that passes through a point $M_1(x_1, y_1)$ and is perpendicular to the straight line $Ax+By+C=0$ is given by the equation

$$A(y-y_1)-B(x-x_1)=0 \tag{2}$$

Example 2. Solving Example 1 ($A=4$, $B=-9$) by formula (2), we find $4(y+1)+9(x-2)=0$ or $9x+4y-14=0$.

Example 3. Form the equation of a straight line passing through the point $(-3, -2)$ perpendicular to the straight line

$$2y+1=0$$

Solution. Here, $A=0$, $B=2$ Formula (2) yields $-2(x+3)=0$ or $x+3=0$. Formula (1) cannot be used because $a=0$ (cf. Sec. 20, Note 1).

27. The Mutual Positions of a Straight Line and a Pair of Points

The mutual positions of points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ and a straight line

$$Ax+By+C=0 \qquad (1)$$

may be determined from the following characteristics:

(a) points M_1 and M_2 lie on one side of the line (1) when the numbers $Ax_1+By_1+C_1$, $Ax_2+By_2+C_2$ have the same sign;

(b) M_1 and M_2 are on different sides of line (1) when these numbers have opposite signs;

(c) one of the points M_1, M_2 (or both) lies on the line (1) if one of these numbers is zero or if both are zero.

Example 1. The points $(2, -6)$, $(-4, -2)$ lie on the same side of the straight line

$$3x+5y-1=0$$

since the numbers $3\cdot2+5\cdot(-6)-1=-25$ and $3\cdot(-4)+5\cdot(-2)-1=-23$ are both negative.

Example 2. The origin of coordinates $(0, 0)$ and the point $(5, 5)$ lie on different sides of the straight line $x+y-8=0$ since the numbers $0+0-8=-8$ and $5+5-8=+2$ have different signs.

28. The Distance From a Point to a Straight Line

The distance d from a point $M_1(x_1, y_1)$ to a straight line

$$Ax+By+C=0 \qquad (1)$$

is equal to the absolute value of

$$\delta=\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}} \qquad (2)$$

that is, [1]

$$d=|\delta|=\left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right| \qquad (3)$$

Example. Find the distance from the point $(-1, +1)$ to the straight line

$$3x-4y+5=0$$

Solution.

$$\delta=\frac{3x_1-4y_1+5}{\sqrt{3^2+4^2}}=\frac{3\cdot(-1)-4\cdot1+5}{\sqrt{3^2+4^2}}=-\frac{2}{5}$$

$$d=|\delta|=\left|-\frac{2}{5}\right|=\frac{2}{5}$$

---

[1] Formula (3) is ordinarily derived by means of an artificial construction. Below (See Note 2) is given a purely analytical derivation.

Note 1. Suppose the line (1) does not pass through the origin 0 and, hence, $C \neq 0$ (Sec. 16). Then, if the signs of δ and C are the same, the points M_1 and O lie to one side of the line (1); if the signs are opposite, then they lie on different sides (cf. Sec. 27). But if $\delta=0$ (this is only possible if $Ax_1+By_1+C=0$), then M_1 lies on the given straight line (Sec. 8).

The quantity δ is called the *oriented distance* from the point M_1 to the line (1). In the example above, the oriented distance δ is equal to $-\frac{2}{5}$, and $C=5$. The quantities δ and C have opposite signs, hence, the points $M_1(-1, +1)$ and O lie on different sides of the straight line $3x-4y+5=0$.

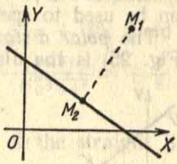

Fig. 24

Note 2. The simplest way to derive formula (3) is as follows.

Let $M_2(x_2, y_2)$ (Fig. 24) be the foot of a perpendicular dropped from the point $M_1(x_1, y_1)$ onto the straight line (1). Then

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \tag{4}$$

The coordinates x_2, y_2 are found as the solution of the following system of equations:

$$Ax+By+C=0, \tag{1}$$

$$A(y-y_1)-B(x-x_1)=0 \tag{5}$$

where the latter equation defines a straight line M_1M_2 (Sec. 26). To simplify computations, transform the first equation of the system to the form

$$A(x-x_1)+B(y-y_1)+Ax_1+By_1+C=0 \tag{6}$$

Solving (5) and (6) for $(x-x_1)$, $(y-y_1)$, we find

$$x-x_1=-\frac{A}{A^2+B^2}(Ax_1+By_1+C), \tag{7}$$

$$y-y_1=-\frac{B}{A^2+B^2}(Ax_1+By_1+C) \tag{8}$$

Putting (7) and (8) into (4), we get

$$d=\left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$$

29. The Polar Parameters (Coordinates) of a Straight Line[1]

The position of a straight line in a plane may be given by two numbers called the *parameters (coordinates)* of the line. For example, the numbers b (initial ordinate) and a

---

[1] This section serves as an introduction to Secs. 30 and 31.

(slope) are (cf. Sec. 14) the parameters of the straight line. However, the parameters b and a are not suitable for all straight lines; they do not specify a straight line parallel to OY (Sec. 15). In contrast, polar parameters (see below) can be used to specify the position of *any* straight line.

The *polar distance* (or *radius vector*) of a straight line UV (Fig. 25) is the distance p of the perpendicular OK drawn from the origin O to the straight line. The polar distance is positive or zero ($p \geqslant 0$).

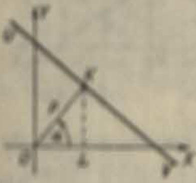

Fig. 25

The *polar angle* of the straight line UV is the angle $\alpha = \angle XOK$ between the rays OX and OK (taken in that order; cf. Sec. 21). If the line UV does not pass through the origin (as in Fig. 25), then the direction of the second ray is quite definite (from O to K); but if UV passes through O (then O and K coincide), the ray perpendicular to UV is drawn in any one of two possible directions.

The polar distance and the polar angle are termed the *polar parameters (or polar coordinates) of a straight line.*

If the straight line UV is given by the equation

$$Ax + By + C = 0$$

then its polar distance is defined by the formula

$$p = \frac{|C|}{\sqrt{A^2+B^2}} \tag{1}$$

and the polar angle α by the formulas

$$\cos\alpha = \pm\frac{A}{\sqrt{A^2+B^2}}, \quad \sin\alpha = \pm\frac{B}{\sqrt{A^2+B^2}} \tag{2}$$

where the upper signs are taken for $C > 0$, and the lower signs for $C < 0$; but if $C = 0$, then either only the upper signs or only the lower signs[1] are taken at will.

---

[1] Formula (1) is obtained from (1), Sec. 24 (for $x_1 = y_1 = 0$). Formulas (2) are obtained as follows: from Fig. 25

$$\cos\alpha = \frac{OL}{OK} = \frac{x}{p}, \quad \sin\alpha = \frac{LK}{OK} = \frac{y}{p} \tag{3}$$

According to (7), (8), Sec. 28 (for $x_1 = y_1 = 0$), we have

$$x = -\frac{AC}{A^2+B^2}, \quad y = -\frac{BC}{A^2+B^2} \text{ (cont'd on p. 47)} \tag{4}$$

Example 1. Find the polar parameters of the straight line

$$3x-4y+10=0$$

Solution. Formula (1) yields $p=\frac{10}{\sqrt{3^2+4^2}}=2$. Formulas (2), where the upper signs are taken (because $C=+10$), yield

$$\cos\alpha=-\frac{3}{\sqrt{3^2+4^2}}=-\frac{3}{5},\quad \sin\alpha=-\frac{(-4)}{\sqrt{3^2+4^2}}=+\frac{4}{5}$$

Hence, $\alpha\approx127^\circ$ (or $\alpha\approx487^\circ$, etc.).

Example 2. Find the polar parameters of the straight line

$$3x-4y=0$$

Formula (1) yields $p=0$; in formulas (2) we can take either only the upper or only the lower signs. In the former case, $\cos\alpha=-\frac{3}{5}$, $\sin\alpha=\frac{4}{5}$ and, hence, $\alpha\approx127^\circ$; in the latter case, $\cos\alpha=\frac{3}{5}$, $\sin\alpha=-\frac{4}{5}$ and, hence, $\alpha\approx-53^\circ$.

30. The Normal Equation of a Straight Line

A straight line with polar distance p (Sec. 29) and polar angle α is given by the equation

$$x\cos\alpha+y\sin\alpha-p=0 \qquad (1)$$

This is the *normal form of the equation of a straight line.*

Example. Let a straight line UV be distant from the origin

$$OK=\sqrt{2}$$

(Fig. 26) and let the ray OK make an angle $\alpha=225^\circ$ with the ray OX. Then the normal equation of UV is

$$x\cos225^\circ+y\sin225^\circ-\sqrt{2}=0$$

that is,

$$-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y-\sqrt{2}=0$$

From (1), (3) and (4), it follows that

$$\cos\alpha=-\frac{C}{|C|}\cdot\frac{A}{\sqrt{A^2+B^2}},\quad \sin\alpha=-\frac{C}{|C|}\cdot\frac{B}{\sqrt{A^2+B^2}} \qquad (5)$$

Formulas (5) coincide with (2) because $\frac{C}{|C|}=+1$ for $C>0$ and $\frac{C}{|C|}=-1$ for $C<0$.

Multiplying by $-\sqrt{2}$, we get the equation of UV in the form $x+y+2=0$, but this equation is no longer in the normal form.

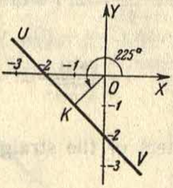

Fig. 26

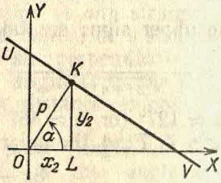

Fig. 27

Derivation of equation (1). Denote the coordinates of the point K (Fig. 27) by x_2, y_2. Then $x_2=OL=p\cos\alpha$, $y_2=LK=p\sin\alpha$. The straight line OK that passes through the points $O\ (0,\ 0)$ and $K\ (x_2,\ y_2)$ is given (Sec. 23) by the equation $\begin{vmatrix} x_2 & y_2 \\ x & y \end{vmatrix}=0$, that is, $(\sin\alpha)\,x-(\cos\alpha)\,y=0$. The line UV passes through $K\ (x_2,\ y_2)$ and is perpendicular to the straight line OK. Hence, (Sec. 26, Item 2), it is given by the equation $\sin\alpha\,(y-y_2)-(-\cos\alpha)\,(x-x_2)=0$. Substituting $x_2=p\cos\alpha$ and $y_2=p\sin\alpha$, we get $x\cos\alpha+y\sin\alpha-p=0$.

31. Reducing the Equation of a Straight Line to the Normal Form

In order to find the normal equation of a straight line given by the equation $Ax+By+C=0$, it is sufficient to divide the given equation by $\mp\sqrt{A^2+B^2}$, the upper sign being taken when $C>0$ and the lower sign when $C<0$; but if $C=0$, any sign is valid. We get the equation

$$\mp\frac{A}{\sqrt{A^2+B^2}}x\mp\frac{B}{\sqrt{A^2+B^2}}y-\frac{|C|}{\sqrt{A^2+B^2}}=0$$

It will be normal.[1]

Example 1. Reduce the equation $3x-4y+10=0$ to the normal form.

Here, $A=3$, $B=-4$ and $C=10>0$. Therefore, divide by $-\sqrt{3^2+4^2}=-5$ to get

$$-\frac{3}{5}x+\frac{4}{5}y-2=0$$

---

[1] Because the coefficients of x and y are, respectively, $\cos\alpha$ and $\sin\alpha$ by virtue of (2), Sec. 29, and the constant term is equal to $(-p)$ by (1), Sec. 29.

This is an equation of the form $x\cos\alpha+y\sin\alpha-p=0$. Namely, $p=2$, $\cos\alpha=-\frac{3}{5}$, $\sin\alpha=+\frac{4}{5}$ (hence, $\alpha\approx 127°$).

Example 2. Reduce the equation $3x-4y=0$ to the normal form.

Since $C=0$ here, it is possible to divide either by 5 or -5. In the former case we get

$$\frac{3}{5}x-\frac{4}{5}y=0$$

($p=0$, $\alpha\approx 307°$), in the latter case,

$$-\frac{3}{5}x+\frac{4}{5}y=0$$

($p=0$, $\alpha\approx 127°$). To the two values of α there correspond two methods of choosing the positive direction on the ray OK (see Sec. 29).

32. Intercepts

To find the line segment $OL=a$ (Fig. 28) intercepted on the x-axis by the straight line UV, it is sufficient to put $y=0$ in the equation of the straight line and solve the equation for x. In similar fashion we find the line segment $ON=b$ on the y-axis. The values of a and b can be either positive or negative. If the straight line is parallel to one of the axes, the corresponding line segment does not exist (becomes infinite). If the straight line passes through the origin, each line segment degenerates into a point ($a=b=0$).

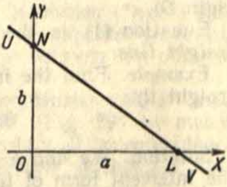

Fig. 28

Example 1. Find the line segments a, b intercepted by the straight line $3x-2y+12=0$ on the axes.

Solution. Set $y=0$ and from the equation $3x+12=0$ find $x=-4$. Putting $x=0$, we get $y=6$ from $-2y+12=0$. Thus, $a=-4$, $b=6$.

Example 2. Find the line segments a and b intercepted on the axes by the straight line

$$5y+15=0$$

Solution. This line is parallel to the axis of abscissas (Sec. 15). The line segment a is nonexistent (putting $y=0$, we get a contradictory relation: $15=0$). The segment b is equal to -3.

Example 3. Find the line segments a and b intercepted on the axes by the straight line

$$3y-2x=0$$

Solution. Using the method given here, we find $a=0$, $b=0$. The end of each of the "segments" coincides with its beginning, which means the line segment has degenerated into a point. The straight line passes through the origin (cf. Sec. 14).

Sec. 33. Intercept Form of the Equation of a Straight Line

If a straight line intercepts, on the coordinate axes, line segments a, b (not equal to zero), then it may be given by the equation

$$\frac{x}{a}+\frac{y}{b}=1 \tag{1}$$

Conversely, Eq. (1) describes a straight line intercepting on the axes the line segments a, b (reckoning from the origin O).

Equation (1) is the *intercept form of the equation of a straight line*.

Example. Find the intercept form of the equation of the straight line

$$3x-2y+12=0 \tag{2}$$

Solution. We find $a=-4$, $b=6$ (see Sec. 32, Example 1). The intercept form of the equation is

$$\frac{x}{-4}+\frac{y}{6}=1 \tag{3}$$

It is equivalent to Eq. (2).

Note 1. A straight line that intercepts on the axes line segments equal to zero (that is, such that passes through the origin: see Example 3 in Sec. 32) cannot be represented by the intercept form of the equation of a straight line.

Note 2. A straight line parallel to the x-axis (Example 2, Sec. 32) can be represented by the equation $\frac{y}{b}=1$, where b is the y-intercept. Similarly, a straight line parallel to the y-axis may be given by the equation $\frac{x}{a}=1$. There is no generally accepted opinion in the literature as to whether to regard these equations as intercept forms or not.[1]

34. Transformation of Coordinates (Statement of the Problem)

One and the same line is described by different equations in different coordinate systems. Frequently, if we know the equation of some line in one ("*old*") coordinate system, it is required to find the equation of the line in another ("*new*") system. *Formulas for the transformation of coordinates* serve this purpose. They establish a relationship between the old and new coordinates of some point M.

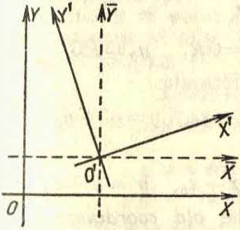

Fig. 29

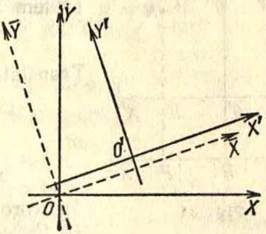

Fig. 30

Any new system of rectangular coordinates $X'O'Y'$ may be obtained from any old system XOY (Fig. 29) by means of two motions: (1) first bring the origin O to coincidence with O', holding the directions of the axes unchanged; this yields an auxiliary system $\bar{X}O'\bar{Y}$ (shown dashed); (2) then rotate the auxiliary system about the point O' to coincidence with the new system $X'O'Y'$.

These two motions may be executed in reverse order (first a rotation about O yielding the auxiliary system $\bar{X}O\bar{Y}$ and then a translation of the origin to the point O', which gives the new system $X'O'Y'$; Fig. 30).

---

[1] The essential thing is that the equation $\frac{x}{a}=1$ or $\frac{y}{b}=1$ may be obtained from the equation $\frac{x}{a}+\frac{y}{b}=1$; however not as a particular case but by passing to the limit as b or a go to infinity.

Thus, it is sufficient to know the formulas of coordinate transformation in translation of the origin (Sec. 35) and rotation of the axes (Sec. 36).

35. Translation of the Origin

Notation (Fig. 31):
old coordinates of point $M: x=OP$, $y=PM$;
new coordinates of point $M: x'=O'P'$, $y'=P'M$;
coordinates of new origin O' in old system XOY:

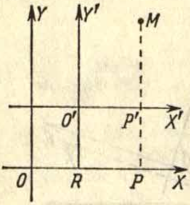

Fig. 31

$$x_0=OR, \quad y_0=RO'$$

Translation formulas:

$$x=x'+x_0, \quad y=y'+y_0 \qquad (1)$$

or

$$x'=x-x_0, \quad y'=y-y_0 \qquad (2)$$

In words, the *old coordinate is equal to the new one combined with the coordinate of the new origin* (in the old system).[1]

Example 1. The coordinate origin is translated to the point (2, —5). Find the new coordinates of the point $M(-3, 4)$.

Solution. We have

$$x_0=2, \quad y_0=-5; \quad x=-3, \quad y=4$$

From formulas (2) we find

$$x'=-3-2=-5, \quad y'=4+5=9$$

Example 2. The equation of some line is

$$x^2+y^2-4x+6y=36$$

What will the equation of the line be after a translation of the origin to the point $O'(2, -3)$?

Solution. According to formulas (1) we have

$$x=x'+2 \quad \text{and} \quad y=y'-3$$

Putting these expressions in the given equation, we get

$$(x'+2)^2+(y'-3)^2-4(x'+2)+6(y'-3)=36$$

---

[1] When memorizing the rule, leave out the words in brackets; they are essential but can readily be restored.

or, after simplifications,

$$x'^2+y'^2=49$$

This is the new equation of the line. It will be recalled (Sec. 38) that this line is a circle of radius $R=7$ with centre at O'.

36. Rotation of the Axes

Notation (Fig. 32):
old coordinates of point $M: x=OP,\ \ y=PM$;
new coordinates of point $M: x'=OP',\ \ y'=P'M$;
angle of rotation of axes [1] $\alpha=\angle XOX'=\angle YOY'$
Formulas of rotation: [2]

$$\left.\begin{aligned} x&=x'\cos\alpha-y'\sin\alpha,\\ y&=x'\sin\alpha+y'\cos\alpha \end{aligned}\right\} \qquad (1)$$

or

$$\left.\begin{aligned} x'&=x\cos\alpha+y\sin\alpha,\\ y'&=-x\sin\alpha+y\cos\alpha \end{aligned}\right\} \qquad (2)$$

Example 1. The equation $2xy=49$ is a curve consisting of two branches: LAN and $L'A'N'$ (Fig. 33). It is called an equilateral (equiangular) hyperbola. Find the equation of the curve after a rotation of the axes through an angle of 45°

Solution. For $\alpha=45°$, the formulas (1) take the form

$$x=x'\frac{\sqrt{2}}{2}-y'\frac{\sqrt{2}}{2},$$

$$y=x'\frac{\sqrt{2}}{2}+y'\frac{\sqrt{2}}{2}$$

Substitute these expressions into the given equation. This yields

$$2\times\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}(x'-y')(x'+y')=49$$

or, after simplifications,

$$x'^2-y'^2=49$$

---

[1] See Sec. 14 for the sign of the angle α (first footnote).

[2] When memorizing formulas (1) note the lack of order in the expression for x (cosine in front of sine, minus sign between terms on the right). On the contrary, there is complete "order" in the expression for y (first the sine, then the cosine, and a plus sign between them).

Formulas (2) are obtained from (1) if one replaces α by $-\alpha$ and x, y by x', y' and vice versa.

Example 2. Prior to a rotation of the axes through an angle of —20°, the point M had an abscissa $x=6$ and an ordinate $y=0$. Find the coordinates of M after a rotation of the axes.

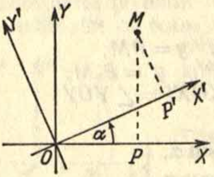

Fig. 32

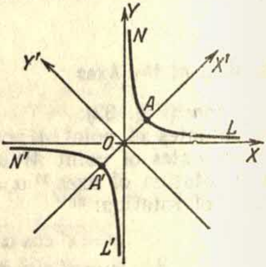

Fig. 33

Solution. The new coordinates x', y' of the point M may be found from formulas (2), where we have to put $x=6$, $y=0$, $x=0$, $\alpha=-20°$ This yields

$$x'=6\cos(-20°)\approx 5.64,$$
$$y'=-6\sin(-20°)\approx 2.05$$

37. Algebraic Curves and Their Order

An equation of the form

$$Ax+By+C=0 \qquad (1)$$

where at least one of the quantities A and B is not zero is an *algebraic equation of the first degree* (in two unknowns x, y). It always represents a straight line.

An *algebraic equation of the second degree* is any equation of the form

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \qquad (2)$$

where at least one of the quantities A, B, C is nonzero.

An equation that is equivalent to Eq. (2) is also called algebraic.

Example 1. The equation $y=5x^2$, which is equivalent to the equation $5x^2-y=0$, is an algebraic equation of the second degree ($A=5$, $B=0$, $C=0$, $D=0$, $E=-1$, $F=0$).

Example 2. The equation $xy=1$, which is equivalent to $xy-1=0$, is an algebraic equation of the second degree ($A=0$, $B=1$, $C=0$, $D=0$, $E=0$, $F=-1$).

Example 3. The equation $(x+y+2)^2-(x+y+1)^2=0$ is an equation of the first degree since it is equivalent to $2x+2y+3=0$.

In similar fashion we define algebraic equations of the third, fourth, fifth, etc. degrees. The quantities A, B, C, D and so forth (including the absolute term) are called the *coefficients* of the algebraic equation.

If some curve L is described in a cartesian coordinate system by an algebraic equation of the nth degree, then in any other cartesian system it will be given by an algebraic equation of the same degree. However, the coefficients (some or all) of the equation will then change their values; in a particular case, some of them can vanish.

A curve L given (in a cartesian system) by an nth degree equation is termed an *algebraic curve of the nth order* (or *of degree n*).

Example 4. In a rectangular coordinate system, a straight line is described by an algebraic equation of the first degree of the form $Ax+By+C=0$ (Sec. 16). Therefore, a straight line is a first-order algebraic curve. In different coordinate systems, the coefficients A, B, C have different values for one and the same straight line. For instance, in an "old" system, let a straight line be given by the equation $2x+3y-5=0$ ($A=2$, $B=3$, $C=-5$). If we rotate the axes through 45°, then (Sec. 36) the same line will, in the "new" system, be described by the equation

$$2\left(x'\frac{\sqrt{2}}{2}-y'\frac{\sqrt{2}}{2}\right)+3\left(x'\frac{\sqrt{2}}{2}+y'\frac{\sqrt{2}}{2}\right)-5=0$$

that is,

$$\frac{5\sqrt{2}}{2}x'+\frac{\sqrt{2}}{2}y'-5=0\left(A=\frac{5\sqrt{2}}{2},\quad B=\frac{\sqrt{2}}{2},\quad C=-5\right)$$

Example 5. If the coordinate origin coincides with the centre of a circle of radius $R=3$, the circle is described by the equation (Sec. 38) $x^2+y^2-9=0$. This is an algebraic equation of the second degree ($A=1$, $B=0$, $C=1$, $D=0$, $E=0$, $F=-9$). Hence, a circle is a second-order (quadric) curve. If the origin is translated to the point $(-5, -2)$, then in the new system the same circle will be given (Sec. 35) by the equation $(x'-5)^2+(y'-2)^2-9=0$, or $x'^2+y'^2-10x'-4y'-20=0$. This is also a second-degree equation; the coefficients A, B and C remain the same, but D, E and F have changed.

Example 6. The curve given by the equation $y=\sin x$ (sine curve) is not algebraic.

38. The Circle

A circle of radius R with centre at the origin of coordinates is given by the equation

$$x^2+y^2=R^2$$

It states that the square of the distance OA (see Fig. 9, p. 24) from the origin to any point A lying on the circle is equal to R^2.

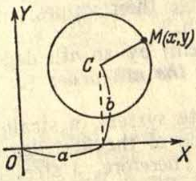

Fig. 34

A circle of radius R with centre at the point $C\,(a, b)$ is described by the equation

$$(x-a)^2+(y-b)^2=R^2 \qquad (1)$$

It states that the square of the distance MC (Fig. 34) between the points $M\,(x, y)$ and $C\,(a, b)$ (Sec. 10) is equal to R^2.

Eq. (1) may be rewritten as

$$x^2+y^2-2ax-2by+a^2+b^2-R^2=0 \qquad (2)$$

Eq. (2) may be multiplied by any number A to give

$$Ax^2+Ay^2-2Aax-2Aby+A\,(a^2+b^2-R^2)=0 \qquad (3)$$

Example 1. A circle of radius $R=7$ with centre at $C\,(4, -6)$ is described by the equation

$$(x-4)^2+(y+6)^2=49 \quad \text{or} \quad x^2+y^2-8x+12y+3=0$$

or (after being multiplied by 3)

$$3x^2+3y^2-24x+36y+9=0$$

Note. A circle is a second-order (or quadric) curve (Sec. 37) since it is described by a second-degree equation. However, an equation of the second degree does not always represent a circle. For this, it is necessary that

(1) it should not have a term with the product xy;
(2) the coefficients of x^2 and y^2 should be equal [cf. Eq. (3)].

These conditions however are not quite sufficient (see Sec. 39).

Example 2. The second-degree equation $x^2+3xy+y^2=1$ is not a circle because it has the term $3xy$.

Example 3. The second-degree equation $9x^2+4y^2=49$ is not a circle because the coefficients of x^2 and y^2 are not equal.

Example 4. The equation

$$5x^2-10x+5y^2+20y-20=0$$

satisfies the conditions (1) and (2). In Sec. 39 it is shown that this is a circle.

39. Finding the Centre and Radius of a Circle

The equation

$$Ax^2+Bx+Ay^2+Cy+D=0 \tag{1}$$

[which satisfies the conditions (1) and (2), Sec. 38] is a circle provided that the coefficients A, B, C, D satisfy the inequality

$$B^2+C^2-4AD>0 \tag{2}$$

Then the centre (a, b) and the radius R of the circle may be found from the formulas (which need not be remembered: see Example 1, second method)

$$a=-\frac{B}{2A}, \quad b=-\frac{C}{2A}, \quad R^2=\frac{B^2+C^2-4AD}{4A^2} \tag{3}$$

Note. The inequality (2) states that the square of the radius must be a positive number; cf. the last formula of (3). If inequality (2) is not fulfilled, then Eq. (1) does not represent any curve at all (see Example 2, below).

Example 1. The equation

$$5x^2-10x+5y^2+20y-20=0 \tag{4}$$

fits (1); here,

$$A=5, \quad B=-10, \quad C=20, \quad D=-20$$

Inequality (2) is fulfilled. Hence, Eq. (4) is a circle. Using formulas (3), we find

$$a=1, \quad b=-2, \quad R^2=9$$

Thus the centre is $(1, -2)$ and the radius $R=3$.

Alternative method. Divide Eq (4) by the coefficient of the second-degree terms (i. e., 5):

$$x^2-2x+y^2+4y-4=0$$

Complete the squares in x^2-2x and y^2+4y by adding 1 to the first sum and 4 to the second. Add the same numbers to the right side of the equation by way of compensation. We then have

$$(x^2-2x+1)+(y^2+4y+4)-4=1+4$$

or

$$(x-1)^2+(y+2)^2=9$$

Example 2. The equation

$$x^2-2x+y^2+2=0 \tag{5}$$

fits the case (1), but inequality (2) is not fulfilled. Which means that Eq. (5) does not describe any curve.

The same conclusion may be arrived at in the following manner (cf. Example 1):

Complete the square in x^2-2x by adding 1; also add 1 to the right side. This yields $(x-1)^2+y^2+2=1$ or $(x-1)^2+$ $+y^2=-1$. But the sum of the squares of (real) numbers cannot be equal to a negative number. For this reason there is no point whose coordinates can satisfy this equation.

40. The Ellipse as a Compressed Circle

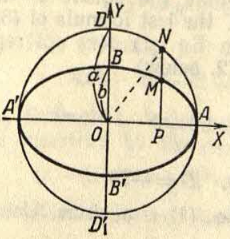

Fig. 35

Through the centre O of a circle of radius a (Fig. 35) draw two mutually perpendicular diameters $A'A$, $D'D$. On the radii OD, OD' lay off from O equal line-segments OB, OB' of length b (less than a). From each point N of the circle drop a perpendicular NP onto the diameter $A'A$ and on this perpendicular lay off a segment PM from the foot P so that

$$PM:PN=b:a \tag{1}$$

This construction transforms every point N into a corresponding point M lying on the same perpendicular NP; PM is obtained from PN by reduction in the same ratio $k=\frac{b}{a}$. A transformation of this kind is termed *uniform compression*. The straight line $A'A$ is called the *axis of compression*.

The line $ABA'B'$ into which the circle has been transformed by uniform compression is called an *ellipse* (see Sec. 41 for an alternative definition).

The line segment $A'A=2a$ (and frequently the straight line $A'A$, i. e. the axis of compression) is called the *major axis* of the ellipse.

The line segment $B'B=2b$ (and often also the straight line $B'B$) is called the *minor axis* of the ellipse ($2a > 2b$, by construction). The point O is the *centre* of the ellipse. The points A, A', B, B' are termed the *vertices* of the ellipse.

The ratio $k=b:a$ is called the *coefficient of compression* of the ellipse. The quantity $1-k=\frac{a-b}{a}$ (the ratio $BD:OD$) is called the *compression* of the ellipse and is denoted by α.

An ellipse is symmetric about the major and minor axes and, hence, about the centre.

A circle may be regarded as an ellipse with a coefficient of compression $k=1$.

Standard form of the equation of the ellipse. If the axes of the ellipse are taken as the coordinate axes, then the ellipse is described by the equation [1]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{2}$$

This is the *standard (canonical) form of the equation* of the ellipse.

Example 1. A circle of radius $a=10$ cm is subject to uniform compression with coefficient of compression 3:5. This produces an ellipse with major axis $2a=20$ cm and minor axis $2b=12$ cm (semi-axes $a=10$ cm, $b=6$ cm). The compression of the ellipse $\alpha=1-k=\frac{10-6}{10}=0.4$. The stan-

---

[1] We have

$$OP^2+PN^2=ON^2=a^2 \tag{3}$$

By (1) we get

$$PN=\frac{a}{b}PM \tag{4}$$

Putting this into (3) yields

$$OP^2+\frac{a^2}{b^2}PM^2=a^2 \tag{5}$$

that is,

$$x^2+\frac{a^2}{b^2}y^2=a^2 \tag{6}$$

Dividing by a^2 yields the equivalent equation (2). Thus, if $M(x, y)$ lies on the ellipse $ABA'B'$, then x, y satisfy Eq. (2). But if M does not lie on the ellipse, then equality (4) and, hence, Eq. (6) are not satisfied (cf. Sec. 7).

dard form of the equation is then

$$\frac{x^2}{100}+\frac{y^2}{36}=1$$

Example 2. In projecting a circle on some plane P, the diameter $A_1'A_1$ (Fig. 36) parallel to the plane is projected full size and all the chords perpendicular to the diameter are reduced in a ratio equal to $\cos\varphi$, where φ is the angle between the plane P_1 of the circle and the plane P. For this reason, the projection of a circle is an ellipse with major axis $2a=A'A$ and coefficient of compression $k=\cos\varphi$.

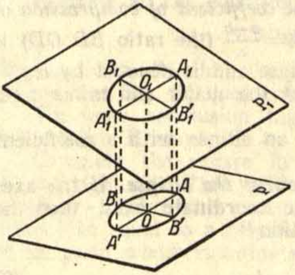

Fig. 36

Example 3. A terrestrial meridian is more accurately taken as an ellipse and not a circle. The axis of the earth is the minor axis of the ellipse. It has an approximate length of 12,712 km. The length of the major axis is roughly 12,754 km. Find the coefficient of compression k and the compression α of this ellipse.

Solution.

$$\alpha=\frac{a-b}{a}=\frac{2a-2b}{2a}=\frac{12{,}754-12{,}712}{12{,}754}\approx 0.003,$$

$$k=1-\alpha\approx 0.997.$$

41. An Alternative Definition of the Ellipse

Definition. An *ellipse* is the locus of points (M), the sum of the distances of which from two given points F', F (Fig. 37) is a constant, $2a$:

$$F'M+FM=2a \qquad (1)$$

The points F' and F are called the *foci*[1] of the ellipse, the distance $F'F$ is the *focal length*, denoted by $2c$:

$$F'F=2c \qquad (2)$$

---

[1] If a light source is placed at F (or F'), the rays of light are reflected from the ellipse and come together at F' (or F) (the other focus).

Since $F'F < F'M + FM$, it follows that $2c < 2a$, or

$$c < a \tag{3}$$

The definition given in this section is equivalent to that of Sec. 40 [cf. Eq. (7) with Eq. (2), Sec. 40].

Standard form of the equation of the ellipse. Take the straight line $F'F$ (Fig. 38) as the axis of abscissas and the midpoind O of the line segment $F'F$ as the origin of coordinates. According to the definition of an ellipse and to (1), Sec. 10, we have $F'(-c, 0)$, $F(c, 0)$. By Sec. 10

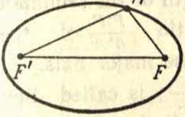

Fig. 37

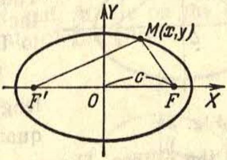

Fig. 38

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a \tag{4}$$

On elimination of the radicals,[1] we obtain an equivalent equation:

$$(a^2-c^2)\,x^2 + a^2y^2 = a^2\,(a^2-c^2) \tag{5}$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1 \tag{6}$$

Because of (3), the quantity a^2-c^2 *is positive.* Therefore we can write (6) as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7}$$

where

$$b^2 = a^2 - c^2 \tag{8}$$

Eq. (7) coincides with Eq. (2) of Sec. 40, and so the curve, called an ellipse in this section, is indeed identical with the curve described as an ellipse in Sec. 40. It then turns out that the centre O of the ellipse (Fig. 39) coincides with the midpoint of the line segment $F'F$, that is, $OF = c$. By equality (1), the major axis $2a = A'A$ of the ellipse turns

---

[1] Transpose one of the radicals to the right side and square. There will be only one radical in the new equation. Separating it and again squaring, we simplify to (5).

out equal to the constant sum of the distances $F'M+FM$ (Fig. 38). The semiminor axis $b=OB$ (Fig. 39) and the line segment $c=OF$ are sides of the right triangle BOF; the hypotenuse BF of this triangle is a. This is evident from (8) and also from the fact that the equal segments $F'B$ and FB add to $2a$ (by the definition of an ellipse). Thus, the distance from a focus to the end of the minor axis is equal to the length of the semimajor axis.

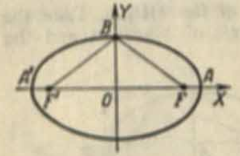

Fig. 39

The ratio $\frac{F'F}{A'A}$ of the focal length to the major axis, i.e. the quantity $\frac{c}{a}$, is called the *eccentricity* of the ellipse. The eccentricity is denoted by the Greek letter ε (epsilon):

$$\varepsilon=\frac{c}{a} \tag{9}$$

Because of (3), the eccentricity of an ellipse is less than unity. By virtue of (8), the eccentricity ε and the coefficient of compression k of an ellipse (Sec. 40) are connected by the relation

$$k^2=1-\varepsilon^2 \tag{10}$$

Example. Let the focal length of the ellipse $2c=8$cm and the sum of the distances of an arbitrary point from the loci be 10 cm. Then the major axis $2a=10$ cm, the eccentricity $\varepsilon=\frac{c}{a}=0.8$. The coefficient of compression $k=\sqrt{1-\varepsilon^2}=0.6$. The minor axis $2b=2ak=2\sqrt{a^2-c^2}=6$cm. The standard form of the equation of the ellipse is

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

Note. If the circle is regarded as a special kind of ellipse, $b=a$, then $c=0$, and the foci F' and F must be taken to coincide. The eccentricity of the circle is zero.

42. Construction of an Ellipse from the Axes

First method. On the perpendicular straight lines $X'X$ and $Y'Y$ (Fig. 40) lay off the line segments $OA'=OA=a$ and $OB'=OB=b$ [halves of the given axes $2a$, $2b$ $(a>b)$]. The points A', A, B', B will be the vertices of the ellipse.

From point B, strike an arc uv with radius a; it will intersect the line segment $A'A$ at the points F', F; these will be the foci of the ellipse [by (8), Sec. 41]. Divide $A'A=2a$ into two parts in arbitrary fashion: $A'K=r'$ and $KA=r$, so that $r'+r=2a$. From the point F draw a circle of radius r and from F' a circle of radius r'. These circles intersect at two points M and M'; by construction, we have $F'M+FM=2a$ and $F'M'+FM'=2a$. By the definition given in Sec. 41 the points M and M' lie on the ellipse. By varying r we obtain new points of the ellipse.

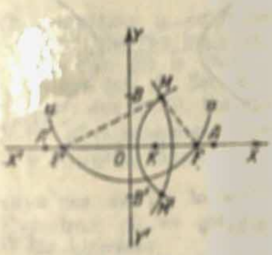

Fig. 40

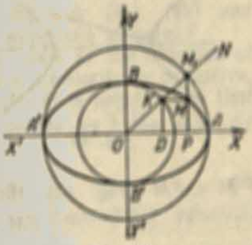

Fig. 41

Second method. Draw two concentric circles of radius $OA=a$ and $OB=b$ (Fig. 41). Through the centre O draw an arbitrary ray ON. Through the points K and M_1, at which ON meets the two circles, draw straight lines that are respectively parallel to the axes $X'X$, $Y'Y$. These straight lines will intersect at the point M. Its ordinate $PM\ (=KD)$ is shorter than the ordinate PM_1 of the point M_1 which lies on the circle of radius a; we have $PM:PM_1=b:a$. Therefore (Sec. 40) the point M lies on the desired ellipse. Varying the direction of the ray ON, we get new points of the ellipse.

42. The Hyperbola

Definition. The *hyperbola* (Fig. 42) is the locus of points (M) whose distances from two fixed points F', F have a constant difference (cf. definition of the ellipse in Sec. 41):

$$|F'M-FM|=2a \tag{1}$$

The points F' and F are called the *foci*[1] of the hyperbola, and the distance $F'F$ is the *focal length* denoted by $2c$:

$$F'F = 2c \tag{2}$$

Since $F'F > |F'M - FM|$, it follows that [cf. formula (3), Sec. 41]

$$c > a \tag{3}$$

If M is closer to the focus F' than to the focus F, i. e. if

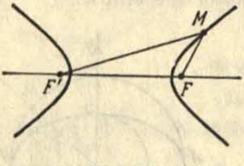

Fig. 42

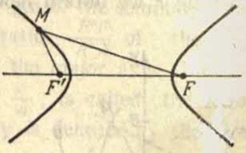

Fig. 43

$F'M < FM$ (Fig. 43), then in place of (1) we can write

$$FM - F'M = 2a \tag{1a}$$

But if M is closer to F than F', i. e. $F'M > FM$ (Fig. 42), then we have

$$F'M - FM = 2a \tag{1b}$$

Those points for which $F'M - FM = 2a$ form one branch of the hyperbola (usually the right branch); those points for which $FM - F'M = 2a$ form the other branch (the left branch).

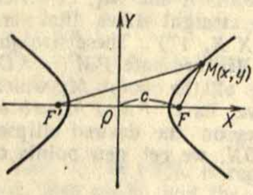

Fig. 44

Standard form of the equation of the hyperbola. In Fig. 44, for the x-axis we take the line $F'F$ and for the origin, the midpoint O of $F'F$. By (2) we have $F(c, 0)$, $F'(-c, 0)$. By (1b) and Sec. 10 the right branch is given by the equation

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = 2a \tag{4a}$$

---

[1] If a light source is placed at one of the foci, the light rays reflected from the hyperbola will form a divergent beam with the centre in the other focus. Cf. footnote on p. 60.

For the left branch, by (1a) and Sec. 10, we have the equation

$$\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}=2a \qquad (4b)$$

On elimination of the radicals we get, in both cases,

$$(a^2-c^2)\,x^2+a^2y^2=a^2\,(a^2-c^2) \qquad (5)$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1 \qquad (6)$$

This equation is equivalent to the pair (4a), (4b) and represents the two branches of the hyperbola at once.[1]

Equation (6) is outwardly the same as the equation of the ellipse [cf. (6), Sec. 41] but this similarity is deceptive, for now, due to (3), the quantity a^2-c^2 is negative, so that $\sqrt{a^2-c^2}$ is imaginary. Therefore, denote by b the quantity $+\sqrt{c^2-a^2}$ so that[2]

$$b^2=c^2-a^2 \qquad (7)$$

Then from (6) we get the *standard (canonical) equation of the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (8)$$

Example. If the magnitude of the difference $F'M-FM$ is $2a=20$ cm and the focal length is $2c=25$ cm, then $b\sqrt{c^2-a^2}=\frac{15}{2}$ (cm). The standard form of the equation of the hyperbola is $\frac{x^2}{100}-\frac{y^2}{\frac{225}{4}}=1$.

44. The Shape of the Hyperbola, Its Vertices and Axes

The hyperbola is symmetric about .the point O—the midpoint of the segment $F'F$ (Fig. 45); it is symmetric about the straight line $F'F$ and about the straight line $Y'Y$ drawn through O perpendicular to $F'F$. The point O is called the

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[1] The two branches of the hyperbola might be taken as two curves and not one. But then neither of the curves, separately, would be a second-degree algebraic equation.

[2] See Sec. 46 on the geometrical meaning of the quantity b.

centre of the hyperbola. The straight line $F'F$ intersects the hyperbola at two points $A(+a, 0)$ and $A'(-a, 0)$. These points are the *vertices* of the hyperbola. The segment $A'A=2a$ (and also frequently the straight line $A'A$) is called the *real (transverse) axis* of the hyperbola

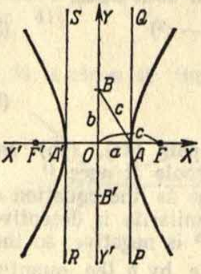

Fig. 45

The straight line $Y'Y$ does not intersect the hyperbola. Nevertheless, it is customary to lay off on this line the segments $B'O=OB=b$ and call $B'B=2b$ (and also $Y'Y$) the *imaginary (conjugate) axis* of the hyperbola.

Since $AB^2=OA^2+OB^2=a^2+b^2$, it follows from (7), Sec. 43, that $AB=c$, i.e. the distance from a vertex of the hyperbola to the end of the conjugate axis is equal to half the focal length.

The conjugate axis $2b$ may be greater than (Fig. 45), less than (Fig. 46), or equal to (Fig. 47) the transverse axis $2a$. If the transverse and conjugate axes are equal $(a=b)$, then the hyperbola is termed *equiangular*, *equilateral*, or *rectangular*.

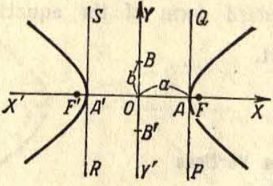

Fig. 46

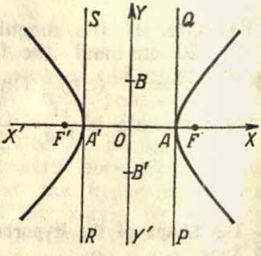

Fig. 47

The ratio $\frac{F'F}{A'A}=\frac{c}{a}$ of the focal length to the transverse axis is called the *eccentricity* of the hyperbola and is denoted by ε [cf. (9), Sec. 41)]. Because of (3), Sec. 43, the eccentricity of the hyperbola is greater than unity. The eccentricity of an equilateral hyperbola is $\sqrt{2}$.

The hyperbola lies completely outside the strip bounded by the straight lines PQ and RS parallel to $Y'Y$ and distant from $Y'Y$ by $OA = A'O = a$ (Figs. 45, 46, 47). To the right and left of this strip the hyperbola goes off without bound.

45. Construction of a Hyperbola from Its Axes

On the perpendicular straight lines $X'X$ and $Y'Y$ (Fig. 48) lay off segments $OA = OA' = a$ and $OB = OB' = b$ (semitransverse axes and semiconjugate axes). Then lay off the segments OF and OF' equal to AB. The points F' and F are foci [according to (7), Sec. 43]. Take an arbitrary point K on the extension of the segment $A'A$. From F draw a circle of radius $r = AK$. From F' describe a circle of radius $r' = A'K = 2a + r$. These circles will intersect in two points M, M'; note that by construction $F'M - FM = 2a$ and $F'M' - FM' = 2a$. By the definition given in Sec. 43, the points M and M' lie on the hyperbola. By varying r we get other points on the "right" branch. Similarly, we can obtain points on the "left" branch.

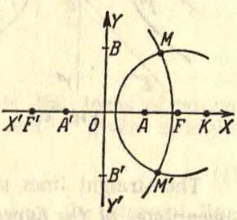

Fig. 48

46. The Asymptotes of a Hyperbola

For $|k| < \frac{b}{a}$, the straight line $y = kx$ (it passes through the centre O of the hyperbola) intersects the hyperbola in two points D', D (Fig. 49) which are symmetric about O. But if $|k| \geqslant \frac{b}{a}$, then the straight line $y = kx$ ($E'E$ in Fig. 50) has no common points with the hyperbola.

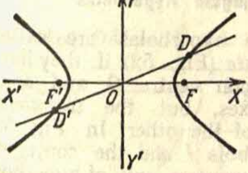

Fig. 49

The straight lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ ($U'U$ and $V'V$ in Fig. 51), for which $|k| = \frac{b}{a}$, have the following unique property: each line when extended indefinitely approaches indefinitely near to the hyperbola.

More precisely: *if the straight line $Q'Q$, parallel to the axis of ordinates, is made to recede to an infinite distance from the centre O (to the right or to the left), the line segments QS, $Q'S'$ between the hyperbola and each of the straight lines $U'U$, $V'V$ become small without bound.*

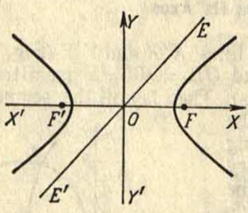

Fig. 50

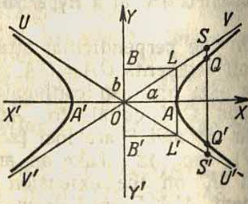

Fig. 51

The straight lines $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$ are called the *asymptotes of the hyperbola.* [1]

The asymptotes to an equilateral hyperbola are mutually perpendicular.

The geometrical meaning of the conjugate axis. Through the vertex A of a hyperbola (Fig. 51) draw a straight line $L'L$ perpendicular to the transverse axis. Then the segment $L'L$ (of this straight line) between the asymptotes to the hyperbola is equal to the conjugate axis $B'B=2b$ of the hyperbola.

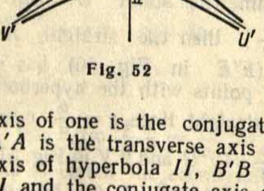

Fig. 52

47. Conjugate Hyperbolas

Two hyperbolas are called *conjugate* (Fig. 52) if they have a common centre O and common axes, but the transverse axis of one is the conjugate axis of the other. In Fig. 52, $A'A$ is the transverse axis of hyperbola I and the conjugate axis of hyperbola II, $B'B$ is the transverse axis of hyperbola II and the conjugate axis of hyperbola I.

---

[1] Asymptote is from the Greek meaning "not meeting."

If

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is the equation of one of the conjugate hyperbolas, then the other one is given by the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

Conjugate hyperbolas have common asymptotes ($U'U$ and $V'V$ in Fig. 52).

48. The Parabola

Definition. The *parabola* (Fig. 53) is the locus of points (M) equidistant from a given point F and a given straight line PQ:

$$FM=KM \tag{1}$$

The point F is called the *focus*,[1] and the straight line PQ the *directrix* of the parabola. The distance $FC=p$ from the focus to the directrix is the *parameter* of the parabola.

For the coordinate origin, take the midpoint O of the line FC so that

$$CO=OF=\frac{p}{2} \tag{2}$$

The straight line CF will be the axis of abscissas and the positive direction will be from O to F.

We then have: $F\left(\frac{p}{2},\ 0\right)$, $KM=KD+DM=\frac{p}{2}+x$ and (Sec. 10) $FM=\sqrt{\left(\frac{p}{2}-x\right)^2+y^2}$. Because of (1), we have

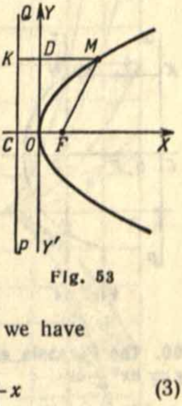

Fig. 53

$$\sqrt{\left(\frac{p}{2}-x\right)^2+y^2}=\frac{p}{2}+x \tag{3}$$

---

[1] After reflection from a parabola, a parallel beam of rays perpendicular to the directrix will become a central beam with centre in the focus. See footnote on p. 60.

On elimination of the radical sign, we get the equivalent equation

$$y^2 = 2px \tag{4}$$

This is the *standard canonical equation* of the parabola.

The equation of the directrix PQ (in the same system of coordinates) is $x + \frac{p}{2} = 0$.

The parabola is symmetric about the straight line FC (the x-axis in our coordinate system). This line is termed the *axis* of the parabola. The parabola passes through the midpoint O of the segment FC. The point O is called the *vertex* of the parabola (which we took for the coordinate origin).

The parabola lies entirely on one side of the straight line $Y'Y$ (tangent at the vertex) and goes off to infinity on that side.

49. Construction of a Parabola from a Given Parameter p

In Fig. 54, draw a straight line PQ (the directrix) and at a given distance $p = CF$ from it take a point F (the focus). The midpoint O of CF will be the vertex and the straight line CF will be the axis of the parabola. On the ray OF take an arbitrary point R and through it draw a straight line RS perpendicular to the axis. From the focus F as centre describe a circle of radius CR. It will intersect RS in two points M and M'. M and M' belong to the sought-for parabola, since it is given (see definition, Sec. 48) that $FM = CR = KM$. By varying the position of the point R we obtain other points of the parabola.

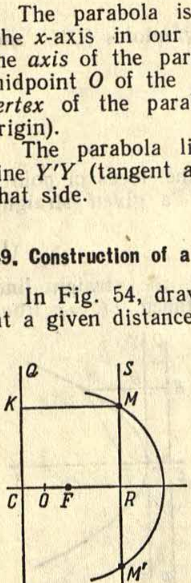

Fig. 54

50. The Parabola as the Graph of the Equation $y = ax^2 + bx + c$

The equation

$$x^2 = 2py \tag{1}$$

represents the same parabola as the equation $y^2 = 2px$ (cf. Sec. 48), only in this case the axis of the parabola coincides with the axis of ordinates; the origin, as before, coincides with the vertex of the parabola (Fig. 55). The focus lies in the point $F\left(0,\ \frac{p}{2}\right)$. The directrix PQ is given by the equation $y+\frac{p}{2}=0$.

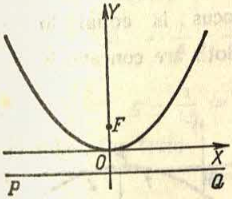

Fig. 55

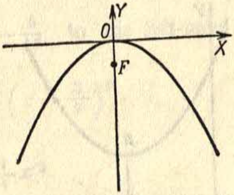

Fig. 56

If for the positive direction of the y-axis we take the direction FO (Fig. 56) instead of OF, then the equation of the parabola will be

$$-x^2=2py \qquad (2)$$

(see Fig. 56, where the coordinate axes have the customary directions). Accordingly, the graphs of the functions

$$y=ax^2 \qquad (3)$$

are parabolas which are concave up when $a>0$ and concave down when $a<0$. The smaller the absolute value of a $\left(\text{in Fig. 57 we have } a=2,\ a=\pm 1,\ a=\pm\frac{1}{2},\ a=\frac{1}{5}\right)$, the closer the focus to the vertex and the more spread out the parabola is.

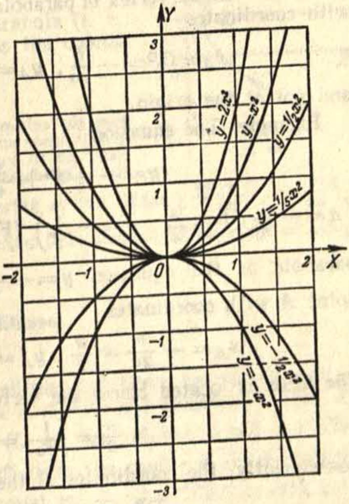

Fig. 57

Graphically, any equation

$$y = ax^2 + bx + c \tag{4}$$

is depicted by the same parabola as the equation $y = ax^2$ $\left(\text{for both parabolas the distance } \frac{p}{2} \text{ from the vertex to the focus is equal to } \frac{1}{|4a|}\right)$. Both are concave in the same direction. But the vertex of parabola (4) lies in point A (Fig. 58) with coordinates

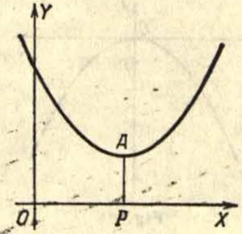

Fig. 58

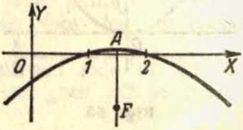

Fig. 59

$$x_A = OP = -\frac{b}{2a}, \quad y_A = PA = \frac{4ac - b^2}{4a} \tag{5}$$

and not at the origin.

Example. The equation

$$y = -\frac{1}{4}x^2 + \frac{3}{4}x - \frac{1}{2} \tag{4a}$$

$\left(a = -\frac{1}{4},\ b = \frac{3}{4},\ c = -\frac{1}{2}\right)$ (Fig. 59) represents the same parabola as the equation $y = -\frac{1}{4}x^2$. The vertex lies in point A with coordinates

$$x_A = -\frac{b}{2a} = \frac{3}{2}, \quad y_A = \frac{4ac - b^2}{4a} = \frac{1}{16} \tag{5a}$$

The focus is located below the vertex at a distance

$$\frac{p}{2} = \frac{1}{|4a|} = 1$$

Consequently, the coordinates of the focus are

$$x_F = \frac{3}{2}, \quad y_F = \frac{1}{16} - 1 = -\frac{15}{16}$$

Note 1. The formulas (5) need not be memorized. The following device may be used to compute x_A, y_A. Rewrite equation (4a) as

$$y+\frac{1}{2}=-\frac{1}{4}(x^2-3x) \tag{6}$$

Complete the square in the brackets by adding $\frac{9}{4}$. To compensate, add $-\frac{1}{4}\cdot\frac{9}{4}=-\frac{9}{16}$ to the left-hand side. This yields

$$y-\frac{1}{16}=-\frac{1}{4}\left(x-\frac{3}{2}\right)^2 \tag{7}$$

Eq. (7) takes the form

$$y'=-\frac{1}{4}x'^2 \tag{8}$$

if we perform a translation of the axes (Sec. 35):

$$y'=y-\frac{1}{16},\quad x'=x-\frac{3}{2} \tag{9}$$

The vertex of the parabola (i. e., the point $x'=0$, $y'=0$) has the coordinates $x=\frac{3}{2}$, $y=\frac{1}{16}$.

Note 2. The general formulas (5) may be derived from (4) by the same technique as was used in Note 1 with respect to Eq. (4a).

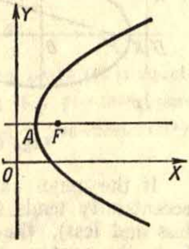

Fig. 60

Note 3. The equation $x=ay^2+by+c$ is a parabola (Fig. 60) with vertex at the point $\left(\frac{4ac-b^2}{4a},\ -\frac{b}{2a}\right)$. Its axis is parallel to the x-axis; it is concave to the right if $a>0$ and to the left if $a<0$.

51. The Directrices of the Ellipse and of the Hyperbola

(a) **Directrices of the ellipse.** Let there be given an ellipse (Fig. 61) with major axis $A'A=2a$ and eccentricity (Sec. 41) $\frac{OF}{OA}=\frac{c}{a}=\varepsilon$. Let $\varepsilon\neq 0$ (i. e., the ellipse is not a circle). On the major axis, lay off from the centre O of the ellipse the segments $OD=OD'$ equal to $\frac{a}{\varepsilon}$ (i.e. $OD:OA=OA:OF$). The straight lines PQ, $P'Q'$, which pass through D and D', respectively, and are parallel to the minor axis, are called the *directrices of the ellipse*.

With each of the directrices we associate the focus of the ellipse which lies on that side of the centre; for example, focus F is associated with directrix PQ, focus F' with directrix $P'Q'$. Then, *for any point M of the ellipse the ratio of its distance from the focus to the distance from the corresponding directrix is equal to the eccentricity* ε:

$$MF:MK=MF':MK'=\varepsilon \tag{1}$$

Since for the ellipse $\varepsilon < 1$, any point of the ellipse is closer to a focus than to the corresponding directrix.

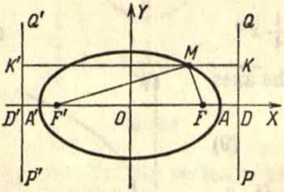

Fig. 61

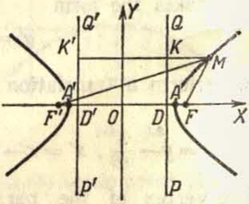

Fig. 62

If the major axis of an ellipse remains the same and the eccentricity tends to zero (i. e. the ellipse differs from a circle less and less), the directrices move off an infinite distance from the centre.

The circle has no directrices.

(b) **Directrices of the hyperbola.** Let $A'A$ (Fig. 62) be the transverse (real) axis of a hyperbola and let $\varepsilon=\frac{OF}{OA}=\frac{c}{a}$ be its eccentricity (Sec. 44). Lay off

$$OD=OD'=\frac{a}{\varepsilon}$$

(i. e. $OD:OA=OA:OF$). The straight lines PQ, $P'Q'$ that pass through D and D' respectively and are parallel to the conjugate axis are called *directrices of the hyperbola. For any point M of a hyperbola the ratio of the distance of M to a focus to the distance to the corresponding directrix* [see Item (*a*)] *is equal to the eccentricity*, or

$$MF:MK=MF':MK'=\varepsilon \tag{2}$$

Since $\varepsilon > 1$ for a hyperbola, any point of a hyperbola is closer to a directrix than to the associated focus.

52. A General Definition of the Ellipse, Hyperbola and Parabola

All ellipses (except circles), hyperbolas and parabolas have the property that for each of them the following ratio is invariant (Fig. 63):

$$FM:MK \tag{1}$$

where FM is the distance from an arbitrary point M to a given point F (focus), and MK is the distance of M to a given straight line PQ (directrix).

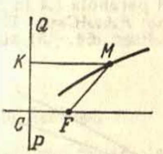

Fig. 63

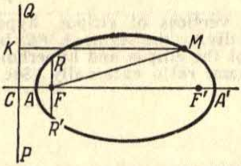

Fig. 64

For the ellipse (Fig. 64) this ratio is less than unity (it is equal to the eccentricity of the ellipse $\frac{c}{a}$; cf. Secs. 41, 51) For the hyperbola (Fig. 65) it is greater than unity (it is equal to the eccentricity of the hyperbola $\frac{c}{a}$; cf. Secs. 43, 51); for the parabola (Fig. 66) it is unity (Sec. 48.).

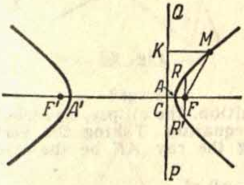

Fig. 65

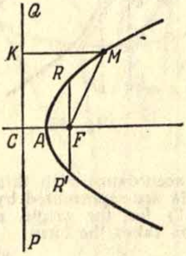

Fig. 66

Conversely, every line having the indicated property is either an ellipse (if $FM:MK < 1$), or a hyperbola (if $FM:MK > 1$), or a parabola (if $FM:MK = 1$). Therefore, this property may be taken as the general definition of an ellipse, hyperbola, and parabola, and the invariant ratio $FM:MK = \varepsilon$ is called the *eccentricity*. The eccentricity of the parabola is equal to unity, that of the ellipse $\varepsilon < 1$, that of the hyperbola $\varepsilon > 1$.

By specifying the eccentricity ε and the distance $FC = d$ from a focus to its directrix we fully define the size and shape of an ellipse, hyperbola and parabola. If for a given ε we vary d, then all the curves will be similar.

The chord RR' of an ellipse, hyperbola or parabola (Figs. 64, 65, 66) passing through a focus F perpendicular to the axis FC is called a *latus rectum* and is denoted by $2p$:

$$RR'=2p \tag{2}$$

The quantity $p=FR=FR'$ (i. e. half the length of the latus rectum) is called the *parameter* of the ellipse, hyperbola or parabola. It is connected with d by the relation

$$p=d\varepsilon \tag{3}$$

so that for the parabola ($\varepsilon=1$)

$$p=d \tag{3a}$$

The vertices of ellipse, hyperbola and parabola (A in Figs. 64, 65, 66) divide the segment FC in the ratio $FA:AC=\varepsilon$. The second vertex of the ellipse and hyperbola (A' in Figs. 64, 65) divides FC in the same ratio externally (Sec. 11).

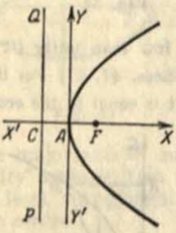

Fig. 67

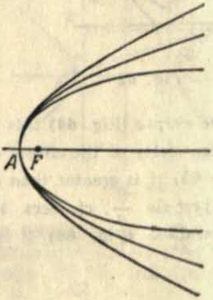

Fig. 68

In accordance with this new definition, the ellipse, hyperbola and parabola are represented by a single equation. Taking the vertex A (Fig. 67) for the origin and letting the ray AF be the axis, this equation takes the form

$$y^2=2px-(1-\varepsilon^2)\,x^2 \tag{4}$$

where p is the parameter and ε the eccentricity.

Near a vertex, the parabola even in shape differs but little from an ellipse and a hyperbola with eccentricity close to unity. Fig. 68 depicts an ellipse with eccentricity $\varepsilon=0.9$, a hyperbola [1] with eccentricity $\varepsilon=1.1$ and a parabola ($\varepsilon=1$) all having a common focus F and a common vertex A.

The semiaxes a, b and the semifocal length c of the ellipse and hyperbola are expressed in terms of ε as follows:

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[1] The closer ε is to unity, the farther the second vertex of the ellipse and hyperbola (and also the entire second branch of the hyperbola) is from the first vertex.

Ellipse	$a=\frac{p}{1-\varepsilon^2}$	$b=\frac{p}{\sqrt{1-\varepsilon^2}}$	$c=a\varepsilon=p\,\frac{\varepsilon}{1-\varepsilon^2}$
Hyperbola	$a=\frac{p}{\varepsilon^2-1}$	$b=\frac{p}{\sqrt{\varepsilon^2-1}}$	$c=a\varepsilon=p\,\frac{\varepsilon}{\varepsilon^2-1}$

In all three cases, the distance $\delta=AF$ from focus F to vertex A is expressed by the formula

$$\delta=\frac{d\varepsilon}{1+\varepsilon}=\frac{p}{1+\varepsilon} \qquad (5)$$

53. Conic Sections

The ellipse, hyperbola and parabola are called *conic sections* (conics) since they are obtainable on the surface of a circular cone (also on the surface of a noncircular cone) at the intersection with a plane P that does not pass through the vertex of the cone. The

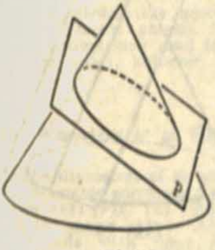


Fig. 69

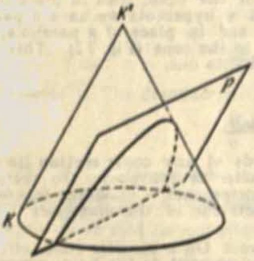


Fig. 70

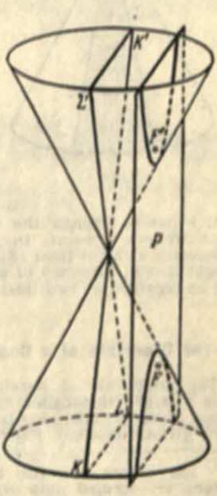


Fig. 71

surface of the cone is visualized as extending indefinitely from the vertex in both directions.

If the plane P is not parallel to any generatrix of the cone (Fig. 69), the conic section is an ellipse.[1]

If the plane P is parallel to only one of the generatrices of the cone (KK' in Fig. 70), then the conic section is a parabola.

If the plane P is parallel to two generatrices of the cone (KK' and LL' in Fig. 71), then the conic section is a hyperbola.

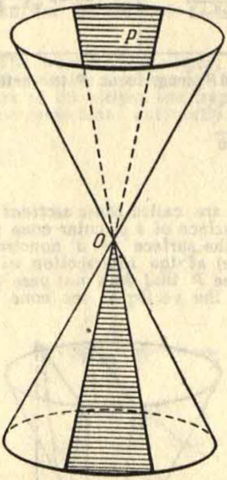


Fig. 72

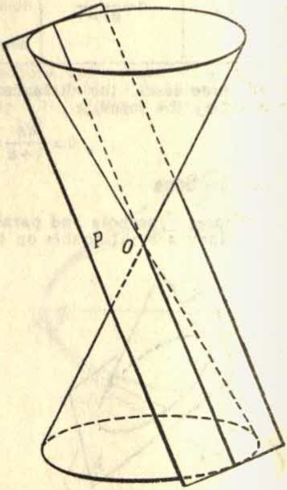


Fig. 73

If P passes through the vertex of the cone, then in place of an ellipse we get a point, in place of a hyperbola we have a pair of intersecting straight lines (Fig. 72), and in place of a parabola, the straight line of tangency of plane P to the cone (Fig. 73). This line may be regarded as two lines merged into one.

54. The Diameters of a Conic Section

The midpoints of parallel chords of any conic section lie on a single straight line called the *diameter* of the conic. To every direction of the parallel chords there corresponds a diameter (*conjugate* to the given direction). Fig. 74 depicts one of the diameters $U'U$ of

---

[1] The ellipse can also be a circle. On a circular cone, circular sections are formed only by planes parallel to the base, whereas a noncircular cone has in addition a family of circular sections.

an ellipse. On it lie the midpoints K_1, K_2, ... of the parallel chords $M_1M'_1$, $M_2M'_2$, ... The locus of these midpoints is the segment $L'L$ of the diameter $U'U$.

Fig. 75 shows a diameter $U'U$ of a hyperbola corresponding to parallel chords $M_1M'_1$, $M_2M'_2$, etc. It contains the midpoints K_1 K_2, ... of these chords The locus of the points K_1, K_2, ... is a pair of rays $L'U'$ and LU.

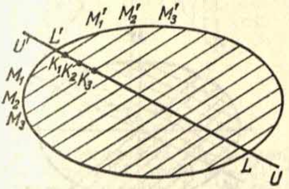


Fig. 74

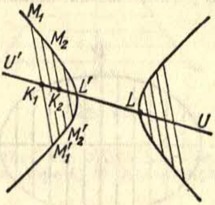


Fig. 75

Note. In elementary geometry the diameter of a circle is a *line segment* (the largest chord). In analytic geometry, the term "diameter" is also sometimes used to denote the *segment* LL' (Figs. 74, 75). It is more usual, however, to use this term to denote the *entire* line UU'.

55. The Diameters of an Ellipse

All the diameters of an ellipse pass through its centre.

The diameter corresponding to chords parallel to the minor axis is the major axis (Fig. 76). The diameter corresponding to chords parallel to the major axis is the minor axis.

To chords with slope $k\,(k \neq 0)$ there corresponds a diameter $y=k_1x$, where k_1 is determined from the relation

$$kk_1=\varepsilon^2-1 \tag{1}$$

i. e.

$$kk_1=-\frac{b^2}{a^2} \tag{1a}$$

Example 1. The diameter $U'U$ of the ellipse

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

(Fig. 77), which corresponds to chords with slope $k=-\frac{8}{9}$, is given by the equation $y=k_1x$; the value of k_1 is found from the relation $-\frac{8}{9}k_1=-\frac{4}{9}$ so that the equation of the diameter $U'U$ is

$$y=\frac{1}{2}x$$

Example 2. The diameter $V'V$ (Fig. 77) of the same ellipse, which corresponds to chords with slope $k=\frac{1}{2}$, is given by the equation $y=-\frac{8}{9}x$.

If the diameter $U'U$ of the ellipse bisects the chords which are parallel to the diameter $V'V$, then the diameter $V'V$ always bisects the chords parallel to the diameter $U'U$.

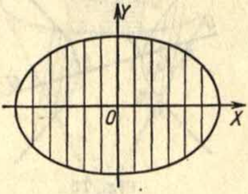


Fig. 76

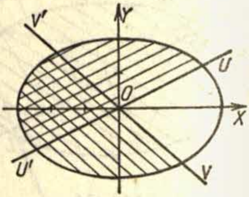


Fig. 77

Example 3. The diameter $y=-\frac{8}{9}x$ of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ (cf. Examples 1 and 2) bisects the chords parallel to the diameter $y=\frac{1}{2}x$. In turn, the diameter $y=\frac{1}{2}x$ bisects the chords parallel to the diameter $y=-\frac{8}{9}x$.

Diameters such that each one bisects the chords parallel to the other one are called *conjugate diameters*.

Two diameters conjugate to one another and mutually perpendicular are termed *principal diameters*. In the circle, any diameter is the principal diameter. The ellipse which differs from a circle has only one pair of principal diameters: the major axis and minor axis.

The slopes of the nonprincipal conjugate directions have [in accordance with (1a)] opposite signs; i. e. *two conjugate diameters of an ellipse belong to different pairs of vertical angles formed by the axes* (in Fig. 77, the diameter $V'V$ lies in the second and fourth quadrants, while $U'U$ lies in the first and third quadrants). The diameter $U'U$ and the conjugate diameter $V'V$ rotate in the same sense.

56. The Diameters of a Hyperbola

All the diameters of a hyperbola pass through its centre.

The diameter corresponding to chords parallel to the conjugate axis (Fig. 78) is the transverse axis (the locus of midpoints of chords is the pair of rays $A'X'$ and AX); the diameter corresponding to chords parallel to the transverse axis (Fig. 79) is the conjugate axis (the midpoints of chords fill the $Y'Y$ axis completely).

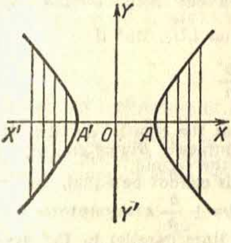


Fig. 78

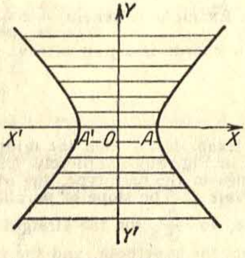


Fig. 79

As in the case of the ellipse, in the case of the hyperbola the slope k of parallel chords ($k \neq 0$) and the slope k_1 of the corresponding diameter are connected by the relation

$$kk_1 = e^2 - 1 \tag{1}$$

However, the relation (1a), Sec. 55, is replaced by the relation

$$kk_1 = +\frac{b^2}{a^2} \tag{1b}$$

Example 1. The diameter $U'U$ of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ (Fig. 80), which corresponds to chords with slope $k = \frac{10}{9}$, is given by the equation $y = k_1 x$; the value of k_1 is determined from the relation $kk_1 = \frac{4}{9}$, so that the equation of the diameter $U'U$ is $y = \frac{2}{5}x$.

Example 2. The diameter $V'V$ (Fig. 80) of the same hyperbola, corresponding to chords with slope $k = \frac{2}{5}$, is given by the equation $y = \frac{10}{9}x$.

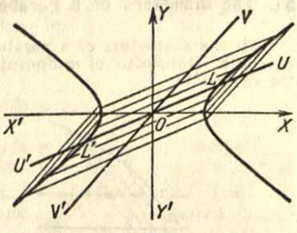


Fig. 80

If the diameter $U'U$ bisects chords parallel to the diameter $V'V$, then $V'V$ will always bisect chords parallel to $U'U$. Two such diameters are termed *conjugate*.

Each hyperbola has only one pair of *principal* (i. e. conjugate and mutually perpendicular) diameters: the transverse axis and the conjugate axis.

If the slope of the parallel chords is greater in absolute value than the slope of the asymptote, i. e.

$$|k| > \frac{b}{a}$$

$\left(\text{see Example 1, where } \frac{b}{a}=\frac{2}{3}\right)$, then the locus of the midpoints of the chords is a pair of rays ($L'U'$ and LU). But if

$$|k|<\frac{b}{a}$$

(see Example 2), then the midpoints of the chords fill the diameter ($V'V$ in Fig. 80) completely. Of two conjugate diameters, one always belongs to the first type, the other to the second.

Note 1. The slope of parallel chords cannot be equal, in absolute value, to $\frac{b}{a}$, for the straight lines $y=\pm\frac{b}{a}x$ (asymptotes) do not intersect the hyperbola, and the straight lines parallel to the asymptote intersect the hyperbola at only one point.

According to (1b), the slopes of the nonprincipal conjugate directions have the same signs; i. e. *two conjugate diameters of a hyperbola belong to one and the same pair of vertical angles formed by the axes.*

Contrariwise, with respect to asymptotes, two conjugate diameters belong to *different pairs* of vertical angles.

Note 2. When the diameter $U'U$ of a hyperbola is rotated, the conjugate diameter $V'V$ rotates in the opposite sense. When, in the process, $U'U$ approaches one of the asymptotes without bound, $V'V$ unboundedly approaches *the same* asymptote. We therefore say that an asymptote is a diameter *conjugate to itself*. This statement is, strictly speaking, not true because an asymptote is not a diameter (cf. Note 1). Aside from the asymptotes, any straight line passing through the centre of the hyperbola is one of its diameters.

57. The Diameters of a Parabola

All the diameters of a parabola are parallel to its axis; see Figs. 81 and 82 (the locus of midpoints of parallel chords of the parabola is the ray LU).

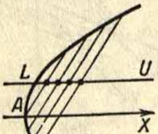


Fig. 81

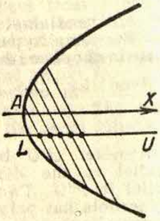


Fig. 82

The diameter corresponding to chords perpendicular to the axis of the parabola is the axis itself (Fig. 83).

The diameter of the parabola $y^2=2px$, which corresponds to chords with slope k ($k \neq 0$), is given by the equation

$$y=\frac{p}{k}$$

(the greater the inclination of the chord to the axis, the farther is the diameter from the axis). [1]

Example. The diameter of the parabola $y^2=2px$, which corresponds to chords inclined to the axis at an angle of +45° ($k=1$) is given by the equation $y=p$; in other words, its distance from the axis AX (Fig. 84) is equal to half the latus rectum FR (Sec. 52). This means the diameter cuts the parabola at the point R above the focus F.

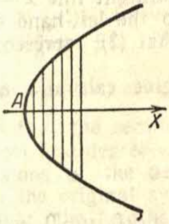


Fig. 83

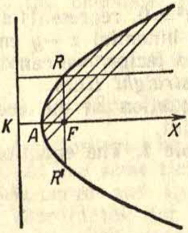


Fig. 84

All straight lines parallel to some diameter of a parabola cut the parabola at only one point. That is why the parabola does not have conjugate diameters.

58. Second-Order Curves (Quadric Curves)

The ellipse (the circle, as a special case), the hyperbola, and the parabola are second-order (or quadric) curves; i. e. in any system of cartesian coordinates they are defined by second-degree equations. However, not every second-degree equation represents one of these curves. It may happen that an equation of the second degree represents a pair of straight lines.

Example 1. The equation

$$4x^2-9y^2=0 \tag{1}$$

---

[1] The slope of any diameter of a parabola is zero, i. e. it satisfies the equation $kk_1=\varepsilon^2-1$, which holds (Secs. 55, 56) for the ellipse and the hyperbola (for the parabola $\varepsilon=1$).

which decomposes into two equations $2x-3y=0$ and $2x+3y=0$, represents a pair of straight lines which intersect at the origin.

Example 2. The equation

$$x^2-2xy+y^2-9=0 \tag{2}$$

which decomposes into the equations $x-y+3=0$ and $x-y-3=0$, represents a pair of parallel straight lines.

Example 3. The equation

$$x^2-2xy+y^2=0 \tag{3}$$

or $(x-y)^2=0$, represents a single straight line $x-y=0$; but since the binomial $x-y$ enters into the left-hand side of (3) twice as a factor, we can take it that (3) represents *two coincident straight lines.*

An equation of the second degree can also represent a single point.

Example 4. The equation

$$x^2+\frac{1}{4}y^2=0 \tag{4}$$

has only one real solution, namely $x=0$, $y=0$, which represents the point (0, 0). Incidentally, (4) can be decomposed into two equations $x+\frac{1}{2}iy=0$, $x-\frac{1}{2}iy=0$ with imaginary coefficients. For this reason, (4) is said to represent a pair of imaginary straight lines intersecting in a real point.

Finally, it can happen that an equation of the second degree does not represent any locus at all.

Example 5. The equation

$$\frac{x^2}{-9}+\frac{y^2}{-16}=1 \tag{5}$$

does not represent either a line or a point because the quantity $\frac{x^2}{-9}+\frac{y^2}{-16}$ cannot have a positive value. However, because of the external similarity between (5) and the equation of the ellipse, equation (5) is said to represent an imaginary ellipse.

Example 6. The equation

$$x^2-2xy+y^2+9=0 \tag{6}$$

likewise fails to represent either a curve or a point. But since it decomposes into the equations $x-y+3i=0$ and $x-y-3i=0$, we say (cf. Example 2) that (6) represents a pair of imaginary parallel straight lines.

Conic sections and pairs of straight lines exhaust all the curves that can be defined by second-degree equations in a cartesian system of coordinates. Thus, the following theorem is valid.

Theorem. *Any curve of the second order (quadric curve) is either an ellipse, a hyperbola, a parabola or a pair of straight lines (intersecting, parallel or coincident).*

Plan of proof. By means of a transformation of coordinates, the given second-degree equation is reduced to a simpler form. We then either obtain one of the canonical (standard) equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\pm 1 \quad \text{(ellipse, real or imaginary)},$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{(hyperbola)}, \quad y^2=2px \quad \text{(parabola)}$$

or we find that the second-degree equation may be decomposed into two first-degree equations. At the same time we find the dimensions of the second-order curve and its position relative to the original system of coordinates (for example, for an ellipse, the lengths of the axes, their equations, the position of the centre, etc.).

These transformations are given in full in Secs. 61 and 62.

59. General Second-Degree Equation

The general equation of the second degree is usually written as

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \tag{1}$$

The designations $2B$, $2D$, $2E$ (instead of B, D, E) are introduced because many formulas employ half-coefficients of xy, of x and of y. This notation dispenses with fractional expressions.

Example 1. For the equation

$$x^2+xy-2y^2+2x+4y+4=0$$

we have

$$A=1, \quad B=\frac{1}{2}, \quad C=-2, \quad D=1, \quad E=2, \quad F=4$$

Example 2. For the equation $2xy+x+5=0$ we have

$$A=0, \quad B=1, \quad C=0, \quad D=\frac{1}{2}, \quad E=0, \quad F=5$$

Note. The quantities A, B, C, D, E, F may take on any values so long as A, B, C are not all equal to zero at once, for then (1) would be an equation of the first degree.

60. Simplifying a Second-Degree Equation. General Remarks

Transformation of the second-degree equation

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \tag{1}$$

to one of the elementary forms (see Sec. 58) will be done as follows [1]:

(a) *Preliminary transformation.* In this way we eliminate the term containing a product of the coordinates (this is achieved by rotating the axes; see Sec. 61).

(b) *Final transformation.* Here we get rid of terms containing first degrees of the coordinates (this is attained by a translation of the origin; see Sec. 62).

61. Preliminary Transformation of a Second-Degree Equation

(If $B=0$, this transformation becomes unnecessary.)

Turn the coordinate axis through an angle α which satisfies the condition [2]

$$\tan 2\alpha=\frac{2B}{A-C} \tag{2}$$

The transformation formulas will be (Sec. 36)

$$x=x'\cos\alpha-y'\sin\alpha,\quad y=x'\sin\alpha+y'\cos\alpha \tag{3}$$

The terms involving $x'y'$ cancel out, [3] and the new equation takes the form

$$A'x'^2+C'y'^2+2D'x'+2E'y'+F'=0 \tag{4}$$

Example 1. Given the equation

$$2x^2-4xy+5y^2-x+5y-4=0 \tag{1a}$$

---

[1] The method presented here is not the fastest, but it has the advantage of not requiring any auxiliary theorems. A faster method is given in Secs. 69 and 70.

[2] If $A=C$ $\left(\text{the quantity } \frac{2b}{A-C} \text{ becomes infinite}\right)$, then (see Sec. 21, Note) $2\alpha=\pm 90°$, or $\alpha=\pm 45°$.

[3] The coefficient of $x'y'$ is of the form

$$2B'=(C-A)\ 2\sin\alpha\cos\alpha+2B(\cos^2\alpha-\sin^2\alpha)=$$
$$=(C-A)\sin 2\alpha+2B\cos 2\alpha$$

This coefficient is equal to zero by virtue of (2).

Here, $A=2$, $B=-2$, $C=5$, $D=-\frac{1}{2}$, $E=\frac{5}{2}$, $F=-4$. From Condition (2) we find

$$\tan 2\alpha=\frac{-4}{-3}=\frac{4}{3} \tag{2a}$$

If the angle 2α is taken in the first quadrant ($2\alpha \approx 53°8'$, $\alpha \approx 26°34'$), we obtain

$$\cos 2\alpha=\frac{1}{\sqrt{1+\tan^2 2\alpha}}=\frac{3}{5},$$

$$\sin\alpha=\sqrt{\frac{1-\cos 2\alpha}{2}}=\frac{1}{\sqrt{5}},$$

$$\cos\alpha=\sqrt{\frac{1+\cos 2\alpha}{2}}=\frac{2}{\sqrt{5}}$$

Formulas (3) take the form

$$\left.\begin{aligned} x&=\frac{2}{\sqrt{5}}x'-\frac{1}{5}y',\\ y&=\frac{1}{\sqrt{5}}x'+\frac{2}{\sqrt{5}}y' \end{aligned}\right\} \tag{3a}$$

Putting this into (1a), we get a new equation:

$$x'^2+6y'^2+\frac{3}{\sqrt{5}}x'+\frac{11}{\sqrt{5}}y'-4=0 \tag{4a}$$

where

$$A'=1,\quad B'=0,\quad C'=6,\quad D'=\frac{3}{2\sqrt{5}},\quad E'=\frac{11}{2\sqrt{5}},\quad F'=-4$$

If an angle 2α is taken in the third quadrant ($2\alpha \approx 233°8'$, $\alpha \approx 116°34'$) then we get in similar fashion the equation

$$6x'^2+y'^2+\frac{11}{\sqrt{5}}x'-\frac{3}{\sqrt{5}}y'-4=0$$

where

$$A'=6,\ B'=0,\ C'=1,\ D'=\frac{11}{2\sqrt{5}},\ E'=-\frac{3}{2\sqrt{5}},\ F'=-4$$

Example 2. Given the equation

$$x^2+2xy+y^2+2x+y=0 \tag{1b}$$

Here

$$A=1,\ B=1,\ C=1,\ D=1,\ E=\frac{1}{2},\ F=0$$

Since $A=C$, it follows (see footnote No. 2 on page 86) that we can take $\alpha=45°$. Substituting into (1b) the expressions

$$\left.\begin{aligned} x=x'\cos 45°-y'\sin 45°=\frac{1}{\sqrt{2}}(x'-y'), \\ y=x'\sin 45°+y'\cos 45°=\frac{1}{\sqrt{2}}(x'+y') \end{aligned}\right\} \tag{3b}$$

we find

$$2x'^2+\frac{3}{\sqrt{2}}x'-\frac{1}{\sqrt{2}}y'=0 \tag{4b}$$

Here

$$A'=2,\ B'=0,\ C'=0,\ D'=\frac{3}{2\sqrt{2}},\ E'=-\frac{1}{2\sqrt{2}},\ F'=0$$

If we take $\alpha=-45°$, then we get

$$2y'^2+\frac{1}{\sqrt{2}}x'+\frac{3}{\sqrt{2}}y'=0 \tag{4b$'$}$$

Here

$$A'=0,\ B'=0,\ C'=2,\ D'=\frac{1}{2\sqrt{2}},\ E'=\frac{3}{2\sqrt{2}},\ F'=0$$

Example 3. Given the equation

$$2x^2-4xy+2y^2+8x-8y-17=0 \tag{1c}$$

Since $A=C$, it follows that we can take $\alpha=45°$. Substituting into (1c) the expressions (3b), we find

$$4y'^2-8\sqrt{2}y'-17=0 \tag{4c}$$

Taking $\alpha=-45°$, we get

$$4x'^2+8\sqrt{2}x'-17=0 \tag{4c$'$}$$

62. Final Transformation of a Second-Degree Equation

One has to distinguish two cases:

(1) not one of the coefficients A', C' in the equation

$$A'x'^2+C'y'^2+2D'x'+2E'y'+F'=0 \tag{4}$$

is zero (as the case in Example 1);

(2) one of the coefficients A', C' is zero (as in Examples 2 and 3).[1]

---

[1] The coefficients A' and C' cannot both be zero, otherwise Eq. (4) would be of degree one.

Case 1. The equation

$$A'x'^2+C'y'^2+2D'x'+2E'y'+F'=0 \tag{4}$$

is transformed as follows: adjoin the term $\frac{D'^2}{A'}$ to the sum $A'x'^2+2D'x'=A'\left(x'^2+2\frac{D'}{A'}x'\right)$ to yield $A'\left(x'+\frac{D'}{A'}\right)^2$; adjoin the term $\frac{E'^2}{C'}$ to the sum $C'y'^2+2E'y'$ to yield $C'\left(y'+\frac{E'}{C'}\right)^2$. To compensate, add $\frac{D'^2}{A'}+\frac{E'^2}{C'}$ to the right-hand side of (4). The result is an equation of the form

$$A'\left(x'+\frac{D'}{A'}\right)^2+C'\left(y'+\frac{E'}{C'}\right)^2=K' \tag{5}$$

where

$$K'=\frac{D'^2}{A'}+\frac{E'^2}{C'}-F'$$

Carry the origin to the point $\left(-\frac{D'}{A'},\ -\frac{E'}{C'}\right)$, which amounts to transforming the coordinates (Sec. 35) by the formulas

$$x'=\bar{x}-\frac{D'}{A'},\quad y'=\bar{y}-\frac{E'}{C'} \tag{6}$$

This yields the equation

$$A'\bar{x}^2+C'\bar{y}^2=K\ \ (A'\neq 0,\ C'\neq 0) \tag{7}$$

If $K'\neq 0$, then we divide the equation by K' to get

$$\frac{\bar{x}^2}{\frac{K'}{A'}}+\frac{\bar{y}^2}{\frac{K'}{C'}}=1 \tag{8}$$

(a) If both quantities $\frac{K'}{A'}$, $\frac{K}{C'}$ are positive, we have an ellipse.

(b) If both quantities $\frac{K'}{A'}$, $\frac{K'}{C'}$ are negative, we have an imaginary ellipse (cf. Example 5, Sec. 58).

(c) If one of the quantities (which one is immaterial) is positive and the other negative, then we have a hyperbola.

But if $K'=0$, then equation (7) is of the form

$$A'\bar{x}^2+C'\bar{y}^2=0 \tag{7'}$$

Two cases are possible:

(d) If A' and C' have different signs, then $A'\bar{x}^2+C'\bar{y}^2$ may be decomposed into first-degree (linear) factors as a difference of squares. The coefficients of both factors are real and we have a pair of intersecting straight lines (cf. Example 1, Sec. 58).

(e) If A' and C' have the same sign, then $A'\bar{x}^2+C'\bar{y}^2$ also decomposes into linear factors, but both factors contain terms involving imaginary coefficients and we have a pair of imaginary intersecting straight lines, i. e. a single real point (cf. Example 4, Sec. 58).

Example 1. After rotation of the axes, Eq. (1a) of Example 1, Sec. 61, was brought to the form

$$x'^2+6y'^2+\frac{3}{\sqrt{5}}x'+\frac{11}{\sqrt{5}}y'-4=0 \tag{4a}$$

This equation can be written as

$$\left(x'+\frac{3}{2\sqrt{5}}\right)^2+6\left(y'+\frac{11}{12\sqrt{5}}\right)^2=\left(\frac{3}{2\sqrt{5}}\right)^2+ +6\left(\frac{11}{12\sqrt{5}}\right)^2+4 \tag{5a}$$

or

$$\left(x'+\frac{3}{2\sqrt{5}}\right)^2+6\left(y'+\frac{11}{12\sqrt{5}}\right)^2=\frac{131}{24}$$

Going over to the new system with origin at the point $\left(-\frac{3}{2\sqrt{5}}, -\frac{11}{12\sqrt{5}}\right)$, via the formulas

$$x'=\bar{x}-\frac{3}{2\sqrt{5}}, \quad y'=\bar{y}-\frac{11}{12\sqrt{5}} \tag{6a}$$

we have

$$\bar{x}^2+6\bar{y}^2=\frac{131}{24} \tag{7a}$$

or

$$\frac{\bar{x}^2}{\frac{131}{24}}+\frac{\bar{y}^2}{\frac{131}{144}}=1 \tag{8a}$$

The equation under study represents an ellipse with semiaxes $a=\sqrt{\frac{131}{24}}\approx 2.3$, $b=\sqrt{\frac{131}{144}}\approx 1.0$. In Fig. 85 (where OE is the scale unit) $a=O'A$, $b=O'B$.

The centre of the ellipse is at the point O' with coordinates $\bar{x}=0$, $\bar{y}=0$. Using formulas (6a) we find the coordi-

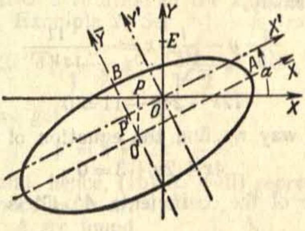


Fig. 85

nates of the centre in the intermediate system $X'OY'$:

$$x' = -\frac{3}{2\sqrt{5}} \approx -0.7,$$

$$y' = -\frac{11}{12\sqrt{5}} \approx -0.4$$

In Fig. 85,

$$x' = OP', \quad y' = P'O'$$

Using formulas (3a), Sec. 61, we find the coordinates of the centre of the original system XOY:

$$x_{\text{cen}} = \frac{2}{\sqrt{5}}\left(-\frac{3}{2\sqrt{5}}\right) - \frac{1}{\sqrt{5}}\left(-\frac{11}{12\sqrt{5}}\right) = -\frac{5}{12} \approx -0.4,$$

$$y_{\text{cen}} = \frac{1}{\sqrt{5}}\left(-\frac{3}{2\sqrt{5}}\right) + \frac{2}{\sqrt{5}}\left(-\frac{11}{12\sqrt{5}}\right) = -\frac{2}{3} \approx -0.7$$

In Fig. 85, $x_{\text{cen}} = OP$, $y_{\text{cen}} = PO'$.

Let us find the equations of the axes of the ellipse in the original system. In the system $\bar{X}O'\bar{Y}$ the major axis is represented by the equation $\bar{y} = 0$, in the system $X'OY'$ the same axis [by virtue of the second equation in (6a)] is given by the equation $y' = -\frac{11}{12\sqrt{5}}$.

Solving the system (3a) for x', y', we find

$$x' = \frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y,$$

$$y' = \frac{2}{\sqrt{5}}y - \frac{1}{\sqrt{5}}x$$

We only need the latter of these equations; putting $y' = -\frac{11}{12\sqrt{5}}$ in it we get the equation of the major axis in the system XOY; namely,

$$\frac{2}{\sqrt{5}}y-\frac{1}{\sqrt{5}}x=-\frac{11}{12\sqrt{5}}$$

or

$$12x-24y-11=0$$

In the same way we find the equation of the minor axis:

$$4x+2y+3=0$$

Case 2. One of the coefficients A', C' is zero. Eq. (4) is of the form

$$A'x'^2+2D'x'+2E'y'+F'=0 \tag{9}$$

or

$$C'y'^2+2D'x'+2E'y'+F'=0 \tag{9'}$$

Let us consider an equation of the type (9) [for equations of type (9′) the calculations are the same but x' and y' are interchanged].

(a) If $E'\neq 0$, then Eq. (9) may be solved for y'; this yields

$$y'=-\frac{A'}{2E'}x'^2-\frac{D'}{E'}x'-\frac{F'}{2E'} \tag{10}$$

We have a parabola. The coordinates of the vertex are defined by the formula (5), Sec. 50, for

$$a=-\frac{A'}{2E'},\quad b=-\frac{D'}{E'},\quad c=-\frac{F'}{2E'}$$

(b) If $E'=0$, then Eq. (9) is of the form

$$A'x'^2+2D'x'+F'=0 \tag{11}$$

Factoring the left-hand side of (11) into linear factors, we get[1]

$$A'\left(x'-\frac{\sqrt{D'^2-A'F'}-D'}{A'}\right)\left(x'+\frac{\sqrt{D'^2-A'F'}+D'}{A'}\right)=0 \tag{12}$$

For $D'^2-A'F'>0$, Eq. (12) [and hence, (11)] represents a pair of parallel lines, for $D'^2-A'F'<0$, a pair of imaginary parallel lines, and for $D'^2-A'F'=0$, two coincident straight lines (Sec. 58, Examples 2, 6, 3).

---

[1] The quantities $\frac{\sqrt{D'^2-A'F'}-D'}{A'}$ and $\frac{-\sqrt{D'^2-A'F'}-D'}{A'}$ are roots of Eq. (11).

Example 2. After a rotation of the axes through an angle of 45°, Eq. (1b), Example 2, Sec. 61, was transformed to

$$2x'^2+\frac{3}{\sqrt{2}}x'-\frac{1}{\sqrt{2}}y'=0 \tag{4b}$$

Solving for y', we get

$$y'=2\sqrt{2}x'^2+3x' \tag{10b}$$

Equation (10b) [and, hence, (1b) as well] represents a parabola (Fig. 86); the coordinates x', y' of its vertex A are found from formulas (5), Sec. 50:

$$x'_A=-\frac{3}{4\sqrt{2}}\approx-0.5,$$

$$y'_A=-\frac{9}{8\sqrt{2}}\approx-0.8$$

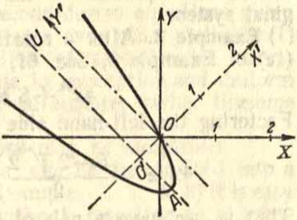


Fig. 86

The coordinates of the vertex may be found without resorting to formulas (5), Sec. 50 (see Sec. 50, Note 1).

Using formulas (3b), Sec. 61, we find the coordinates of the vertex in the original system:

$$x_A=\frac{\sqrt{2}}{2}(x'-y')=\frac{\sqrt{2}}{2}\left(-\frac{3}{4\sqrt{2}}+\frac{9}{8\sqrt{2}}\right)=\frac{3}{16}\approx 0.2,$$

$$y_A=\frac{\sqrt{2}}{2}(x'+y')=\frac{\sqrt{2}}{2}\left(-\frac{3}{4\sqrt{2}}-\frac{9}{8\sqrt{2}}\right)=$$

$$=-\frac{15}{16}\approx-0.9$$

Let us find the equation of the axis AU of the parabola. In the new system, this axis is given by the equation

$$x'=-\frac{3}{4\sqrt{2}}$$

Solving Eq. (3b) for x', y', we find

$$x'=\frac{\sqrt{2}}{2}(x+y),$$

$$y'=\frac{\sqrt{2}}{2}(y-x)$$

Substituting $x' = -\frac{3}{4\sqrt{2}}$ into the first equation (the second one is not needed), we obtain

$$-\frac{3}{4\sqrt{2}} = \frac{\sqrt{2}}{2}(x+y)$$

or

$$4x + 4y + 3 = 0$$

This is the equation of the axis of the parabola in the original system.

Example 3. After a rotation of the axes through −45°, Eq. (1c) of Example 3, Sec. 61, was transformed to

$$4x'^2 + 8\sqrt{2}\,x' - 17 = 0 \tag{4c'}$$

Factoring the left-hand side of Eq. (4c′), we get

$$4\left(x' - \frac{5-2\sqrt{2}}{2}\right)\left(x' + \frac{5+2\sqrt{2}}{2}\right) = 0 \tag{12c}$$

That is, we have a pair of parallel lines (UV and $U'V'$ in Fig. 87):

$$x' = \frac{5-2\sqrt{2}}{2}, \quad x' = -\frac{5+2\sqrt{2}}{2} \tag{13}$$

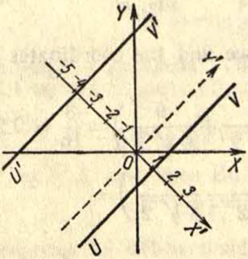


Fig. 87

Let us find the equations of these lines in the XOY system. Since the XOY system is obtained from $X'OY'$ by a rotation through +45°, it follows that

$$x' = \frac{\sqrt{2}}{2}(x-y), \quad y' = \frac{\sqrt{2}}{2}(x+y) \tag{14}$$

Substituting, into the first of these equations, first one and then the other value of (13), we find

$$\frac{5-2\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x-y),$$

$$-\frac{5+2\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x-y)$$

or

$$\sqrt{2}x-\sqrt{2}y-5+2\sqrt{2}=0,$$
$$\sqrt{2}x-\sqrt{2}y+5+2\sqrt{2}=0$$

These are equations of the straight lines UV, $U'V'$ in the original system.

63. Techniques to Facilitate Simplification of a Second-Degree Equation

The method of simplifying second-degree equations given in Secs. 61 and 62 has two advantages over other methods: (1) it provides a complete classification of second-order curves (Theorem, Sec. 58); (2) it is simple in conception and uniform in structure. However, this method requires rather tiresome computations.

In many cases the computations may be simplified.

1. For second-order curves that can be decomposed into a pair of straight lines (Sec. 58, Examples 2, 3, 4, 6) it is easy to find equations of both lines without resorting to a transformation of coordinates. This method is presented in Sec. 65. Sec. 64 gives a decomposition test.

2. A nondecomposable curve of second order may be either an ellipse, a hyperbola or a parabola. The ellipse and hyperbola have centres, while the parabola does not. It is therefore convenient to start simplifying the equations of the ellipse and hyperbola by translating the origin to the centre. We can find out, beforehand, to which of these three types the second-order curve belongs. The appropriate test is given in Sec. 67; the concept of centre is specified in Sec. 68; Sec. 69 explains how to find the coordinates of the centre. A device is explained in Sec. 70 for simplifying the equations of the ellipse and the hyperbola.

3. As for the parabola, the method of simplification given in Sec. 61 remains the best. Incidentally, the dimensions of a parabola (i. e. the magnitude of the parameter p) are readily found by means of so-called invariants (see Sec. 66).

64. Test for Decomposition of Second-Order Curves

If a second-order curve

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \tag{1}$$

can be decomposed into two (different or coincident) straight lines (which can be imaginary as well), then the third-order determinant (Sec. 118)

$$\Delta=\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} \tag{2}$$

(the major discriminant 1)) vanishes. Conversely, if $\Delta=0$, then line (1) decomposes into two straight lines.

For the proof see Note 2, Sec. 65.

Example 1. In Sec. 61 (Example 3) we considered the second-order curve

$$2x^2-4xy+2y^2+8x-8y-17=0$$
$$(A=2,\quad B=-2,\quad C=2,\quad D=4,\quad E=-4,\quad F=-17)$$

In Sec. 62 (Example 3) it was established that this curve may be decomposed into two parallel lines:

$$\sqrt{2}\,x-\sqrt{2}\,y-5+2\sqrt{2}=0 \tag{3}$$

and

$$\sqrt{2}\,x-\sqrt{2}\,y+5+2\sqrt{2}=0 \tag{4}$$

Accordingly, the major discriminant Δ is zero. Indeed,

$$\Delta=\begin{vmatrix} 2 & -2 & 4 \\ -2 & 2 & -4 \\ 4 & -4 & -17 \end{vmatrix}=2\begin{vmatrix} 2 & -4 \\ -4 & -17 \end{vmatrix}+2\begin{vmatrix} -2 & -4 \\ 4 & -17 \end{vmatrix}+$$
$$+4\begin{vmatrix} -2 & 2 \\ 4 & -4 \end{vmatrix}=2\cdot(-50)+2\cdot 50+0=0$$

Example 2. The second-order curve

$$2x^2-4xy+5y^2-x+5y-4=0$$

does not decompose, since the major discriminant

$$\Delta=\begin{vmatrix} 2 & -2 & -\frac{1}{2} \\ -2 & 5 & \frac{5}{2} \\ -\frac{1}{2} & \frac{5}{2} & -4 \end{vmatrix}=-\frac{131}{4}$$

---

1) The discriminant Δ is called *major* in contrast to the *minor* discriminant described in Sec. 66.

is not equal to zero. In Secs. 61, 62 (Example 1) it was shown that this curve is an ellipse.

Rule for memorizing expression (2). The first row contains the letters followed by x in Eq. (1), the second row, the letters followed by y (either directly or after x), the third row, the last three letters.

65. Finding Straight Lines that Constitute a Decomposable Second-Order Curve

In order to find the equations of two straight lines which together form a decomposable second-order curve

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \qquad (1)$$

(see Sec. 64 for the condition of decomposition), it is sufficient to expand the left-hand side of (1) into linear factors. When at least one of the coefficients A, C is nonzero, it is best to solve Eq. (1) directly for the square of x or y. The two solutions (they may coincide) are the two desired straight lines.

Example 1. The second-order curve

$$2x^2-4xy+2y^2+8x-8y-17=0 \qquad (2)$$

is a decomposable line, since the major discriminant

$$\Delta=\begin{vmatrix} 2 & -2 & 4 \\ -2 & 2 & -4 \\ 4 & -4 & -17 \end{vmatrix}$$

is zero. Eq. (2) may be solved for either of the letters x, y (both enter as squares). Representing (2) as

$$y^2-2(x+2)y+\left(x^2+4x-\frac{17}{2}\right)=0$$

we solve for y and obtain

$$y=x+2\pm\sqrt{(x+2)^2-\left(x^2+4x-\frac{17}{2}\right)}$$

i. e.

$$y=x+2\pm\frac{5}{\sqrt{2}}$$

One of the straight lines is given by the equation $y=x+2+\frac{5}{\sqrt{2}}$, the other, by the equation $y=x+2-\frac{5}{\sqrt{2}}$. These lines are parallel (cf. Example 3, Secs. 61, 62).

Example 2. The second-order curve

$$2x^2+7xy-15y^2-10x+54y-48=0 \tag{3}$$

decomposes, since

$$\Delta=\begin{vmatrix} 2 & \frac{7}{2} & -5 \\ \frac{7}{2} & -15 & 27 \\ -5 & 27 & -48 \end{vmatrix}=0$$

Representing (3) as

$$15y^2-(7x+54)\,y-(2x^2-10x-48)=0$$

we find

$$y=\frac{7x+54\pm\sqrt{(7x+54)^2+4\cdot 15\,(2x^2-10x-48)}}{30}$$

The radicand is equal to $169x^2+156x+36=(13x+6)^2$. Consequently, $y=\frac{7x+54\,\pm\,(13x+6)}{30}$. One of the straight lines is given by the equation $y=\frac{2x+6}{3}$, the other by the equation $y=\frac{-x+8}{5}$. These lines intersect in the point $\left(-\frac{6}{13},\ \frac{22}{13}\right)$.

Example 3. The curve

$$10xy-14x+15y-21=0 \tag{4}$$

decomposes, since

$$\Delta=\begin{vmatrix} 0 & 5 & -7 \\ 5 & 0 & \frac{15}{2} \\ -7 & \frac{15}{2} & -21 \end{vmatrix}=0$$

Both x and y are linear in Eq. (4), and so we factor the left-hand side of (4) and group terms:

$$10xy-14x+15y-21=2x\,(5y-7)+3\,(5y-7)=$$
$$=(2x+3)\,(5y-7)$$

The curve (4) decomposes into the straight lines $2x+3=0$ and $5y-7=0$.

Note 1. If $A=C=0$ we can also solve the equation for x or y; in Example 3 we get $(10x+15)\,y=14x+21$; but it is possible to further divide both sides by $10x+15$ only when $10x+15$ is not equal to zero. We then get $y=\frac{14x+21}{10x+15}=$ $=\frac{7\,(2x+3)}{5\,(2x+3)}=\frac{7}{5}$ and the equation of one of the straight lines is $y=\frac{7}{5}$, i.e. $5y-7=0$. When $10x+15=0$, or $x=-\frac{3}{2}$, the equation $(10x+15)\,y=14x+21$ is satisfied for any value of y; we thus get the other straight line $x=-\frac{3}{2}$ or $2x+$ $+3=0$.

Note 2. The calculations carried out in Examples 1 and 2 may be performed for any equation of type (1), provided $C \neq 0$. Performing these computations in the general form, we get as the radicand the quadratic trinomial

$$(B^2-AC)\,x^2+2\,(BE-CD)\,x+E^2-CF \tag{5}$$

It will be a perfect square if and only if

$$(BE-CD)^2-(B^2-AC)\,(E^2-CF)=0 \tag{6}$$

After simple transformations we see that the left-hand side of (6) is equal to $C\Delta$ where Δ is the major discriminant. Since, by hypothesis, $C \neq 0$, the criterion for decomposition is $\Delta=0$. When $C=0$, but $A \neq 0$, we arrive at the same conclusion by interchanging x and y. Such is the proof of the criterion (test) in Sec. 64 for the general case. In the exceptional case of $A=C=0$ (and, hence, $B \neq 0$), the left-hand side of Eq. (1) is in the form

$$2Bxy+2Dx+2Ey+F$$

We can give this polynomial in the form $2x\,(By+D)+$ $+(2Ey+F)$. This expression may be factored into linear terms only when the appropriate coefficients of the binomials $By+D$ and $2Ey+F$ are equal or proportional (see Example 3); i.e. when $2DE-BF=0$. However, in the case at hand the major discriminant Δ is of the form $\begin{vmatrix} 0 & B & D \\ B & 0 & E \\ D & E & F \end{vmatrix}$, whence it follows that $2DE-BF=\frac{\Delta}{B}$ Such is the proof of the criterion (test) of Sec. 64 for the exceptional case.

66. Invariants of a Second-Degree Equation

When passing from one system of rectangular coordinates to another we replace the equation

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \tag{1}$$

of a second-order curve by the equation

$$A'x'^2+2B'x'y'+C'y'^2+2D'x'+2E'y'+F'=0 \qquad (2)$$

which is obtained from (1) by the formulas of transformation of coordinates (see examples in Secs. 61 and 62). The values of A', B', C', D', E', F' (all or some) differ from the values of the like quantities A, B, C, D, E, F.

However, the three expressions given below which consist of the quantities A', B', C', D', E', F' always remain *equal to the like expressions composed of the quantities* A, B, C, D, E, F. These three expressions are called the *invariants* (meaning that they do not change) of a second-degree equation.

(a) First invariant $A+C$

(b) Second invariant $\delta=\begin{vmatrix} A & B \\ B & C \end{vmatrix}$ (minor discriminant)

(c) Third invariant

$$\Delta=\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} \quad \text{(major discriminant)}$$

Example 1. In Sec. 61 (Example 1) we transformed the equation

$$2x^2-4xy+5y^2-x+5y-4=0$$

$$\left(A=2,\ B=-2,\ C=5,\ D=-\frac{1}{2},\ E=\frac{5}{2},\ F=-4\right)$$

to the form

$$x'^2+6y'^2+\frac{3}{\sqrt{5}}x'+\frac{11}{\sqrt{5}}y'-4=0$$

$$\left(A'=1,\ B'=0,\ C'=6,\ D'=\frac{3}{2\sqrt{5}},\ E'=\frac{11}{2\sqrt{5}},\ F'=-4\right)$$

in accordance with the rotation of axes through the angle $\arcsin\frac{1}{\sqrt{5}}\approx 26°34'$.

(a) The expression $A+C$ in the old system was equal to $2+5=7$; in the new system, the like expression $A'+C'$ is $1+6=7$, so that

$$A+C=A'+C'$$

(b) The minor discriminant in the old system was

$$\delta=\begin{vmatrix} 2 & -2 \\ -2 & 5 \end{vmatrix}=2\cdot5-(-2)\cdot(-2)=6$$

in the new system we have

$$\delta' = \begin{vmatrix} 1 & 0 \\ 0 & 6 \end{vmatrix} = 6$$

so that

$$\delta = \delta'$$

(c) The major discriminant in the old system was

$$\Delta = \begin{vmatrix} 2 & -2 & -\frac{1}{2} \\ -2 & 5 & \frac{5}{2} \\ -\frac{1}{2} & \frac{5}{2} & -4 \end{vmatrix} = -\frac{131}{4}$$

in the new system it is

$$\Delta' = \begin{vmatrix} 1 & 0 & \frac{3}{2\sqrt{5}} \\ 0 & 6 & \frac{11}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} & \frac{11}{2\sqrt{5}} & -4 \end{vmatrix} = -\frac{131}{4}$$

so that

$$\Delta = \Delta'$$

Example 2. In Sec. 62 (Example 1) we transformed the equation

$$x'^2 + 6y'^2 + \frac{3}{\sqrt{5}} x' + \frac{11}{\sqrt{5}} y' - 4 = 0$$

to the form $\bar{x}^2 + 6\bar{y}^2 - \frac{131}{24} = 0$ in accordance with a translation of the origin to the point $x' = -\frac{3}{2\sqrt{5}}$, $y' = -\frac{11}{12\sqrt{5}}$. The major discriminant is now

$$\bar{\Delta} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -\frac{131}{24} \end{vmatrix} = -\frac{131}{4}$$

so that

$$\bar{\Delta} = \Delta' = \Delta$$

The two other invariants have obviously also retained their earlier values.

To prove the invariance of each of the quantities (a), (b), and (c), it is sufficient to form expressions of the quantities $A', B', C', \ldots$ in terms of $A, B, C, \ldots$ (these expressions will also contain the angle of rotation α and the coordinates of the new origin). Substituting them, for example, into the expression $A'+C'$, we get (after simplifications) $A+C$ and so forth. However, these computations are very cumbersome.[1]

Note. If both sides of Eq. (1) are multiplied (or divided) by some number k, the new equation will represent the same second-order curve. However, the quantities (a), (b), (c) will be changed: the first will be multiplied by k, the second by k^2 and the third by k^3. That is why the quantities (a), (b), and (c) are termed invariants of a quadratic (second-degree) *equation* and not invariants of a quadric (second-order) *curve*.

67. Three Types of Second-Order Curves

The minor discriminant δ (Sec. 66) for the ellipse is positive (see Example 1, Sec. 66), for the hyperbola it is negative, and for the parabola it is zero.

Proof. The ellipse is given by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. The minor discriminant of this equation $\delta=\frac{1}{a^2}\cdot\frac{1}{b^2}>0$. In a transformation of coordinates, δ retains its magnitude, but in multiplication of both sides of the equation by some number k the discriminant is multiplied by k^2 (Sec. 66, note). Hence, the discriminant of the ellipse is positive in any system of coordinates. The proof is similar for the hyperbola and the parabola.

We accordingly distinguish three types of second-order curves (and quadratic equations):

(a) *Elliptic type* characterized by the condition

$$\delta=AC-B^2>0$$

This type includes (in addition to the real ellipse) the imaginary ellipse (Sec. 58, Example 5) and a pair of imaginary straight lines intersecting in a real point (Sec. 58, Example 4).

(b) *Hyperbolic type* characterized by the condition

$$\delta=AC-B^2<0$$

This type includes a pair of real intersecting straight lines (Sec. 58, Example 1) in addition to the hyperbola.

---

[1] There are artificial techniques which facilitate the proof.

(c) *Parabolic type* characterized by the condition

$$\delta = AC - B^2 = 0$$

This type includes, besides the parabola, a pair of parallel (real or imaginary) straight lines, which are possibly coincident.

Example 1. The equation

$$x^2 + 2xy + y^2 + 2x + y = 0 \tag{1}$$

is of the parabolic type because

$$\delta = AC - B^2 = 1 \cdot 1 - 1^2 = 0$$

Since the major discriminant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{4}$$

is nonzero, Eq. (1) represents a nondecomposable curve, i.e. a parabola (cf. Secs. 61, 62, Example 2).

Example 2. The equation

$$8x^2 + 24xy + y^2 - 56x + 18y - 55 = 0 \tag{2}$$

is of the hyperbolic type because

$$\delta = AC - B^2 = 8 \cdot 1 - 12^2 = -136 < 0$$

Since

$$\Delta = \begin{vmatrix} 8 & 12 & -28 \\ 12 & 1 & 9 \\ -28 & 9 & -55 \end{vmatrix} = 0$$

Eq. (2) represents a pair of intersecting straight lines. Their equations may be found by the method given in Sec. 65.

Example 3. The equation

$$2x^2 - 4xy + 5y^2 - x + 5y - 4 = 0$$

is of the elliptic type because

$$\delta = AC - B^2 = 5 \cdot 2 - 2^2 = 6 > 0$$

Since

$$\Delta = \begin{vmatrix} 2 & -2 & -\frac{1}{2} \\ -2 & 5 & \frac{5}{2} \\ -\frac{1}{2} & \frac{5}{2} & -4 \end{vmatrix} \neq 0$$

the curve does not decompose and, hence, is an ellipse.

Note. Curves of the same type are geometrically related as follows: a pair of intersecting imaginary straight lines (i.e. one real point) is the limiting case of an ellipse shrinking to a point (Fig. 88); a pair of intersecting real straight lines is the limiting case of a hyperbola approaching its asymptotes (Fig. 89); a pair of parallel lines is the limiting case of a parabola in which the axis and one pair of points M, M', symmetric about the axis (Fig. 90), are fixed while the vertex recedes to infinity.

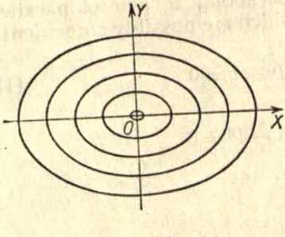


Fig. 88

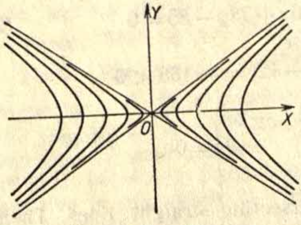


Fig. 89

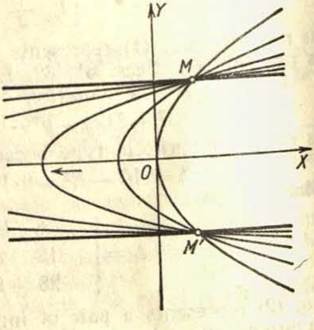


Fig. 90

68. Central and Noncentral Second-Order Curves (Conics)

Definition. The points A and B (Fig. 91) are termed *symmetric* about a point C if C bisects the segment AB. The point C is called the *centre of symmetry* (or, simply, the *centre*) of the figure if the figure has, in addition to each point M, another point N symmetric with respect to M about C.

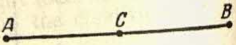


Fig. 91

The point which we called the centre of an ellipse (Sec. 40) and also the point called the centre of a hyperbola (Sec. 44) obviously fit this definition. The centre of a second-order curve (conic) that decomposes into two intersecting straight lines (Sec. 58) is, by the definition given in this section, the point of intersection of these straight lines (L in Fig. 92).

Each of the above-considered conics has a unique centre. But if the conic consists of two parallel straight lines (AB and CD in Fig. 93), then *any* point of MN equidistant from AB and CD will be suitable as centre.

The parabola has no centre.

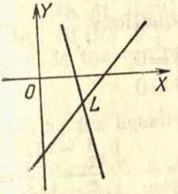


Fig. 92

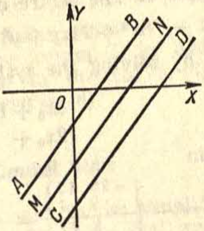


Fig. 93

Conics having a unique centre (ellipse, hyperbola, a pair of intersecting straight lines) are termed *central* conics; conics having a multiplicity of centres or none at all (parabola, a pair of parallel lines) are called *noncentral* conics.

Note. Imaginary ellipses and pairs of imaginary straight lines intersecting at a real point (see Sec. 58) are included in the group of central conics. This inclusion is symbolic as regards the imaginary ellipse, while a figure consisting of one real point fits the definition of a central "conic" (this point is itself the centre). Pairs of imaginary parallel lines (Sec. 58) are included in the group of noncentral conics.

Thus, conics belonging to the elliptic and hyperbolic types (for them $AC-B^2 \neq 0$, see Sec. 67) are central conics; conics of the parabolic type ($AC-B^2=0$) are noncentral conics.

69. Finding the Centre of a Central Conic

To find the coordinates x_0, y_0 of the centre of the central conic

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \tag{1}$$

we have to solve the system of equations

$$\left.\begin{aligned} Ax_0+By_0+D=0, \\ Bx_0+Cy_0+E=0 \end{aligned}\right\} \tag{2}$$

This is a simultaneous system and it has a unique solution (Sec. 187)

$$x_0 = -\frac{\begin{vmatrix} D & B \\ E & C \end{vmatrix}}{\begin{vmatrix} A & B \\ B & C \end{vmatrix}}, \quad y_0 = -\frac{\begin{vmatrix} A & D \\ B & E \end{vmatrix}}{\begin{vmatrix} A & B \\ B & C \end{vmatrix}} \tag{3}$$

since $\begin{vmatrix} A & B \\ B & C \end{vmatrix} \neq 0$ (this is the condition of centrality; Sec. 68).

Example 1. The centre of the conic (Example 2, Sec. 67)

$$8x^2 + 24xy + y^2 - 56x + 18y - 55 = 0 \tag{4}$$

is found by solving the system of equations

$$\begin{aligned} 8x_0 + 12y_0 - 28 &= 0, \\ 12x_0 + \quad y_0 + 9 &= 0 \end{aligned}$$

We obtain

$$x_0 = -\frac{\begin{vmatrix} -28 & 12 \\ 9 & 1 \end{vmatrix}}{\begin{vmatrix} 8 & 12 \\ 12 & 1 \end{vmatrix}} = -1, \quad y_0 = -\frac{\begin{vmatrix} 8 & -28 \\ 12 & 9 \end{vmatrix}}{\begin{vmatrix} 8 & 12 \\ 12 & 1 \end{vmatrix}} = 3$$

Since (4) is a decomposable conic of the hyperbolic type, the point $(-1, 3)$ is the point of intersection of the straight lines forming the conic (4).

Example 2. The centre of the conic (Example 1, Sec. 61)

$$2x^2 - 4xy + 5y^2 - x + 5y - 4 = 0 \tag{5}$$

is found by solving the system

$$\begin{aligned} 2x_0 - 2y_0 - \frac{1}{2} &= 0, \\ -2x_0 + 5y_0 + \frac{5}{2} &= 0 \end{aligned}$$

We obtain

$$x_0 = -\frac{5}{12} \quad y_0 = -\frac{2}{3}$$

The conic (5) is an ellipse (since $\delta > 0$ and $\Delta \neq 0$).

Derivation of equations (2). If the origin is translated to the desired centre $C(x_0, y_0)$, then Eq. (1) is transformed by means of the formulas of translation

$$x = x_0 + x', \quad y = y_0 + y' \tag{6}$$

to

$$\begin{aligned} Ax'^2 + 2Bx'y' + Cy'^2 + 2(Ax_0 + By_0 + D)x' + \\ + 2(Bx_0 + Cy_0 + E)y' + F' = 0 \end{aligned} \tag{7}$$

where, for brevity, we put

$$F' = Ax_0^2 + 2Bx_0y_0 + Cy_0^2 + 2Dx_0 + 2Ey_0 + F$$

If x_0, y_0 satisfy Eqs. (2), then (7) will take the form

$$Ax'^2 + 2Bx'y' + Cy'^2 + F' = 0 \tag{8}$$

This equation may be rewritten in the form

$$A(-x')^2 + 2B(-x')(-y') + C(-y')^2 + F' = 0$$

For this reason, this curve contains point $N(-x', -y')$, symmetric with M about the new origin C, in addition to every point $M(x', y')$ belonging to the curve (8). Hence (Sec. 68), C is the centre of the curve (8).

70. Simplifying the Equation of a Central Conic

The equation of a central conic can be simplified faster than by the general method (Sec. 60) if we first translate the origin to the centre (thus eliminating linear terms; see Sec. 69) and then rotate the axes (thus eliminating the term in xy). The angle α of this rotation is known beforehand (Sec. 61) and is found from the equation

$$\tan 2\alpha = \frac{2B}{A-C} \tag{1}$$

Note. This method is applicable to any central conic, but for a decomposable curve it is better to use the method given in Sec. 65.

Example. Given the equation (Example 1, Secs. 61, 62)

$$2x^2 - 4xy + 5y^2 - x + 5y - 4 = 0 \tag{2}$$

Translate the origin to the centre $x_0 = -\frac{5}{12}$, $y_0 = -\frac{2}{3}$ (Sec. 69, Example 2).

Using the translation formulas

$$x = x_0 + x', \quad y = y_0 + y' \tag{3}$$

we get [cf. (8), Sec. 69]

$$2x'^2 - 4x'y' + 5y'^2 - \frac{131}{24} = 0 \tag{4}$$

From (1) we find $\tan 2\alpha = \frac{4}{3}$, and if we take an angle α in the first quadrant (cf. Sec. 61), we obtain the rotation formulas

$$\left.\begin{aligned} x' &= \frac{2}{\sqrt{5}}\bar{x} - \frac{1}{\sqrt{5}}\bar{y}, \\ y' &= \frac{1}{\sqrt{5}}\bar{x} + \frac{2}{\sqrt{5}}\bar{y} \end{aligned}\right\} \tag{5}$$

Substituting into (4) yields

$$\bar{x}^2 + 6\bar{y}^2 = \frac{131}{24} \tag{6}$$

or

$$\frac{\bar{x}^2}{\frac{131}{24}} + \frac{\bar{y}^2}{\frac{131}{144}} = 1 \tag{7}$$

This curve is an ellipse with semiaxes $a = \sqrt{\frac{131}{24}} \approx 2.3$ and $b = \sqrt{\frac{131}{144}} \approx 1.0$. In the original system, its centre has the coordinates $x_0 = -\frac{5}{12}$, $y_0 = -\frac{2}{3}$, the major axis (it is the x-axis in the $\bar{x}$, $\bar{y}$ system) is given by the equation $y - y_0 = \tan\alpha\,(x - x_0)$ or $y + \frac{2}{3} = \frac{1}{2}\left(x + \frac{5}{12}\right)$; i.e. $12x - 24y - 11 = 0$ (cf. Sec. 62, Example 1).

Note. The dimensions of the ellipse may be found without performing a transformation of coordinates. We know beforehand that a transformation has to yield an equation of the type $\bar{A}\bar{x}^2 + \bar{C}\bar{y}^2 + \bar{F} = 0$. The quantities $\bar{A}$, $\bar{C}$ and $\bar{F}$ may be found with the aid of invariants (Sec. 66). In the original equation they are

$$A + C = 2 + 5 = 7, \quad \delta = AC - B^2 = 2\cdot 5 - (-2)^2 = 6,$$

$$\Delta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = -\frac{131}{4}$$

They must have the same values in the simplified equation. Hence,

$$\bar{A} + \bar{C} = 7, \quad \bar{A}\bar{C} = 6,$$

$$\begin{vmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{C} & 0 \\ 0 & 0 & \bar{F} \end{vmatrix} = \bar{A}\bar{C}\bar{F} = -\frac{131}{4}$$

whence

$$\overline{A}=1,\quad \overline{C}=6,\quad \overline{F}=-\frac{131}{24}$$

and we again get Eq. (6).

71. The Equilateral Hyperbola as the Graph of the Equation $y=\frac{k}{x}$

The equation

$$y=\frac{k}{x} \tag{1}$$

($k \neq 0$) represents an equilateral hyperbola (Sec. 44); its asymptotes coincide with the coordinate axes. The semiaxes are

$$a=b=\sqrt{2|k|} \tag{2}$$

If $k>0$, the branches of the hyperbola are arranged as follows: one in the first quadrant, the other in the third quadrant. But if $k<0$, then they lie in the second and fourth quadrants (Fig. 94). In the first case, the real axis of the hyperbola makes an angle of 45° with the axis of abscissas, in the second case, an angle of $-45°$.

This is obtained by the method of Sec. 61 if Eq. (1) is written as

$$xy=k \tag{3}$$

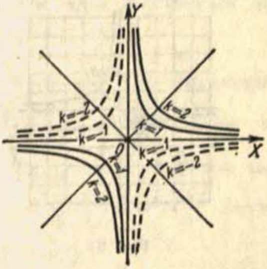


Fig. 94

Note. When $k=0$, Eq. (3) represents a pair of straight lines $y=0$ (axis of abscissas) and $x=0$ (axis of ordinates). When $|k|$ decreases without bound, the hyperbolas (3) come closer and closer to these lines (so that a pair of perpendicular straight lines may be regarded as a degenerate equilateral hyperbola).

For $k=0$, Eq. (1) represents only one straight line $y=0$ (axis of abscissas), and not in its entirety but without the origin of coordinates because for $k=0$ and $x=0$ the expression $y=\frac{k}{x}$ becomes indeterminate. But if we give this indeterminate quantity all possible values, we get the "lost" axis of ordinates.

72. The Equilateral Hyperbola as the Graph of the Equation $y=\frac{mx+n}{px+q}$

Consider the equation

$$y=\frac{mx+n}{px+q} \tag{1}$$

for $p \neq 0$ $\left(\text{for } p=0 \text{ we have the straight line } y=\frac{m}{q}x+\frac{n}{q}\right)$.

If the determinant

$$D=\begin{vmatrix} m & n \\ p & q \end{vmatrix}=mq-np$$

is nonzero, then Eq. (1) represents the same equilateral hyperbola as Eq. (1) of Sec. 71:

$$y=\frac{k}{x}$$

where $k=-\frac{D}{p^2}$, with the sole difference that the centre is displaced from the origin to the point $C\left(-\frac{q}{p}, \frac{m}{p}\right)$

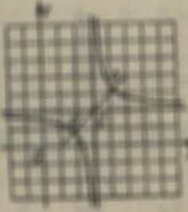


Fig. 95

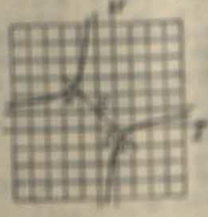


Fig. 96

(Figs. 95, 96). This means that (Sec. 71) the semi-axes are $a=b=\sqrt{\frac{2|D|}{p^2}}$.

When $D<0$ (then $k>0$), the real axis makes an angle of $+45°$ with the axis of abscissas (Fig. 95), but if $D>0$, then the angle is $-45°$ (Fig. 96).

Example 1. The equation

$$y=\frac{3x+5}{2x+2}$$

$\left(\text{here } m=4,\ n=-9,\ p=2,\ q=-6,\ D=\begin{vmatrix} 4 & -9 \\ 2 & -6 \end{vmatrix}=-6\right)$ represents an equilateral hyperbola (Fig. 59) with centre $C(3, 2)$ and with semiaxes $a=b=\sqrt{\frac{2\cdot 6}{2^2}}=\sqrt{3}\approx 1.73$. The axis $A'A$ forms a 45° angle with Ox since $D<0$. The coordinates of the vertex A will then be

$$x_A=x_C+a\cos 45^\circ=3+\sqrt{3}\,\frac{\sqrt{2}}{2}\approx 4.2,$$

$$y_A=y_C+a\sin 45^\circ=2+\sqrt{3}\,\frac{\sqrt{2}}{2}\approx 3.2$$

Similarly, we find

$$x_{A'}=3-\sqrt{3}\,\frac{\sqrt{2}}{2}\approx 1.8,\quad y_{A'}=2-\sqrt{3}\,\frac{\sqrt{2}}{2}\approx 0.8$$

Example 3. The equation

$$y=\frac{x-1}{x+1}$$

(here, $m=1$, $n=-1$, $p=1$, $q=1$, $D=2$) represents an equilateral hyperbola (Fig. 60) with centre $C(-1, 1)$ and with semiaxes $a=b=\sqrt{\frac{2\cdot 2}{1^2}}=2$. The axis $A'A$ makes an angle of $-45°$ with Ox since $D>0$.

Note 1. If the determinant $D=\begin{vmatrix} m & n \\ p & q \end{vmatrix}$ is zero, then the quantities m, n and p, q are proportional $\left(\frac{m}{p}=\frac{n}{q}\right)$ so that $mx+n$ is divisible by $px+q$; the quotient is $\frac{m}{p}$. Eq. (1) then represents the straight line $y=\frac{m}{p}$ devoid of the point $x=-\frac{q}{p}$ $\left[\text{for } x=-\frac{q}{p} \text{ expression (1) is indeterminate: see Sec. [illegible], Note}\right]$.

For example, the equation $y=\frac{3x+6}{x+2}$ $\left(m=3,\ n=6,\ p=1,\ q=2,\ D=\begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix}=0\right)$ represents a straight line $y=3$ devoid of the point $x=-2$. If the indeterminate quantity y is given all possible values, we then get another straight line $x=-2$ (in addition to the straight line $y=3$).

Note 2. We can visualize the "deletion" of point $x=-2$ from the straight line $y=3$ as follows. Consider the equation $y=\frac{3x+6.01}{x+2}$.

here $D=\begin{vmatrix}3 & 6\beta\\ 1 & 2\end{vmatrix}=6(1-\beta)$ so that for $\beta\neq 1$ we have a hyperbola with asymptotes $x=-2$ and $y=3$. But when the quantity β is close to 1, the hyperbola (Fig. 97, where $\beta=1.1$) comes very close to its asymptotes $U'U$ and $V'V$ which intersect at the point $K(-2, 3)$. We might expect that for $\beta=1$ we would get a pair of straight lines $U'U$ ($y=3$) and $V'V$ ($x=-2$). However, the line $V'V$ "falls out" since it is parallel to the y-axis and, hence (Sec. 14, Note 2), cannot be represented by an equation solved for the ordinate. The point K is also omitted since it lies on the line $V'V$.

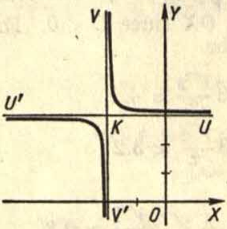


Fig. 97

73. Polar Coordinates

In a plane (Fig. 98) take an arbitrary point O (*pole*) and draw a ray OX (*polar axis*). Take some segment OA for the unit of length and some angle (it is customary to take the radian) for the unit of angular measurement. Then the position of any point M in the plane may be specified by two numbers: (1) a positive number ρ expressing the length of the line segment OM (*radius vector*), (2) a number φ expressing the magnitude of the angle XOM (*polar angle*). The numbers ρ and φ are termed the *polar coordinates* of the point M.

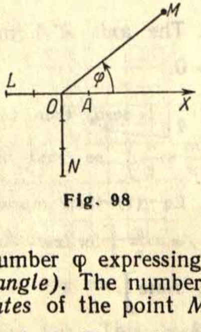


Fig. 98

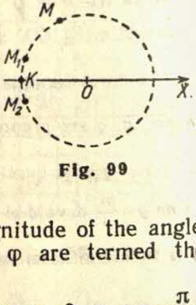


Fig. 99

Example 1. The polar coordinates $\rho=3$, $\varphi=-\frac{\pi}{2}$ define a point N (Fig. 98), the polar coordinates $\rho=3$, $\varphi=\frac{3\pi}{2}$ define the same point N, the polar coordinates $\rho=1$, $\varphi=0$ (and also $\rho=1$, $\varphi=2\pi$ or $\rho=1$, $\varphi=-2\pi$, etc.) define the point A.

Each pair of values ρ, φ is associated with a unique point; but one and the same point M is associated with an infinity of values of the polar angle which differ by a multiple of 2π (cf. Example 1). But if the point M coincides with the pole, the value of the polar angle is completely arbitrary.

We can agree to take only one of the values of the polar angle, say we take (φ) within the limits

$$-\pi < \varphi \leqslant \pi \tag{1}$$

This value of the polar angle is called the *principal* value.

Example 2. The point N (Fig. 98) is associated with the polar coordinates $\rho=3$, $\varphi=-\frac{\pi}{2}+2k\pi$; the principal value of the polar angle is $-\frac{\pi}{2}$.

The point L is associated with the polar coordinates $\rho=2$, $\varphi=\pi+2k\pi$; the principal value of φ is, according to Condition (1), π (not $-\pi$).

When dealing with principal values, every point (except the pole) is associated with one pair of polar coordinates. For the pole, $\rho=0$, and φ is arbitrary.

Note 1. When point M describes a circle centred at the pole O (Fig. 99) and intersects, at point K, the extension of the polar axis, the principal value of the polar angle changes abruptly experiencing a jump (at the point M_1 it is close to π, at M_2, it is close to $-\pi$). In many cases, it is not advisable, therefore, to confine oneself to the principal values of φ.

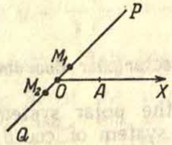


Fig. 100

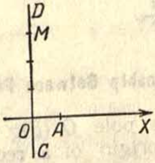


Fig. 101

Note 2. When the point M describes a straight line PQ (Fig. 100) and passes through the pole O, the value of φ changes abruptly (a jump). For instance, if $\angle XOP=\frac{\pi}{4}$, then for the point M_1 (on the ray OP) $\varphi=\frac{\pi}{4}+2k\pi$, and for the point M_2 (on the ray OQ) $\varphi=-\frac{3\pi}{4}+2n\pi$ (k and n are integers). To avoid this situation, we can ascribe to *all points* of the straight line PQ one and the same value of φ, for example, $\varphi=\angle XOP$ and consider the radius vectors as positive on the OP ray and negative on OQ. For example, the polar coordinates

$$\varphi=\frac{\pi}{4}, \quad \rho=\frac{1}{2}$$

define the point M_1, and the polar coordinates

$$\varphi=\frac{\pi}{4}, \quad \rho=-\frac{1}{2}$$

define the point M_2.

The same points may be specified by the coordinates

$$\varphi=-\frac{3}{4}\pi, \quad \rho=\frac{1}{2}$$

(point M_2) and

$$\varphi=-\frac{3}{4}\pi, \quad \rho=-\frac{1}{2}$$

(point M_1). We thus ascribe to all points of the straight line PQ the value $\varphi=\angle XOQ$, so that ρ is positive on the ray OQ and negative on OP.

Example 3. Construct a point M with polar coordinates

$$\rho=-3, \quad \varphi=-\frac{\pi}{2}$$

The polar angle $\varphi=-\frac{\pi}{2}$ is associated with the ray OC (Fig. 101). Lay off $OM=3OA$ on its extension OD. This yields the desired point M. To the same point there correspond the polar coordinates $\rho=3$, $\varphi=\frac{\pi}{2}$.

74. Relationship Between Polar and Rectangular Coordinates

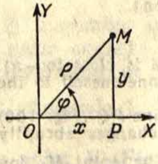


Fig. 102

Let the pole O (Fig. 102) of the polar system coincide with the origin of a rectangular system of coordinates and let the polar axis OX coincide with the positive direction of the axis of abscissas. Let M be an arbitrary point in the plane, x and y its rectangular coordinates, and ρ, φ its polar coordinates. Then

$$x=\rho\cos\varphi, \quad y=\rho\sin\varphi \tag{1}$$

Conversely, [1]

$$\rho=\sqrt{x^2+y^2} \tag{2}$$

---

[1] It is assumed in formulas (2) and (3) that the radius vector ρ is always positive. If, however, we consider the negative values of ρ as well (Sec. 73, Note 2), then in place of (2) and (3) we will have to

$$\cos\varphi=\frac{x}{\sqrt{x^2+y^2}}, \quad \sin\varphi=\frac{y}{\sqrt{x^2+y^2}} \tag{3}$$

and

$$\tan\varphi=\frac{y}{x} \tag{4}$$

However, alone, formula (4) [likewise, only one of the formulas (3)] is not sufficient for a determination of the angle φ (see Example 1).

Example 1. The rectangular coordinates of a point are $x=2$, $y=-2$. Find its polar coordinates (for the above-indicated mutual arrangement of the two systems).

Solution. By formula (2),

$$\rho=\sqrt{2^2+(-2)^2}=2\sqrt{2}$$

By formula (4), $\tan\varphi=\frac{-2}{2}=-1$. Hence, either $\varphi=-\frac{\pi}{4}+2k\pi$ or $\varphi=\frac{3\pi}{4}+2k\pi$. Since the point lies in the fourth quadrant, only the first value is correct. The principal value of φ is $-\frac{\pi}{4}$.

If we take advantage of the formula $\cos\varphi=\frac{x}{\sqrt{x^2+y^2}}$, we get $\cos\varphi=\frac{2}{2\sqrt{2}}=\frac{\sqrt{2}}{2}$. Hence, either $\varphi=\frac{\pi}{4}+2k\pi$ or $\varphi=-\frac{\pi}{4}+2k\pi$. Only the second value is correct.

Example 2. In the rectangular system XOY, the circle depicted in Fig. 103 is given by the equation (Sec. 38) $(x-R)^2+y^2=R^2$. Formulas (1) and (2) permit finding its equation in the polar system (O is the pole and OX is the polar axis). We get $\rho^2-2R\rho\cos\varphi=0$. This equation may be decomposed into two: (1) $\rho=0.2$, (2) $\rho-2R\cos\varphi=0$. The first (for any value of φ) represents the pole O. The second yields all points of the circle including the pole (for $\varphi=\frac{\pi}{2}$ and $\varphi=-\frac{\pi}{2}$). Therefore, the first equation may be discarded. We then have

$$\rho=2R\cos\varphi \tag{5}$$

---

write $\rho=\pm\sqrt{x^2+y^2}$, $\cos\varphi=\frac{x}{\pm\sqrt{x^2+y^2}}$, $\sin\varphi=\frac{y}{\pm\sqrt{x^2+y^2}}$ (the signs either all upper or all lower). The formulas (1) and (4) remain unchanged.

This equation is obtained directly from the triangle OMK with right angle at the vertex M ($OK = 2R$, $OM = \rho$, $\angle KOM = \varphi$).

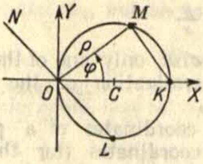


Fig. 103

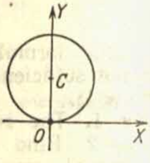


Fig. 104

Note. If negative values of ρ are not introduced, then in Eq. (5) we can take the angle φ in the fourth and first quadrants, but not in the second and third quadrants. Thus, for $\varphi = \frac{3}{4}\pi$ Eq. (5) gives $\rho = -R\sqrt{2}$. Indeed, the ray ON (Fig. 103) does not have any points in common with the circle, with the exception of the pole. Now if we introduce negative values of ρ (Sec. 73, Note 2), then the coordinates $\rho = -R\sqrt{2}$, $\varphi = \frac{3}{4}\pi$ yield the point L on the extension of the straight line ON.

Example 3. Determine which curve is defined by the equation

$$\rho = 2a \sin \varphi \tag{6}$$

Solution. Passing to the rectangular system, we find

$$\sqrt{x^2 + y^2} = 2a \frac{y}{\sqrt{x^2 + y^2}}$$

or

$$x^2 + y^2 - 2ay = 0$$

or

$$x^2 + (y - a)^2 = a^2$$

Eq. (6) is a circle of radius a (Fig. 104) passing through the pole O and tangent to the polar axis OX.

75. The Spiral of Archimedes [1]

1. **Definition.** Let the straight line UV (Fig. 105) emanate from an initial position $X'X$ and uniformly rotate about a fixed point O and let the point M emanate from an initial

---

[1] This curve is discussed in detail in Sec. 511.

position O and uniformly move along UV. The curve described by the point M is called the *spiral of Archimedes* in honour of the great Greek scholar Archimedes (third century B. C.) who first studied that curve.

Note. The kinematic concepts that enter into this definition may be removed by replacing them by the condition that the distance $\rho = OM$ be proportional to the angle of rotation φ of the straight line UV.

The rotation of the line UV from any position through the given angle is associated with the same increment in the distance ρ. For instance, a complete revolution is associated with the same displacement $MM_1 = a$. The segment a is called the *lead* of the spiral of Archimedes.

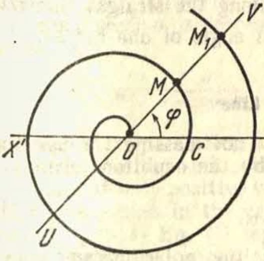


Fig. 105

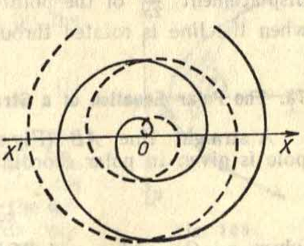


Fig. 106

To a given lead a there correspond two Archimedean spirals which differ in the direction of rotation of the line UV. Counterclockwise rotation generates a right-handed spiral (Fig. 106, solid line); clockwise rotation generates a *left-handed* spiral (Fig. 106, dashed line).

Right and left spirals with the same lead may be brought to coincidence. To do this, one of them has to be turned over (reverse up).

From Fig. 106, it will be seen that right and left spirals of one and the same lead may be regarded as two branches of a curve described by a point M when the point traverses the entire straight line UV, passing through point O in so doing.

2. **The polar equation** (O is the pole, the direction of the polar axis OX coincides with the direction of motion of M

when it passes through the point O; a is the lead of the spiral):

$$\frac{\rho}{a}=\frac{\varphi}{2\pi} \qquad (1)$$

The right branch corresponds to positive values of φ, the left to negative values.

Eq. (1) may be written as

$$\rho=k\varphi$$

where k (the *parameter* of the spiral of Archimedes) is the displacement $\frac{a}{2\pi}$ of the point M along the straight line UV when the line is rotated through an angle of one radian.

76. The Polar Equation of a Straight Line

A straight line AB (Fig. 107) not passing through the pole is given in polar coordinates by the equation

$$\rho=\frac{p}{\cos(\varphi-\alpha)} \qquad (1)$$

where $p=OK$ and $\alpha=\angle XOK$ are the polar parameters of the straight line AB (Sec. 29).

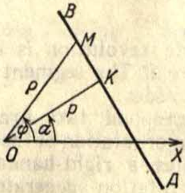


Fig. 107

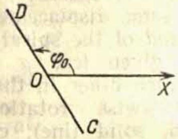


Fig. 108

Eq. (1) is obtained from the triangle OKM (where $OM=\rho$ and $\angle KOM=\varphi-\alpha$).

The straight line CD (Fig. 108) passing through the pole cannot be represented by an equation of the type (1) $\left[\text{for such a line } p=0 \text{ and } \varphi-\alpha=\pm\frac{\pi}{2}, \text{ so that } \cos(\varphi-\alpha)=0\right]$.

Its ray OD is represented by the equation $\varphi=\varphi_0$ (where $\varphi_0=\angle XOD$), and ray OC, by the equation $\varphi=\varphi_1$ (where

$\varphi_1 = \angle XOC$). Each of these equations can represent the entire straight line if negative values of ρ are introduced (Sec. 73 Note 2).

77. The Polar Equation of a Conic Section

Put the pole in the focus F (Fig. 109) of a conic section (ellipse, hyperbola or parabola) and bring the polar axis to coincidence with the axis FX of the conic section in the direction opposite to that in which the corresponding directrix PQ lies. Then the conic section is represented by the equation

$$\rho = \frac{p}{1 - \varepsilon \cos \varphi} \qquad (1)$$

where p is a parameter and ε is the eccentricity of the conic section (Sec. 52).

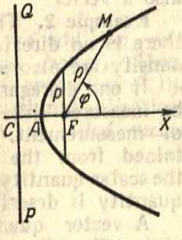


Fig. 109

Note. If only positive values of ρ are considered, then in the case of the hyperbola ($\varepsilon > 1$) Eq. (1) represents only one branch, that enclosing the focus. Also, for φ the inequality $1 - \varepsilon \cos \varphi > 0$ must hold. Now if negative values of ρ are considered, then φ may have any value, and for $1 - \varepsilon \cos \varphi < 0$ we get the second branch.

SOLID ANALYTIC GEOMETRY

78. Vectors and Scalars. Fundamentals

A *vector quantity*, or a *vector* (in the broad sense of the word), is any quantity possessing direction. A *scalar quantity* (or *scalar*) is a quantity that does not possess direction.

Example 1. A force acting on a mass point is a vector because it has direction. The velocity of a mass point is also a vector.

Example 2. The temperature of a body is a scalar since there is no direction involved. The mass of a body and its density are also scalar quantities.

If one disregards the direction of a vector, then it may be measured (like a scalar) by choosing an appropriate unit of measurement. However, the number obtained from the measurement characterizes the scalar quantity entirely, whereas the vector quantity is described only partially.

A vector quantity is fully specified by giving the *direction of a line segment* and indicating a linear scale unit.

Example 3. The directed segment AB in Fig. 110 with scale unit MN depicting unit force (1 Newton) characterizes a force of 3.5 Newtons, the direction of which coincides with the direction of the segment AB (indicated by the arrow).

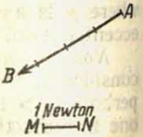


Fig. 110

79. The Vector in Geometry

In geometry, a *vector* (in the narrow sense) is any directed line-segment.

A vector with initial point A and terminal point B is denoted as $\overrightarrow{AB}$ (Fig. 110).

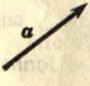


Fig. 111

A vector can also be denoted by a single letter as in Fig. 111. In printing this letter is given in boldface type ($\boldsymbol{a}$), in writing it is given with a bar ($\overline{a}$).

The length of a vector is also called the *absolute value* (or *modulus*) of the vector. The absolute value of a vector is a *scalar quantity*.

The absolute value of a vector is denoted by two vertical lines: $|\overrightarrow{AB}|$ or $|\boldsymbol{a}|$ or $|\overline{a}|$.

In the two-letter notation of a vector, its absolute value is sometimes denoted by the same letters without an arrow (AB is the absolute value of the vector $\overrightarrow{AB}$), in the single-letter notation, the absolute value is denoted by a normal weight letter (b is the absolute value of the vector $\boldsymbol{b}$).

80. Vector Algebra

Operations involving vectors are called the addition, subtraction and multiplication of vectors (see below). These operations have much in common with the properties of the algebraic operations of addition, subtraction and multiplication. Therefore, the study of vector operations is called *vector algebra*.

81. Collinear Vectors

Vectors lying on parallel straight lines (or on one and the same straight line) are termed *collinear*. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in Fig. 112 are collinear. The vectors $\overrightarrow{AC}$, $\overrightarrow{BD}$ and $\overrightarrow{CB}$ in Fig. 113 are collinear.

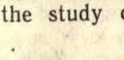


Fig. 112

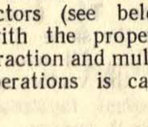


Fig. 113

Collinear vectors can have the same direction or they can have opposite directions. Thus, the vectors $\boldsymbol{a}$ and $\boldsymbol{c}$ (Fig. 112) are in the same direction, vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (and also $\boldsymbol{b}$ and $\boldsymbol{c}$) are in opposite directions. The vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in Fig. 113 are in the same direction, vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ are in opposite directions.

82. The Null Vector

If the origin A and the terminus B of a segment AB coincide, then the segment AB becomes a point and loses direction. However, for the purpose of generality of the rules of vector algebra it is agreed that a pair of coincident points is to be regarded as a vector, the *null vector*. It is considered collinear with any vector.

The null vector is symbolized by 0, the number zero.

83. Equality of Vectors

Definition. Two (nonzero) vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal if they are in the same direction and have one and the same absolute value. All zero vectors are taken to be equal. In all other cases, the vectors are not equal.

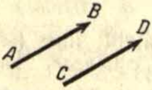


Fig. 114

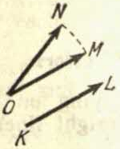


Fig. 115

Example 1. The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ (Fig. 114) are equal.

Example 2. The vectors $\overrightarrow{OM}$ and $\overrightarrow{ON}$ (Fig. 115) are not equal (although they are of the same length) because they have different directions. The vectors $\overrightarrow{ON}$ and $\overrightarrow{KL}$ are likewise not equal, while the vectors $\overrightarrow{OM}$ and $\overrightarrow{KL}$ are equal.

Warning. Do not confuse the concept of "equality of vectors" with that of "equality of line segments". When we say that the line segments ON and KL are equal, we assert that one of them can be brought to coincidence with the other. But this may require a rotation of the segment being brought to coincidence (as in Fig. 115). In that case, the *vectors* $\overrightarrow{ON}$ *and* $\overrightarrow{KL}$ *are*, by definition, *not equal*. The two vectors will be equal only when they *can be brought to coincidence without a rotation.*

Notation. The notation $\boldsymbol{a}=\boldsymbol{b}$ expresses the fact that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal. The notation $\boldsymbol{a} \neq \boldsymbol{b}$ expresses the fact that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not equal. The notation $|\boldsymbol{a}|=|\boldsymbol{b}|$ expresses the fact that the absolute values (lengths) of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal; here, the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ may (or may not) be equal to one another.

Example 3. $\overrightarrow{AB}=\overrightarrow{CD}$ (Fig. 114), $\overrightarrow{ON} \neq \overrightarrow{KL}$ (Fig. 115), $|\overrightarrow{ON}|=|\overrightarrow{KL}|$ (Fig. 115), $\overrightarrow{OM}=\overrightarrow{KL}$ (Fig. 115).

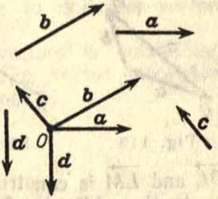


Fig. 116

84. Reduction of Vectors to a Common Origin

Two vectors (or any number of vectors) can be reduced to a common origin; i. e. it is possible to construct vectors that are equal to the given ones and have a common origin at some point O. This reduction is shown in Fig. 116.

85. Opposite Vectors

Definition. Two vectors having the same absolute values and opposite directions are called *opposite vectors*.

A vector which is in the direction opposite to a vector $\boldsymbol{a}$ is denoted by $-\boldsymbol{a}$.

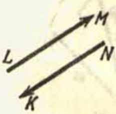


Fig. 117

Example 1. The vectors $\overrightarrow{LM}$ and $\overrightarrow{NK}$ in Fig. 117 are in opposite directions.

Example 2. If the vector $\overrightarrow{LM}$ (Fig. 117) is denoted by $\boldsymbol{a}$, then $\overrightarrow{NK}=-\boldsymbol{a}$, $\overrightarrow{ML}=-\boldsymbol{a}$, $\overrightarrow{KN}=\boldsymbol{a}$.

From the definition it follows that $-(-\boldsymbol{a})=\boldsymbol{a}$, $|-\boldsymbol{a}|=|\boldsymbol{a}|$.

86. Addition of Vectors

Definition. The *sum* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a third vector $\boldsymbol{c}$ obtained by the following construction: from an arbitrary origin O (Fig. 118) construct a vector $\overrightarrow{OL}$ equal to $\boldsymbol{a}$ (Sec. 83); from the point L, as origin, construct the vector $\overrightarrow{LM}$ equal to $\boldsymbol{b}$. The vector $\boldsymbol{c}=\overrightarrow{OM}$ is the sum of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (triangle rule).

Notation: $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{c}$.

Warning. Do not confuse the concept of a "sum of line segments" with that of a "sum of vectors". The sum of the line segments OL and LM is obtained by the following construction: extend the straight line OL (Fig. 119), lay off a segment LN equal to LM. The segment ON is the sum of the segments OL and LM. The sum of the *vectors* $\overrightarrow{OL}$ and $\overrightarrow{LM}$ is constructed differently (see definition).

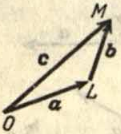


Fig. 118

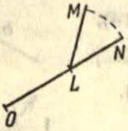


Fig. 119

In the addition of vectors we have the following inequalities:

$$|\boldsymbol{a}+\boldsymbol{b}|\leqslant|\boldsymbol{a}|+|\boldsymbol{b}|, \tag{1}$$

$$|\boldsymbol{a}+\boldsymbol{b}|\geqslant||\boldsymbol{a}|-|\boldsymbol{b}|| \tag{2}$$

which state that the side OM of the triangle OML (Fig. 118) is less than the sum and greater than the difference of the other two sides. In formula (1) the equality sign is valid only for vectors in the same direction (Fig. 120); in formula (2), only for vectors in opposite directions (Fig. 121).

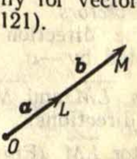


Fig. 120

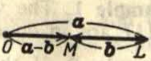


Fig. 121

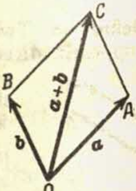


Fig. 122

The sum of, opposite vectors. From the definition it follows that the *sum of opposite vectors is equal to the null vector*:

$$\boldsymbol{a}+(-\boldsymbol{a})=0$$

Commutative property. The order in which vectors may be added is immaterial:

$$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$$

Parallelogram rule. If the summands $\boldsymbol{a}$ and $\boldsymbol{b}$ are *not collinear*, then the sum $\boldsymbol{a}+\boldsymbol{b}$ may be found by the following construction: from any origin O (Fig. 122) construct the vectors $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$; on the segments OA, OB construct a parallelogram $OACB$. The vector of the diagonal $\overrightarrow{OC}=\mathbf{c}$ is the sum of the vectors $\mathbf{a}$ and $\mathbf{b}$ (since $\overrightarrow{AC}=\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}$).

This construction is *not applicable* to collinear vectors (Figs. 120, 121).

Note. The definition of addition of vectors is established in accord with the physical laws of adding vector quantities (for example, forces applied to a mass point).

87. The Sum of Several Vectors

Definition. The *sum* of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, ..., $\mathbf{a}_n$ is a vector obtained as the result of a sequence of additions: to the vector $\mathbf{a}_1$ add the vector $\mathbf{a}_2$, to the resultant vector add the vector $\mathbf{a}_3$, etc.

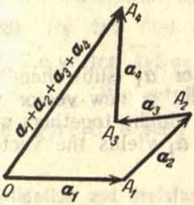


Fig. 123

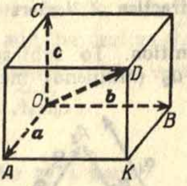


Fig. 124

From the definition there follows the following construction (*rule of the polygon*, or *chain rule*).

Starting from an arbitrary origin O (Fig. 123) construct a vector $\overrightarrow{OA_1}=\mathbf{a}_1$, from the point A_1 (as origin) construct a vector $\overrightarrow{A_1A_2}=\mathbf{a}_2$, from the point A_2 construct a vector $\overrightarrow{A_2A_3}=\mathbf{a}_3$, and so forth. The vector $\overrightarrow{OA_n}$ (Fig. 123, $n=4$) is the sum of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$.

The sum of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ is denoted $\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4$.

Associative property. In the addition of vectors, the terms may be grouped in any way whatsoever. For example, if one first finds the sum of the vectors $\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4$ (it is equal to the vector $\overrightarrow{A_1A_4}$, not depicted in Fig. 123), and then adds the vector $\boldsymbol{a}_1 (=\overrightarrow{OA_1})$, we get the same vector $\boldsymbol{a}_1+\boldsymbol{a}_2+$ $+\boldsymbol{a}_3+\boldsymbol{a}_4 (=\overrightarrow{OA_4})$:

$$\boldsymbol{a}_1+(\boldsymbol{a}_2+\boldsymbol{a}_3+\boldsymbol{a}_4)=\boldsymbol{a}_1+\boldsymbol{a}_2+\boldsymbol{a}_3+\boldsymbol{a}_4$$

Rule of the parallelepiped. If three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are reduced to a common origin (Sec. 84) and do not lie in the same plane, then the sum $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ may be found by the following construction. From any origin O (Fig. 124) construct the vectors $\overrightarrow{OA}=\boldsymbol{a}$, $\overrightarrow{OB}=\boldsymbol{b}$, $\overrightarrow{OC}=\boldsymbol{c}$. On the segments OA, OB, OC (as edges) construct a parallelepiped. The vector of the diagonal $\overrightarrow{OD}$ is the sum of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ (since $\overrightarrow{OA}=\boldsymbol{a}$, $\overrightarrow{AK}=\overrightarrow{OB}=\boldsymbol{b}$, $\overrightarrow{KD}=\overrightarrow{OC}=\boldsymbol{c}$ and $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AK}+\overrightarrow{KD}$).

This construction is *not applicable* to vectors which (after reduction to a common origin) lie in the same plane.

88. Subtraction of Vectors

Definition. To subtract a vector $\boldsymbol{a}_1$ (subtrahend) from a vector $\boldsymbol{a}_2$ (minuend) means to find a new vector $\boldsymbol{x}$ (difference) which together with the vector $\boldsymbol{a}_1$ yields the vector $\boldsymbol{a}_2$.

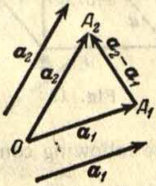


Fig. 125

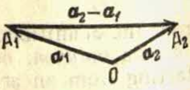


Fig. 126

Briefly, subtraction of vectors is the inverse operation of addition.

Notation: $\boldsymbol{a}_2-\boldsymbol{a}_1$.

From the definition follows the construction: from an arbitrary origin O (Figs. 125, 126) construct the vectors $\overrightarrow{OA_1}=\boldsymbol{a}_1$, $\overrightarrow{OA_2}=\boldsymbol{a}_2$. The vector A_1A_2 (drawn from the terminus of the subtrahend vector to the terminus of the minuend) is the difference $\boldsymbol{a}_2-\boldsymbol{a}_1$:

$$\overrightarrow{A_1A_2}=\overrightarrow{OA_2}-\overrightarrow{OA_1}$$

Indeed, the sum $\overrightarrow{OA_1}+\overrightarrow{A_1A_2}$ is equal to $\overrightarrow{OA_2}$.

Note. The absolute value of the difference (the length of the vector $\overrightarrow{A_1A_2}$) may be less than the absolute value of the "minuend" but may also be greater than or equal to it. These three cases are shown in Figs. 125, 126, 127.

Alternative construction. To construct the difference $\boldsymbol{a}_2-\boldsymbol{a}_1$ of the vectors $\boldsymbol{a}_2$ and $\boldsymbol{a}_1$ we can take the sum of the vectors $\boldsymbol{a}_2$ and $-\boldsymbol{a}_1$, i. e.

$$\boldsymbol{a}_2-\boldsymbol{a}_1=\boldsymbol{a}_2+(-\boldsymbol{a}_1)$$

Example. Let it be required to find the difference $\boldsymbol{a}_2-\boldsymbol{a}_1$ (Fig. 128). By the first construction $\boldsymbol{a}_2-\boldsymbol{a}_1=\overrightarrow{A_1A_2}$. Now construct the vector $\overrightarrow{A_2L}=-\boldsymbol{a}_1$ and add the vectors $\overrightarrow{OA_2}=\boldsymbol{a}_2$ and $\overrightarrow{A_2L}=-\boldsymbol{a}_1$. We get (Sec. 86, definition) the vector $\overrightarrow{OL}$. From the figure it is seen that $\overrightarrow{OL}=\overrightarrow{A_1A_2}$.

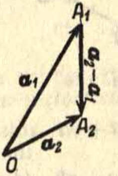


Fig. 127

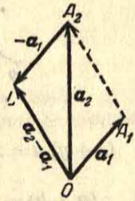


Fig 128

89. Multiplication and Division of a Vector by a Number

Definition 1. To multiply a vector $\boldsymbol{a}$ (multiplicand) by a number x (multiplier) means to construct a new vector (product) the absolute value of which is obtained by multiplying the absolute value of the vector $\boldsymbol{a}$ by the absolute value of the number x, the direction coinciding with the direction of the vector $\boldsymbol{a}$ or being in the opposite sense, depending on whether the number x is positive or negative. If $x=0$, the product is the null vector.

Notation: $\boldsymbol{a}x$ or $x\boldsymbol{a}$.

Examples. $\overrightarrow{OB}=\overrightarrow{OA}\cdot 4$ or $\overrightarrow{OB}=4\overrightarrow{OA}$ (Fig. 129), $\overrightarrow{OC}=3\frac{1}{2}\overrightarrow{OA}$, $\overrightarrow{OD}=-2\overrightarrow{OA}$, $\overrightarrow{OE}=-1.5\overrightarrow{OA}$ (Fig. 130).

Definition 2. To divide a vector $\boldsymbol{a}$ by a number x means to find a vector such that when it is multiplied by the number x it yields the vector $\boldsymbol{a}$ as a product.

Notation: $\boldsymbol{a}:x$ or $\frac{\boldsymbol{a}}{x}$.

Instead of the division $\frac{\boldsymbol{a}}{x}$ we can perform the multiplication $\boldsymbol{a}\cdot\frac{1}{x}$.

The multiplication of a vector by a number obeys the same laws as the multiplication of numbers:

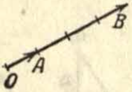


Fig. 129

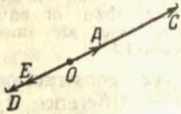


Fig. 130

1. $(x+y)\boldsymbol{a}=x\boldsymbol{a}+y\boldsymbol{a}$ (distributive property with respect to the numerical factor)
2. $x(\boldsymbol{a}+\boldsymbol{b})=x\boldsymbol{a}+x\boldsymbol{b}$ (distributive property with respect to the vector factor)
3. $x(y\boldsymbol{a})=(xy)\boldsymbol{a}$ (associative property).

By virtue of these properties it is possible to construct vector expressions having the same external aspect as polynomials of the first degree in algebra; these expressions can be manipulated in the same fashion as the corresponding algebraic expressions (collect like terms, remove parentheses, take outside of parentheses, transpose terms from one side of an equality to the other with opposite sign, etc.).

Examples. $2\boldsymbol{a}+3\boldsymbol{a}=5\boldsymbol{a}$ (by Property 1),

$2(\boldsymbol{a}+\boldsymbol{b})=2\boldsymbol{a}+2\boldsymbol{b}$ (by Property 2),

$5\cdot 12\boldsymbol{c}=60\boldsymbol{c}$ (by Property 3);

$$4(2\boldsymbol{a}-3\boldsymbol{b})=4[2\boldsymbol{a}+(-3\boldsymbol{b})]=4[2\boldsymbol{a}+(-3)\boldsymbol{b}]=4\cdot 2\boldsymbol{a}+4(-3)\boldsymbol{b}=$$
$$=8\boldsymbol{a}+(-12)\boldsymbol{b}=8\boldsymbol{a}-12\boldsymbol{b},$$
$$2(3\boldsymbol{a}-4\boldsymbol{b}+\boldsymbol{c})-3(2\boldsymbol{a}+\boldsymbol{b}-3\boldsymbol{c})=6\boldsymbol{a}-8\boldsymbol{b}+2\boldsymbol{c}-6\boldsymbol{a}-3\boldsymbol{b}+$$
$$+9\boldsymbol{c}=-11\boldsymbol{b}+11\boldsymbol{c}=11(\boldsymbol{c}-\boldsymbol{b})$$

90. Mutual Relationship of Collinear Vectors (Division of a Vector by a Vector)

If a vector $\boldsymbol{a}$ is nonzero, then any vector $\boldsymbol{b}$ *collinear* with it may be represented in the form $x\boldsymbol{a}$, where x is a number obtained as follows: it has an adsolute value $|\boldsymbol{b}|:|\boldsymbol{a}|$ (ratio of absolute values); it is positive if the vector $\boldsymbol{b}$ is in the same direction as the vector $\boldsymbol{a}$, it is negative if $\boldsymbol{b}$ and $\boldsymbol{a}$ are oppositely directed, and is zero if $\boldsymbol{b}$ is a null vector.

Examples. For the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in Fig. 131 we have $\boldsymbol{b}=2\boldsymbol{a}\,(x=2)$, in Fig. 132 we have $\boldsymbol{b}=-2\boldsymbol{a}$.

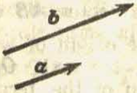


Fig. 131

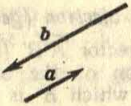


Fig. 132

Note. Finding the number x is termed the *division of a vector* $\boldsymbol{b}$ *by a vector* $\boldsymbol{a}$. *Noncollinear vectors cannot be divided by each other*.

91. The Projection of a Point on an Axis

An *axis* is any straight line on which one of its directions (no matter which) has been selected. This direction is called *positive* (indicated by an arrow in drawings); the opposite direction is the *negative* direction.

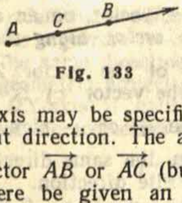


Fig. 133

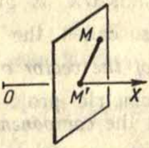


Fig. 134

Each axis may be specified by any vector lying on it and having that direction. The axis in Fig. 133 may be specified by the vector $\overrightarrow{AB}$ or $\overrightarrow{AC}$ (but not by the vector $\overrightarrow{BA}$).

Let there be given an axis OX (Fig. 134) and some point M (exterior to the axis or lying on it). Draw through M a plane perpendicular to the axis; it will intersect the axis at some point M'. The point M' is termed the *projection of the point* M *on the axis* OX (if M lies on the axis, then it is its own projection).

Note. In other words, the projection of the point M on the axis OX is the foot of a perpendicular drawn from M to OX. The above definition stresses the fact that the construction is performed in space.

92. The Projection of a Vector on an Axis

The expression "the projection of a vector $\overrightarrow{AB}$ on an axis OX" is used in two different meanings: geometrical and algebraic (arithmetical).

1. The *projection (geometric)* of a vector $\overrightarrow{AB}$ on an axis OX is the vector $\overrightarrow{A'B'}$ (Fig. 135), the origin of which A' is the projection of the origin A on the axis OX, and the terminus of which B' is the projection of the terminus B on the same axis.

Notation: $\mathrm{Pr}_{OX}\overrightarrow{AB}$ or, briefly, $\mathrm{Pr}\,\overrightarrow{AB}$.

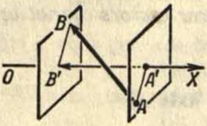


Fig. 135

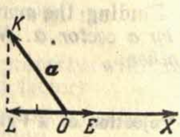


Fig. 136

If the axis OX is given by a vector $\mathbf{c}$, then the vector $\overrightarrow{A'B'}$ is also called the *projection of the vector $\overrightarrow{AB}$ on the direction of the vector* $\mathbf{c}$ and is denoted by $\mathrm{Pr}_{c}\overrightarrow{AB}$.

The geometric projection of a vector on an axis OX is also called the *component of the vector along the OX-axis.*

2. The *projection (algebraic)* of the vector $\overrightarrow{AB}$ on the OX-axis (or on the direction of the vector $\mathbf{c}$) is the *length* of the vector $\overrightarrow{A'B'}$ taken with the + or — sign depending on whether the vector $\overrightarrow{A'B'}$ is in the same direction as the OX-axis (vector $\mathbf{c}$) or in the opposite direction.

Notation:

$$\mathrm{pr}_{OX}\overrightarrow{AB} \text{ or } \mathrm{pr}_{c}\overrightarrow{AB}$$

Note. The geometric projection (component) of a vector is a *vector*, while the algebraic projection of a vector is a *number*.

Example 1. The geometric projection of the vector $\overrightarrow{OK}=\mathbf{a}$ (Fig. 136) on the OX-axis is the vector $\overrightarrow{OL}$. Its direction is opposite to that of the axis, and the length (with scale unit

OE) is equal to 2. Hence, the algebraic projection of the vector $\overrightarrow{OK}$ on the OX-axis is a negative *number*, —2:

$$\mathrm{Pr}\,\overrightarrow{OK}=\overrightarrow{OL},\quad \mathrm{pr}\,\overrightarrow{OK}=-2$$

If the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ (Fig. 137) are equal, then their algebraic projections along the same axis are also equal $\left(\mathrm{pr}\,\overrightarrow{AB}=\mathrm{pr}\,\overrightarrow{CD}=-\frac{1}{2}\right)$. The same holds for geometric projections.

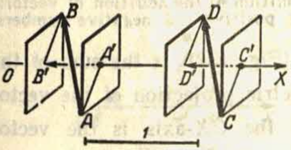


Fig. 137

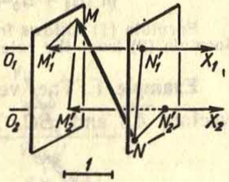


Fig. 138

Algebraic projections of the same vector on two *like directed* axes (O_1X_1 and O_2X_2 in Fig. 138) are the same[1] ($\mathrm{pr}_{O_1X_1}\overrightarrow{NM}=\mathrm{pr}_{O_2X_2}\overrightarrow{NM}=-2$). The same holds for geometric projections.

3. *The relationship between a component* (geometric projection) *and the algebraic projection of a vector.* Let $\boldsymbol{c}_1$ be a vector in the same direction as the OX-axis and of length 1. Then the geometric projection (component) of some vector $\boldsymbol{a}$ along the OX-axis is equal to the product of the vector $\boldsymbol{c}_1$ by the algebraic projection of the vector $\boldsymbol{a}$ along the same axis:

$$\mathrm{Pr}\,\boldsymbol{a}=\mathrm{pr}\,\boldsymbol{a}\cdot\boldsymbol{c}_1$$

Example 2. In the notation of Fig. 136 we have $\boldsymbol{c}_1=\overrightarrow{OE}$. The geometric projection of the vector $\overrightarrow{OK}=\boldsymbol{a}$ on the OX-axis is the vector $\overrightarrow{OL}$, and the algebraic projection of the same vector is the number -2 (see Example 1). We have $\overrightarrow{OL}=-2\overrightarrow{OE}$.

---

[1] If the axes are parallel but in opposite directions, the algebraic projections are not equal; they differ in sign.

93. Principal Theorems on Projections of Vectors

Theorem 1. The projection of a sum of vectors on some axis is equal to the sum of the projections of those vectors on the same axis.

The theorem holds true for both meanings of the term "projection of a vector" and for any number of terms; thus, for three terms

$$\mathrm{Pr}\,(a_1+a_2+a_3)=\mathrm{Pr}\,a_1+\mathrm{Pr}\,a_2+\mathrm{Pr}\,a_3 \tag{1}$$

and

$$\mathrm{pr}\,(a_1+a_2+a_3)=\mathrm{pr}\,a_1+\mathrm{pr}\,a_2+\mathrm{pr}\,a_3 \tag{2}$$

Formula (1) follows from the definition of the addition of vectors, formula (2) from the rule for adding positive and negative numbers.

Example 1. The vector $\overrightarrow{AC}$ (Fig. 139) is the sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. The geometric projection of the vector $\overrightarrow{AC}$ on the OX-axis is the vector $\overrightarrow{AC'}$ and the geometric projections of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are $\overrightarrow{AB'}$ and $\overrightarrow{B'C'}$. Here,

$$\overrightarrow{AC'}=\overrightarrow{AB'}+\overrightarrow{B'C'}$$

so that

$$\mathrm{Pr}\,(\overrightarrow{AB}+\overrightarrow{BC})=\mathrm{Pr}\,\overrightarrow{AB}+\mathrm{Pr}\,\overrightarrow{BC}$$

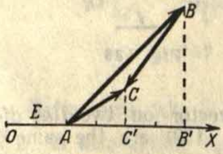


Fig. 139

Example 2. Let OE (Fig. 139) be the scale unit; then the algebraic projection of the vector $\overrightarrow{AB}$ on the OX-axis is equal to 4 (the length of AB' taken with the plus sign); i. e. $\mathrm{pr}\,\overrightarrow{AB}=4$. Further, $\mathrm{pr}\,\overrightarrow{BC}=-2$ (the length of $B'C'$ taken with the minus sign) and $\mathrm{pr}\,\overrightarrow{AC}=+2$ (the length of AC' taken with the plus sign). We have

$$\mathrm{pr}\,\overrightarrow{AB}+\mathrm{pr}\,\overrightarrow{BC}=4-2=2$$

On the other hand,

$$\mathrm{pr}\,(\overrightarrow{AB}+\overrightarrow{BC})=\mathrm{pr}\,\overrightarrow{AC}=2$$

so that

$$\mathrm{pr}\,(\overrightarrow{AB}+\overrightarrow{BC})=\mathrm{pr}\,\overrightarrow{AB}+\mathrm{pr}\,\overrightarrow{BC}$$

Theorem 2. The algebraic projection of a vector on some axis is equal to the product of the length of the vector by the cosine of the angle between the axis and the vector:

$$\mathrm{pr}_{\boldsymbol{a}}\,\boldsymbol{b}=|\,\boldsymbol{b}\,|\cos(\widehat{\boldsymbol{a},\,\boldsymbol{b}}) \tag{3}$$

Example 3. The vector $\boldsymbol{b}=\overrightarrow{MN}$ (Fig. 140) forms with the OX-axis (it is specified by the vector $\boldsymbol{a}$) an angle of 60°. If OE is the scale unit, then $|\,\boldsymbol{b}\,|=4$, so that

$$\mathrm{pr}_{\boldsymbol{a}}\,\boldsymbol{b}=4\cdot\cos 60^\circ=4\cdot\frac{1}{2}=2$$

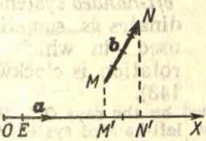


Fig. 140

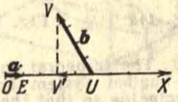


Fig. 141

Indeed, the length of the vector $\overrightarrow{M'N'}$ (geometric projection of the vector $\boldsymbol{b}$) is equal to 2, and the direction coincides with that of the OX-axis (cf. Sec. 92, Item 2).

Example 4. The vector $\boldsymbol{b}=\overrightarrow{UV}$ in Fig. 141 forms with the OX-axis (with the vector $\boldsymbol{a}$) an angle $(\widehat{\boldsymbol{a},\,\boldsymbol{b}})=120^\circ$. The length $|\,\boldsymbol{b}\,|$ of the vector $\boldsymbol{b}$ is 4. Therefore, $\mathrm{pr}_{\boldsymbol{a}}\,\boldsymbol{b}=4\cdot\cos 120^\circ=-2$.

Indeed, the length of the vector $\overrightarrow{UV'}$ is 2 and the direction is opposite to that of the axis.

94. The Rectangular Coordinate System in Space

Base vectors. The three mutually perpendicular axes OX, OY, OZ (Fig. 142) which pass through a certain point O form a *rectangular system of coordinates*. The point O is the *origin*, the straight lines OX, OY, OZ are the *axes of coordinates* (OX is the *axis of abscissas*, or x-axis, OY is the *axis of ordinates*, or y-axis, and OZ is the z-axis), and the planes XOY, YOZ, ZOX are the *coordinate planes*. Some line segment UV is taken as the *scale unit for all three axes*.

Laying off on the x, y, z-axes in the positive direction the segments OA, OB, OC equal to the scale unit, we obtain three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, which are called *base vectors* and are designated by $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, respectively.

It is customary to choose the positive directions on the axes so that a rotation through 90°, which brings the positive ray OX to coincidence with the ray OY (Fig. 142) would appear to be counterclockwise when viewed from the ray OZ. This is the *right-handed* coordinate system. The *left-handed* system of coordinates is sometimes also used, in which case the rotation is clockwise (Fig. 143).

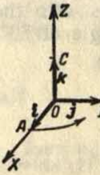


Fig. 142

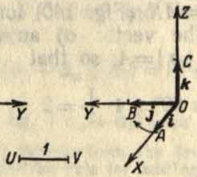


Fig. 143

Note 1. The trihedral angles formed by the rays OX, OY, OZ in the right-handed system and in the left-handed system cannot be made to coincide so that the *corresponding* axes coincide.

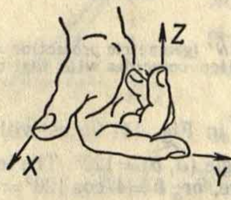


Fig. 144

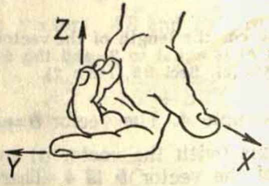


Fig. 145

Note 2. The names "left-handed" and "right-handed" stem from the fact that the right-handed system is generated if one places his thumb, index and middle fingers of the right hand as the axes OX, OY, OZ shown in Fig. 144. The same arrangement for the left hand (Fig. 145) produces the left-handed system.

95. The Coordinates of a Point

The position of any point M in space may be determined by three coordinates in the following manner. Through M draw planes MP, MQ, MR (Fig. 146) parallel, respectively, to the planes YOZ, ZOX, XOY. At the intersections with the axes we obtain the points P, Q, R. The numbers x (abscissa),

y (ordinate), and z (z-coordinate), which measure the line segments OP, OQ, OR to a given scale are called the *(rectangular) coordinates* of the point M. They are positive or negative according as the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ are in the same directions as the base vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, or in opposite directions.

Example. The coordinates of the point M in Fig. 146 are: abscissa

$$x=2$$

ordinate

$$y=-3$$

z-coordinate

$$z=2$$

Notation:

$$M(2, -3, 2).$$

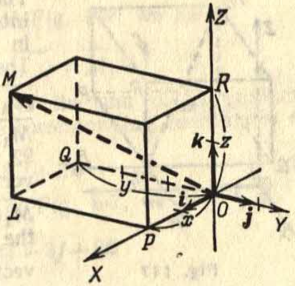


Fig. 146

The vector $\overrightarrow{OM}$ from the origin O to some point M is called the *radius vector* of M and is denoted by the letter $\boldsymbol{r}$; it is customary to use subscripts to distinguish the various radius vectors of different points: $\boldsymbol{r}_M$ for the radius vector of the point M. The radius vectors of the points A_1, A_2, ..., A_n are denoted

$$\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$$

96. The Coordinates of a Vector

Definition. The rectangular coordinates of a vector $\boldsymbol{m}$ are the algebraic projections (Sec. 92) of the vector $\boldsymbol{m}$ on the coordinate axes. The coordinates of a vector are denoted by capital letters X, Y, Z (the coordinates of a point, by lower-case letters).

Notation:

$$\boldsymbol{m}\{X, Y, Z\} \quad \text{or} \quad \boldsymbol{m}=\{X, Y, Z\}$$

Instead of projecting the vector $\boldsymbol{m}$ on the x, y, z-axes, one can project it on the axes M_1A, M_1B, M_1C (Fig. 147) drawn through the origin M_1 of the vector $\boldsymbol{m}$ and having the same directions as the coordinate axes (Sec. 92, Item 2).

Example 1. Find the coordinates of the vector $\overrightarrow{M_1M_2}$ (Fig. 147) with respect to the coordinate system $OXYZ$.

Through the point M_1 draw axes M_1A, M_1B, M_1C in the same directions, respectively, as the x, y, z-axes.

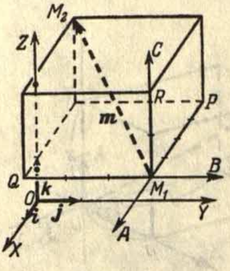


Fig. 147

Through the point M_2 draw the planes M_2P, M_2Q, M_2R parallel to the coordinate planes. The planes M_2P, M_2Q, M_2R will intersect the axes M_1A, M_1B, M_1C in the points P, Q, R, respectively. The abscissa X of the vector $\overrightarrow{M_1M_2}$ is the length of the vector $\overrightarrow{M_1P}$ taken with the minus sign (Sec. 92, Item 2); the ordinate Y of the vector $\boldsymbol{m}$ is the length of the vector $\overrightarrow{M_1Q}$ taken with the minus sign; the z-coordinate is the length of the vector $\overrightarrow{M_1R}$ taken with the plus sign. Given the scale of Fig. 147, $X=-4$, $Y=-3$, $Z=2$.

Notation:

$$\overrightarrow{M_1M_2}\{-4, -3, 2\}$$

or

$$\overrightarrow{M_1M_2}=\{-4, -3, 2\}$$

If two vectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are equal, then their coordinates are respectively the same:

$$X_1=X_2,\ Y_1=Y_2,\ Z_1=Z_2$$

(cf. Sec. 92, Item 2).

The coordinates of a *vector* are *invariant* under a parallel translation of the system of coordinates. This is *not true* of the coordinates of a *point* under the same translation (see below, Sec. 166, Item 1).

If the origin O of a vector $\overrightarrow{OM}$ coincides with the origin of coordinates, then the coordinates of the vector $\overrightarrow{OM}$ are equal, respectively, to the coordinates of the terminus M (Sec. 95).

Example 2. In Fig. 146, the vector $\overrightarrow{OM}$ has abscissa $X=2$, ordinate $Y=-3$, and z-coordinate $Z=2$. The point M has the same coordinates.

Notation: $\overrightarrow{OM}\{2, -3, 2\}$ or $\overrightarrow{OM}=\{2, -3, 2\}$

97. Expressing a Vector in Terms of Components and in Terms of Coordinates

1. Every vector is equal to the sum of its components (geometric projections) on the three coordinate axes:

$$\boldsymbol{m}=\mathrm{Pr}_{OX}\,\boldsymbol{m}+\mathrm{Pr}_{OY}\,\boldsymbol{m}+\mathrm{Pr}_{OZ}\,\boldsymbol{m} \tag{1}$$

Example 1. In the notation of Fig. 147, we have

$$\overrightarrow{M_1M_2}=\overrightarrow{M_1P}+\overrightarrow{M_1Q}+\overrightarrow{M_1R}$$

2. Every vector $\boldsymbol{m}$ is equal to the sum of the products of the three base vectors by the corresponding coordinates of the vector $\boldsymbol{m}$:

$$\boldsymbol{m}=X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k} \tag{2}$$

Example 2. In the notation of Fig. 147, we have

$$\overrightarrow{M_1M_2}=-4\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{k}$$

98. Operations Involving Vectors Specified by Their Coordinates

1. *When vectors are added, their coordinates are also added;* i. e. if $\boldsymbol{a}=\boldsymbol{a}_1+\boldsymbol{a}_2$, then $X=X_1+X_2$, $Y=Y_1+Y_2$, $Z=Z_1+Z_2$.

2. A similar rule holds for the subtraction of vectors: if $\boldsymbol{a}=\boldsymbol{a}_2-\boldsymbol{a}_1$, then $X=X_2-X_1$, $Y=Y_2-Y_1$, $Z=Z_2-Z_1$.

3. *When multiplying a vector by a number, multiply all the coordinates by that number;* i. e. if $\boldsymbol{m}_2=\lambda\boldsymbol{m}_1$, then $X_2=\lambda X_1$, $Y_2=\lambda Y_1$, $Z_2=\lambda Z_1$.

4. A similar rule holds for the division of a vector by a number: if $\boldsymbol{m}_2=\frac{\boldsymbol{m}_1}{\lambda}$, then $X_2=\frac{X_1}{\lambda}$, $Y_2=\frac{Y_1}{\lambda}$, $Z_2=\frac{Z_1}{\lambda}$.

99. Expressing a Vector in Terms of the Radius Vectors of Its Origin and Terminus

Note an important formula:

$$\overrightarrow{A_1A_2}=\boldsymbol{r}_2-\boldsymbol{r}_1 \tag{1}$$

where $\boldsymbol{r}_1=\overrightarrow{OA_1}$ (Fig. 148) is the radius vector (Sec. 95) of the origin A_1 of the vector $\overrightarrow{A_1A_2}$, and $\boldsymbol{r}_2=\overrightarrow{OA_2}$ is the radius vector of its terminus A_2.

From (1), by virtue of Sec. 98, Item 2, we get the following formulas:

$$X=x_2-x_1, \quad Y=y_2-y_1, \quad Z=z_2-z_1 \tag{2}$$

Here, X, Y, Z are the coordinates of the vector $\overrightarrow{A_1A_2}$; x_1, y_1, z_1 are the coordinates of the point A_1 (they are equal respectively to the coordinates of the radius vector $r_1=\overrightarrow{OA_1}$) and x_2, y_2, z_2 are the coordinates of the point A_2 (they are equal respectively to the coordinates of the radius vector $r_2=\overrightarrow{OA_2}$).

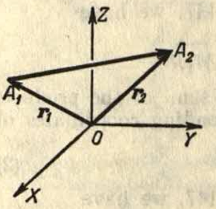


Fig. 148

In words: *to find the x-coordinate (abscissa) of a vector, subtract the abscissa of the origin of the vector from the abscissa of the terminus.*

Similar rules hold for the y-coordinate (ordinate) and the z-coordinate.

Example. Find the coordinates of the vector $\overrightarrow{A_1A_2}$ if $A_1(1, -2, 5)$ and $A_2(-2, 4, 0)$.

Solution. $X=-2-1=-3$, $Y=4-(-2)=6$, $Z=0-5=-5$ so that $\overrightarrow{A_1A_2}=\{-3, 6, -5\}$.

100. The Length of a Vector. The Distance Between Two Points

The length of a vector $a\{X, Y, Z\}$ is expressed in terms of its coordinates by the formula

$$|a|=\sqrt{X^2+Y^2+Z^2} \tag{1}$$

Example 1. The length of the vector $a\{-4, -3, 2\}$ is equal (cf. Fig. 147) to

$$|a|=\sqrt{(-4)^2+(-3)^2+2^2}=\sqrt{29}\approx 5.4$$

The distance d between the points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$ is given by the formula

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \tag{2}$$

which is obtained from (1) by virtue of formulas (2), Sec. 99 (cf. Sec. 10).

Example 2. The distance between points $A_1(8, -3, 8)$, $A_2(6, -1, 9)$ is $d=\sqrt{(6-8)^2+(-1+3)^2+(9-8)^2}=3$.

101. The Angle Between a Coordinate Axis and a Vector

The angles α, β, γ (Fig. 149) formed by the positive directions of OX, OY, OZ with the vector $\boldsymbol{a}\{X, Y, Z\}$ may be found from the formulas[1]

$$\cos\alpha = \frac{X}{\sqrt{X^2+Y^2+Z^2}} \left(=\frac{X}{|\boldsymbol{a}|}\right), \quad (1)$$

$$\cos\beta = \frac{Y}{\sqrt{X^2+Y^2+Z^2}} \left(=\frac{Y}{|\boldsymbol{a}|}\right), \quad (2)$$

$$\cos\gamma = \frac{Z}{\sqrt{X^2+Y^2+Z^2}} \left(=\frac{Z}{|\boldsymbol{a}|}\right) \quad (3)$$

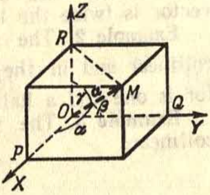


Fig. 149

If the vector $\boldsymbol{a}$ has length egual to the scale unit, i.e. if $|\boldsymbol{a}|=1$, then

$$\cos\alpha = X, \quad \cos\beta = Y, \quad \cos\gamma = Z$$

From (1), (2), (3), it follows that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad (4)$$

Example. Find the angles formed by the coordinate axes with the vector $\{2, -2, -1\}$.

Solution. $\cos\alpha = \frac{2}{\sqrt{2^2+(-2)^2+1}} = \frac{2}{3}$, $\cos\beta = -\frac{2}{3}$, $\cos\gamma = -\frac{1}{3}$, whence $\alpha \approx 48°11'$, $\beta \approx 131°49'$, $\gamma \approx 109°28'$.

102. Criterion of Collinearity (Parallelism) of Vectors

If the vectors $\boldsymbol{a}_1\{X_1, Y_1, Z_1\}$, $\boldsymbol{a}_2\{X_2, Y_2, Z_2\}$ are collinear, then their respective coordinates are proportional:

$$X_2 : X_1 = Y_2 : Y_1 = Z_2 : Z_1 \quad (1)$$

and vice versa.

If the coefficient of proportionality $\lambda = \frac{X_2}{X_1} = \frac{Y_2}{Y_1} = \frac{Z_2}{Z_1}$ is positive, then the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are in the same direction; if it is negative, the directions are opposite. The absolute value of λ expresses the ratio of the lengths $|\boldsymbol{a}_2| : |\boldsymbol{a}_1|$.

---

[1] From the right-angle triangle OMR we have

$$\cos\gamma = \frac{OR}{|\overrightarrow{OM}|} = \frac{Z}{|\boldsymbol{a}|} = \frac{Z}{\sqrt{X^2+Y^2+Z^2}}$$

Formulas (1) and (2) are obtained in similar fashion.

Note. If one of the coordinates of the vector a_1 is zero, then the proportion (1) is to be understood in the meaning that the corresponding coordinate of the vector a_2 is also zero.

Example 1. The vectors $\{-2, 1, 3\}$ and $\{4, -2, -6\}$ are collinear and oppositely directed $(\lambda=-2)$. The second vector is twice the length of the first.

Example 2. The vectors $\{4, 0, 10\}$ and $\{6, 0, 15\}$ are collinear and in the same direction $\left(\lambda=\frac{3}{2}\right)$. The second vector is one and a half times longer than the first.

Example 3. The vectors $\{2, 0, 4\}$ and $\{4, 0, 2\}$ are not collinear.

103. Division of a Segment in a Given Ratio

The radius vector $\boldsymbol{r}$ of a point A, which divides the segment A_1A_2 in the ratio $A_1A:AA_2=m_1:m_2$, is determined by the formula

$$\boldsymbol{r}=\frac{m_2\boldsymbol{r}_1+m_1\boldsymbol{r}_2}{m_1+m_2} \tag{1}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the radius vectors of the points A_1 and A_2.

The coordinates of the point A are found from the formulas

$$x=\frac{m_2x_1+m_1x_2}{m_1+m_2}, \quad y=\frac{m_2y_1+m_1y_2}{m_1+m_2}, \quad z=\frac{m_2z_1+m_1z_2}{m_1+m_2} \tag{2}$$

(cf. Sec. 11).

In particular, the coordinates of the midpoint of the segment A_1A_2 are

$$x=\frac{x_1+x_2}{2}, \quad y=\frac{y_1+y_2}{2}, \quad z=\frac{z_1+z_2}{2} \tag{3}$$

Note. The point A may also be taken on the prolongation of the segment A_1A_2 in either direction; then one of the numbers m_1, m_2 must be taken with the minus sign.

Example. Find the coordinates of the point A which divides the segment A_1A_2 in the ratio $A_1A:AA_2=2:3$ if $A_1(2, 4, -1)$, $A_2(-3, -1, 6)$.

Using formulas (2), we find

$$x=\frac{3\cdot 2+2\cdot(-3)}{2+3}=0, \quad y=\frac{3\cdot 4+2\cdot(-1)}{2+3}=2,$$

$$z=\frac{3\cdot(-1)+2\cdot 6}{2+3}=\frac{9}{5}$$

104. Scalar Product of Two Vectors

Definition. The scalar product of a vector $\mathbf{a}$ by a vector $\mathbf{b}$ is the product of their absolute values by the cosine of the angle between them.

Notation: $\mathbf{a}\cdot\mathbf{b}$ or $\mathbf{ab}$

By definition,

$$\mathbf{ab}=|\mathbf{a}|\cdot|\mathbf{b}|\cos(\widehat{\mathbf{a},\ \mathbf{b}}) \tag{1}$$

By virtue of Theorem 2, Sec. 93

$$|\mathbf{b}|\cos(\widehat{\mathbf{a},\ \mathbf{b}})=\mathrm{pr}_{\mathbf{a}}\mathbf{b}$$

so that instead of (1) we can write

$$\mathbf{ab}=|\mathbf{a}|\,\mathrm{pr}_{\mathbf{a}}\mathbf{b} \tag{2}$$

Analogously

$$\mathbf{ab}=|\mathbf{b}|\,\mathrm{pr}_{\mathbf{b}}\mathbf{a}$$

In words, the *scalar product of two vectors is equal to the absolute value of one of them multiplied by the algebraic projection of the other vector on the direction of the first.*

If the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is acute, then $\mathbf{ab}>0$; if it is obtuse, then $\mathbf{ab}<0$; if it is a right angle, then $\mathbf{ab}=0$.

This follows from formula (1).

Example. The lengths of the vectors $\mathbf{a}$ and $\mathbf{b}$ are respectively equal to 2 metres and 1 metre, and the angle between them is 120°. Find the scalar product $\mathbf{ab}$.

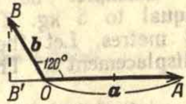


Fig. 150

Using formula (1), we have $\mathbf{ab}=2\cdot1\cdot\cos120^\circ=-1$ (metre squared).

Let us compute the same quantity using formula (2). The algebraic projection of the vector $\mathbf{b}$ (Fig. 150) on the direction of the vector $\mathbf{a}$ is equal to $|\overrightarrow{OB}|\cos120^\circ=-\frac{1}{2}$ (the length of the vector $\overrightarrow{OB'}$ taken with the minus sign). We have

$$\mathbf{ab}=|\mathbf{a}|\,\mathrm{pr}_{\mathbf{a}}\mathbf{b}=2\cdot\left(-\frac{1}{2}\right)=-1 \text{ (metre squared)}$$

Note 1. Let us examine the term "scalar product". The first word states that the result of the operation is a scalar and not a vector (in contrast to a *vector product*; see Sec. 111). The second word stresses the fact that for this operation the basic properties of ordinary multiplication hold (Sec. 105).

Note 2. Scalar multiplication cannot be extended to the case of three factors.

Indeed, the scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a number; if this number is multiplied by a vector $\boldsymbol{c}$ (Sec. 89), then the product will be a *vector:*

$$(\boldsymbol{ab})\,\boldsymbol{c} = |\boldsymbol{a}|\cdot|\boldsymbol{b}|\cos\,(\widehat{\boldsymbol{a},\ \boldsymbol{b}})\,\boldsymbol{c}$$

collinear with the vector $\boldsymbol{c}$.

104a. The Physical Meaning of a Scalar Product

If the vector $\boldsymbol{a} = \overrightarrow{OA}$ (Fig. 151) depicts a displacement of a mass point, and the vector $\boldsymbol{F} = \overrightarrow{OF}$ depicts the force acting on that point, then the scalar product $\boldsymbol{aF}$ is *numerically equal to the work of the force* $\boldsymbol{F}$.

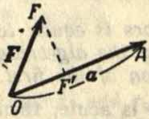


Fig. 151

Indeed, only the component $\overrightarrow{OF'}$ performs work. This means that in absolute value the work is equal to the product of the lengths of the vectors $\boldsymbol{a}$ and $\overrightarrow{OF'}$. It is considered positive if the vectors $\overrightarrow{OF'}$ and $\boldsymbol{a}$ are in the same direction, and negative if they are in opposite directions. Hence, the work is equal to the absolute value of the vector $\boldsymbol{a}$ multiplied by the algebraic projection of the vector $\boldsymbol{F}$ along the direction of the vector $\boldsymbol{a}$; i.e. the work is equal to the scalar product $\boldsymbol{aF}$.

Example. The vector of a force $\boldsymbol{F}$ has an absolute value equal to 5 kg. The length of the displacement vector $\boldsymbol{a}$ is 4 metres. Let the force $\boldsymbol{F}$ act at an angle $\alpha = 45^\circ$ to the displacement $\boldsymbol{a}$. Then the work of the force $\boldsymbol{F}$ is

$$\boldsymbol{Fa} = |\boldsymbol{F}|\cdot|\boldsymbol{a}|\cos\alpha = 5\cdot 4\frac{\sqrt{2}}{2} = 10\sqrt{2} \approx 14.1 \text{ kg-m}$$

105. Properties of a Scalar Product

1. The scalar product $\boldsymbol{ab}$ vanishes if one of the factors is a null vector or if the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

This follows from (1), Sec. 104.

Example. $3\boldsymbol{i}\cdot 2\boldsymbol{j} = 0$, since the base vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and, hence, also the vectors $3\boldsymbol{i}$, $2\boldsymbol{j}$ are perpendicular.

Note. In ordinary algebra, the equality $ab=0$ states that either $a=0$ or $b=0$. For a scalar product this property does not hold true.

2. $\boldsymbol{ab}=\boldsymbol{ba}$ (commutative property).

This follows from (1), Sec. 104.

3. $(\boldsymbol{a}_1+\boldsymbol{a}_2)\boldsymbol{b}=\boldsymbol{a}_1\boldsymbol{b}+\boldsymbol{a}_2\boldsymbol{b}$ (distributive property).

This property holds for any number of terms; for example, for three terms

$$(\boldsymbol{a}_1+\boldsymbol{a}_2+\boldsymbol{a}_3)\boldsymbol{b}=\boldsymbol{a}_1\boldsymbol{b}+\boldsymbol{a}_2\boldsymbol{b}+\boldsymbol{a}_3\boldsymbol{b}$$

This follows from (2), Sec. 104, and from (3), Sec. 93.

4. $(m\boldsymbol{a})\boldsymbol{b}=m(\boldsymbol{ab})$ (associative property with respect to a scalar factor).[1]

Examples.

$$(2\boldsymbol{a})\boldsymbol{b}=2\boldsymbol{ab},\quad (-3\boldsymbol{a})\boldsymbol{b}=-3\boldsymbol{ab},\quad \boldsymbol{p}(-6\boldsymbol{q})=-6\boldsymbol{pq}$$

Property 4 is derived from (1), Sec. 104 (it is convenient to consider separately the cases $m>0$ and $m<0$).

4a. $(m\boldsymbol{a})(n\boldsymbol{b})=(mn)\boldsymbol{ab}$.

Examples.

$$(2\boldsymbol{a})(-3\boldsymbol{b})=-6\boldsymbol{ab},\ (-5\boldsymbol{p})\left(-\frac{2}{3}\boldsymbol{q}\right)=\frac{10}{3}\boldsymbol{pq}$$

This property follows from Property 4.

Properties 2, 3, and 4a permit applying to scalar products the same operations as are performed in algebra on the products of polynomials.

Example 1.

$$2\boldsymbol{ab}+3\boldsymbol{ac}=\boldsymbol{a}(2\boldsymbol{b}+3\boldsymbol{c})$$

(by virtue of Properties 3 and 4).

Example 2.

$$(2\boldsymbol{a}-3\boldsymbol{b})(\boldsymbol{c}+5\boldsymbol{d})=2\boldsymbol{ac}+10\boldsymbol{ad}-3\boldsymbol{bc}-15\boldsymbol{bd}$$

(by virtue of Properties 3 and 4a).

Example 3. Compute the expression $(\boldsymbol{i}+\boldsymbol{k})(\boldsymbol{j}-\boldsymbol{k})$, where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are base vectors.

Solution. Since the vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are mutually perpendicular, it follows that $\boldsymbol{ij}=\boldsymbol{ik}=\boldsymbol{jk}=0$; besides,

$$\boldsymbol{kk}=|\boldsymbol{k}|\,|\boldsymbol{k}|\cos(\widehat{\boldsymbol{k},\boldsymbol{k}})=|\boldsymbol{k}|^2\cos 0=1$$

---

[1] The associative property does not hold with respect to a vector factor: the expression $(\boldsymbol{cb})\boldsymbol{a}$ is a vector collinear with $\boldsymbol{a}$ (Sec. 104, Note 2) whereas $\boldsymbol{c}(\boldsymbol{ba})$ is a vector collinear with $\boldsymbol{c}$ so that

$$(\boldsymbol{cb})\boldsymbol{a}\neq\boldsymbol{c}(\boldsymbol{ba})$$

(the absolute value of the base vector is equal to unity). Therefore

$$(\boldsymbol{i}+\boldsymbol{k})(\boldsymbol{j}-\boldsymbol{k})=\boldsymbol{ij}-\boldsymbol{ik}+\boldsymbol{kj}-\boldsymbol{kk}=-1$$

5. If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are collinear, then $\boldsymbol{ab}=\pm|\boldsymbol{a}|\cdot|\boldsymbol{b}|$; (the plus sign if $\boldsymbol{a}$, $\boldsymbol{b}$ have the same direction; and the minus sign if opposite directions).

5a. In particular, $\boldsymbol{aa}=|\boldsymbol{a}|^2$

The scalar product $\boldsymbol{aa}$ is denoted $\boldsymbol{a}^2$ (*scalar square* of the vector $\boldsymbol{a}$), so that

$$\boldsymbol{a}^2=|\boldsymbol{a}|^2 \tag{1}$$

(the *scalar square of a vector is the square of its absolute value*).

Note 1. In vector algebra there is no scalar cube (higher powers are all the more so absent, cf. Sec. 104, Note 2).

Note 2. $\boldsymbol{a}^2$ is a positive number (the square of the length of the vector); we can extract any *n*th root, for example, the square root $\sqrt{\boldsymbol{a}^2}$ (the length of the vector $\boldsymbol{a}$). *However, one cannot* write $\boldsymbol{a}$ in place of $\sqrt{\boldsymbol{a}^2}$, since $\boldsymbol{a}$ is a vector, while $\sqrt{\boldsymbol{a}^2}$ is a number. The proper result is

$$\sqrt{\boldsymbol{a}^2}=|\boldsymbol{a}| \tag{2}$$

106. The Scalar Products of Base Vectors

From the definition given in Sec. 104 it follows that

$$\boldsymbol{ii}=\boldsymbol{i}^2=1,\quad \boldsymbol{jj}=\boldsymbol{j}^2=1,\quad \boldsymbol{kk}=\boldsymbol{k}^2=1,$$
$$\boldsymbol{ij}=\boldsymbol{ji}=0,\quad \boldsymbol{jk}=\boldsymbol{kj}=0,\quad \boldsymbol{ki}=\boldsymbol{ik}=0$$

(cf. Sec. 105, Example 3).

These relations may be presented in the form of a table of scalar multiplication:

Multiplicand Multiplier	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{i}$	1	0	0
$\boldsymbol{j}$	0	1	0
$\boldsymbol{k}$	0	0	1

107. Expressing a Scalar Product in Terms of the Coordinates of the Factors

If $a_1=\{X_1, Y_1, Z_1\}$ and $a_2=\{X_2, Y_2, Z_2\}$, then[1]

$$a_1a_2=X_1X_2+Y_1Y_2+Z_1Z_2 \tag{1}$$

In particular, if $m=\{X, Y, Z\}$; then

$$m^2=X^2+Y^2+Z^2 \tag{2}$$

whence

$$\sqrt{m^2}=|m|=\sqrt{X^2+Y^2+Z^2} \tag{2a}$$

(cf. Sec. 105, Note 2, and Sec. 100).

Example 1. Find the lengths of the vectors $a_1\{3, 2, 1\}$, $a_2\{2, -3, 0\}$ and the scalar product of these vectors.

Solution. The desired lengths are

$$\sqrt{a_1^2}=\sqrt{3^2+2^2+1^2}=\sqrt{14},$$
$$\sqrt{a_2^2}=\sqrt{2^2+(-3)^2+0^2}=\sqrt{13}$$

The scalar product is

$$a_1a_2=3\cdot 2+2(-3)+1\cdot 0=0$$

Hence (Sec. 105, Item 1), the vectors a_1 and a_2 are perpendicular.

Example 2. Find the angle between the vectors

$$a_1\{-2, 1, 2\} \quad \text{and} \quad a_2\{-2, -2, 1\}$$

Solution. The lengths of the vectors are

$$|a_1|=\sqrt{(-2)^2+1^2+2^2}=3,$$
$$|a_2|=\sqrt{(-2)^2+(-2)^2+1^2}=3$$

The scalar product $a_1a_2=(-2)(-2)+1(-2)+2\cdot 1=4$. Since $a_1a_2=|a_1||a_2|\cos(\widehat{a_1, a_2})$, it follows that

$$\cos(\widehat{a_1, a_2})=\frac{a_1a_2}{|a_1|\cdot|a_2|}=\frac{4}{3\cdot 3}=\frac{4}{9}$$

i. e.

$$(\widehat{a_1, a_2})\approx 63°37'$$

---

[1] We have $a_1=X_1i+Y_1j+Z_1k$, $a_2=X_2i+Y_2j+Z_2k$. Multiply together taking into account Properties 3, 4, Sec. 105 and the table in Sec. 106.

108. The Perpendicularity Condition of Vectors

If the vectors $\boldsymbol{a}_1\{X_1, Y_1, Z_1\}$, $\boldsymbol{a}_2\{X_2, Y_2, Z_2\}$ are mutually perpendicular, then

$$X_1X_2+Y_1Y_2+Z_1Z_2=0$$

Conversely, if $X_1X_2+Y_1Y_2+Z_1Z_2=0$, then the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are perpendicular or one of them (say, $\boldsymbol{a}_1$) is a null vector [1] (then $X_1=Y_1=Z_1=0$).

This is derived from Sec. 105, Item 1, and (1) of Sec. 107.

109. The Angle Between Vectors

The angle φ between the vectors $\boldsymbol{a}_1\{X_1, Y_1, Z_1\}$, $\boldsymbol{a}_2\{X_2, Y_2, Z_2\}$ may be found from the formula (cf. Example 2, Sec. 107)

$$\cos\varphi=\frac{\boldsymbol{a}_1\boldsymbol{a}_2}{|\boldsymbol{a}_1|\cdot|\boldsymbol{a}_2|}=\frac{X_1X_2+Y_1Y_2+Z_1Z_2}{\sqrt{X_1^2+Y_1^2+Z_1^2}\cdot\sqrt{X_2^2+Y_2^2+Z_2^2}} \qquad (1)$$

This is derived from (1) and (2a) of Sec. 107.

Example 1. Find the angle between the vectors $\{1, 1, 1\}$ and $\{2, 0, 3\}$.

Solution.

$$\cos\varphi=\frac{1\cdot 2+1\cdot 0+1\cdot 3}{\sqrt{1^2+1^2+1^2}\cdot\sqrt{2^2+3^2}}=\frac{5}{\sqrt{3}\cdot\sqrt{13}}\approx 0.8006$$

whence $\varphi\approx 36°50'$.

Example 2. The vertices of a triangle ABC are

$$A(1, 2, -3);\ B(0, 1, 2);\ C(2, 1, 1)$$

Find the lengths of the sides AB and AC and the angle A.

Solution.

$$\overrightarrow{AB}=\{(0-1), (1-2), (2+3)\}=\{-1, -1, 5\},$$

$$\overrightarrow{AC}=\{(2-1), (1-2), (1+3)\}=\{1, -1, 4\},$$

$$|\overrightarrow{AB}|=\sqrt{(-1)^2+(-1)^2+5^2}=3\sqrt{3},$$

$$|\overrightarrow{AC}|=\sqrt{1^2+(-1)^2+4^2}=3\sqrt{2},$$

---

[1] The null vector may be regarded as perpendicular to any vector; cf. Sec. 82.

$$\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| \cdot |\overrightarrow{AC}|} = \frac{(-1)\cdot 1 + (-1)\cdot(-1) + 5\cdot 4}{9\sqrt{6}} = \frac{20}{9\sqrt{6}}$$

Note. The formulas (1) to (3), Sec. 101, are special cases of formula (1) of this section.

110. Right-handed and Left-handed Systems of Three Vectors

Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be three (nonzero) vectors that are not parallel to one plane and are taken in the indicated order (i. e. $\boldsymbol{a}$ is the first vector, $\boldsymbol{b}$ the second and $\boldsymbol{c}$ the third). Bringing them to the common origin O (Fig. 152), we get three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ not lying in one plane.

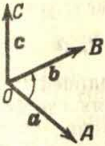


Fig. 152

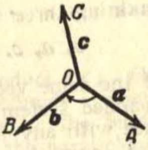


Fig. 153

The system of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, is called *right-handed* (Fig. 152) if a rotation of the vector $\overrightarrow{OA}$ which brings it to coincidence (by the shortest route) with the vector $\overrightarrow{OB}$ is performed in a counterclockwise sense for an observer at point C.

If the rotation is clockwise (Fig. 153), then the system of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ is called *left-handed.* 1)

Example 1. The base vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ in a right-handed coordinate system (Sec. 94) form a right-handed system. However, the system $\boldsymbol{j}$, $\boldsymbol{i}$, $\boldsymbol{k}$ (the vectors are the same, but the order is different) is left-handed.

If we have two systems of three vectors and each of them is right-handed or each is left-handed, then we say that these systems have the *same orientation*; if one of the systems is right-handed and the other is left-handed, then we say that the systems have *opposite orientations*.

---

1) On the origin of the names "right-handed" and "left-handed" see Sec. 94, Note 2.

A system changes its orientation in a single interchange of two vectors (cf. Example 1).

A system maintains its orientation in the case of a *circular permutation* of the vectors as indicated in Fig. 154 (the second vector becomes the first, the third the second, and the first becomes the third, i. e. in place of the system $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ we have the system $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}$).

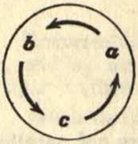


Fig. 154

Example 2. A circular permutation carries the right-handed system $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ into the right-handed system $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{i}$, and from this system to the right-handed system $\boldsymbol{k}, \boldsymbol{i}, \boldsymbol{j}$.

Example 3. If the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ form a right-handed system, then the following three systems are right-handed:

$$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \quad \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}, \quad \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}$$

and the remaining three systems

$$\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}, \quad \boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}, \quad \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{a}$$

composed of the same vectors are left-handed.

A right-handed system of three vectors cannot be brought to coincidence with any left-handed system.

The mirror image of a right-handed system is a left-handed system, and vice versa.

111. The Vector Product of Two Vectors

Definition. The vector product of a vector $\boldsymbol{a}$ (multiplicand) by a noncollinear vector $\boldsymbol{b}$ (multiplier) is a third vector $\boldsymbol{c}$ (product), which is constructed as follows:

(1) its absolute value is numerically equal to the area of a parallelogram (*AOBL* in Fig. 155) constructed on the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$; i. e. it is equal to $|\boldsymbol{a}|\cdot|\boldsymbol{b}|\sin(\widehat{\boldsymbol{a}, \boldsymbol{b}})$;

(2) its direction is perpendicular to the plane of the indicated parallelogram;

(3) the direction of the vector $\boldsymbol{c}$ is chosen (from two possible directions) so that the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ form a right-handed system (Sec. 110).

Notation: $\boldsymbol{c}=\boldsymbol{a}\times\boldsymbol{b}$ or $\boldsymbol{c}=[\boldsymbol{ab}]$

Supplement to definition. If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are collinear, then it is natural to assign a zero area to the figure *AOBL* (conditionally we continue to consider it a parallelogram). Therefore the vector product of collinear vectors is considered equal to the null vector.

Since any direction can be attributed to a null vector, this agreement does not contradict Items 2 and 3 of the definition.

Note 1. In the term "vector product" the first word indicates that the result of the operation is a vector (in contrast to a scalar product; cf. Sec. 104, Note 1).

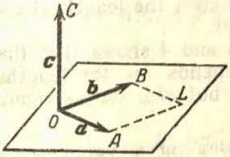


Fig. 155

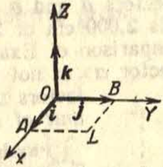


Fig. 156

Example 1. Find the vector product $\boldsymbol{i}\times\boldsymbol{j}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ are base vectors of a right-handed coordinate system (Fig. 156).

Solution. (1) Since the lengths of the base vectors are equal to the scale unit, the area of the parallelogram (square) *AOBL* is numerically equal to unity. Hence, the absolute value of the vector product is unity.

(2) Since the perpendicular to the plane *AOBL* is the axis *OZ*, the desired vector product is a vector collinear with the vector $\boldsymbol{k}$; and since both of them have absolute value 1, the desired vector product is equal either to $\boldsymbol{k}$ or to $-\boldsymbol{k}$.

(3) Of these two possible vectors we have to choose the first, since the vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ form a right-handed system (and the vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $-\boldsymbol{k}$ form a left-handed system).

Thus,

$$\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$$

Example 2. Find the vector product $\boldsymbol{j}\times\boldsymbol{i}$.

Solution. As in Example 1, we conclude that the vector $\boldsymbol{j}\times\boldsymbol{i}$ is equal either to $\boldsymbol{k}$ or to $-\boldsymbol{k}$. But this time we have to choose $-\boldsymbol{k}$, since the vectors $\boldsymbol{j}$, $\boldsymbol{i}$, $-\boldsymbol{k}$ form a right-handed system (and the vectors $\boldsymbol{j}$, $\boldsymbol{i}$, $\boldsymbol{k}$ form a left-handed system).

Thus

$$\boldsymbol{j}\times\boldsymbol{i}=-\boldsymbol{k}$$

Example 3. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have lengths equal to 80 cm and 50 cm, respectively, and form an angle of 30°.

Taking the metre as the unit of length, find the length of the vector product $\boldsymbol{a}\times\boldsymbol{b}$.

Solution. The area of the parallelogram constructed on the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $80\cdot 50\sin 30^\circ = 2{,}000$ (cm^2) or 0.2 m^2. The length of the desired vector product is 0.2 metre.

Example 4. Find the length of the vector product of the same vectors, taking the centimetre as the unit of length.

Solution. Since the area of the parallelogram constructed on the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is 2,000 cm^2, the length of the vector product is 2,000 cm or 20 metres.

A comparison of Examples 3 and 4 shows that the length of the vector $\boldsymbol{a}\times\boldsymbol{b}$ not only depends on the lengths of the factors $\boldsymbol{a}$ and $\boldsymbol{b}$ but *also on the choice of the unit of length.*

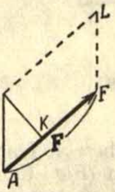


Fig. 157

Physical meaning of a vector product. Out of a multitude of physical quantities depicted by a vector product we consider only the moment of a force.

Let A be the point of application of a force $\boldsymbol{F}$. *The momemt of the force* $\boldsymbol{F}$ relative to the point O is the vector product $\overrightarrow{OA}\times\boldsymbol{F}$. Since the absolute value of this vector product is numerically equal to the area of the parallelogram $AFLO$ (Fig. 157), the absolute value of the moment is equal to the product of the base AF by the altitude OK, i. e. to the force multiplied by the distance from the point O to the straight line along which the force acts.

In mechanics, proof is given to show that for equilibrium of a rigid body it is necessary that not only the sum of the vectors $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, \ldots$, representing the forces applied to the body be equal to zero, but the sum of the moments of the forces as well. When all forces are parallel to a single plane, the addition of the vectors representing the moments may be replaced by the addition and subtraction of their absolute values. This substitution is impossible in the case of arbitrary directions of the forces. Accordingly, the vector product is determined as a vector and not as a number.

112. The Properties of a Vector Product

1. The vector product $\boldsymbol{a}\times\boldsymbol{b}$ vanishes only when the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are collinear (in particular, if one or both of them are null vectors).

This follows from the first item of the definition of Sec. 111.

1a. $\boldsymbol{a}\times\boldsymbol{a}=0$.

The equality $\boldsymbol{a}\times\boldsymbol{a}=0$ makes it unnecessary to introduce the concept of a "vector square" (cf. Sec. 105, Item 5a).

2. If the factors are interchanged, the vector product is multiplied by -1 (reverses sign):

$$\mathbf{b}\times\mathbf{a}=-(\mathbf{a}\times\mathbf{b})$$

(cf. Examples 1 and 2 of Sec. 111).

Thus, a *vector product does not possess the commutative property* (cf. Sec. 105, Item 2).

3. $(\mathbf{a}+\mathbf{b})\times\mathbf{l}=\mathbf{a}\times\mathbf{l}+\mathbf{b}\times\mathbf{l}$ (distributive property).

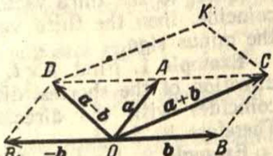


Fig. 158

This property holds for any number of terms; for example, for three terms we have

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})\times\mathbf{l}=\mathbf{a}\times\mathbf{l}+\mathbf{b}\times\mathbf{l}+\mathbf{c}\times\mathbf{l}$$

4. $(m\mathbf{a})\times\mathbf{b}=m(\mathbf{a}\times\mathbf{b})$ (associative property relative to a scalar multiplier).

4a. $(m\mathbf{a})\times(n\mathbf{b})=mn(\mathbf{a}\times\mathbf{b})$.

Examples: (1) $-3\mathbf{a}\times\mathbf{b}=-3(\mathbf{a}\times\mathbf{b})$.

(2) $0.3\mathbf{a}\times 4\mathbf{b}=1.2(\mathbf{a}\times\mathbf{b})$.

(3) $(2\mathbf{a}-3\mathbf{b})\times(\mathbf{c}+5\mathbf{d})=2(\mathbf{a}\times\mathbf{c})+10(\mathbf{a}\times\mathbf{d})-3(\mathbf{b}\times\mathbf{c})-15(\mathbf{b}\times\mathbf{d})=2(\mathbf{a}\times\mathbf{c})+10(\mathbf{a}\times\mathbf{d})+3(\mathbf{c}\times\mathbf{b})+15(\mathbf{d}\times\mathbf{b})=2(\mathbf{a}\times\mathbf{c})-10(\mathbf{d}\times\mathbf{a})+3(\mathbf{c}\times\mathbf{b})+15(\mathbf{d}\times\mathbf{b})$.

(4) $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}-\mathbf{b}\times\mathbf{b}$. The first and fourth terms are equal to zero (Item 1). Besides, $\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$ (Item 2). Hence

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=-2(\mathbf{a}\times\mathbf{b})=2(\mathbf{b}\times\mathbf{a})$$

Thus, the area *OCKD* (Fig. 158) is twice that of *OACB*.

113. The Vector Products of the Base Vectors

From the definition given in Sec. 111 it follows that

$$\begin{aligned}
&\mathbf{i}\times\mathbf{i}=0, &&\mathbf{i}\times\mathbf{j}=\mathbf{k}, &&\mathbf{i}\times\mathbf{k}=-\mathbf{j},\\
&\mathbf{j}\times\mathbf{i}=-\mathbf{k}, &&\mathbf{j}\times\mathbf{j}=0, &&\mathbf{j}\times\mathbf{k}=\mathbf{i},\\
&\mathbf{k}\times\mathbf{i}=\mathbf{j}, &&\mathbf{k}\times\mathbf{j}=-\mathbf{i}, &&\mathbf{k}\times\mathbf{k}=0
\end{aligned}$$

The following mnemonic scheme will help you to avoid making mistakes in the signs (Fig. 159).

If the direction of the shortest distance from the first vector (multiplicand) to the second (multiplier) coincides with the direction of the arrow, the product is equal to the third vector; if it does not coincide, then the third vector is taken with the minus sign.

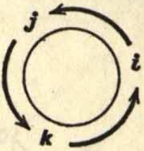


Fig. 159

Example 1. Find $\mathbf{k}\times\mathbf{i}$. See diagram, the direction of the shortest distance from $\mathbf{k}$ to $\mathbf{i}$ coincides with the direction of the arrow. Therefore $\mathbf{k}\times\mathbf{i}=\mathbf{j}$.

Example 2. Find $\mathbf{k}\times\mathbf{j}$. Here, the direction of the shortest distance is opposite to that of the arrow. Therefore $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$.

Example 3. Simplify the expression $(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})\times(4\mathbf{i}-6\mathbf{j}+12\mathbf{k})$. Removing the parentheses and taking advantage of the table or of the scheme, we find

$$\begin{aligned}
(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})\times(4\mathbf{i}-6\mathbf{j}+12\mathbf{k})&=8(\mathbf{i}\times\mathbf{i})-12(\mathbf{i}\times\mathbf{j})+\\
&+24(\mathbf{i}\times\mathbf{k})-12(\mathbf{j}\times\mathbf{i})+18(\mathbf{j}\times\mathbf{j})-36(\mathbf{j}\times\mathbf{k})+\\
&+24(\mathbf{k}\times\mathbf{i})-36(\mathbf{k}\times\mathbf{j})+72(\mathbf{k}\times\mathbf{k})=-12\mathbf{k}-24\mathbf{j}+12\mathbf{k}-\\
&-36\mathbf{i}+24\mathbf{j}+36\mathbf{i}=0
\end{aligned}$$

Since a vector product vanishes only in the case of collinearity of the factors (Sec. 112, Item 1), the vectors $2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$ and $4\mathbf{i}-6\mathbf{j}+12\mathbf{k}$ are collinear. This is also indicated by the criterion of Sec. 102.

114. Expressing a Vector Product in Terms of the Coordinates of the Factors

If $\mathbf{a}_1=\{X_1, Y_1, Z_1\}$ and $\mathbf{a}_2=\{X_2, Y_2, Z_2\}$, then [1]

$$\mathbf{a}_1\times\mathbf{a}_2=\left\{\begin{vmatrix}Y_1 Z_1\\ Y_2 Z_2\end{vmatrix},\ \begin{vmatrix}Z_1 X_1\\ Z_2 X_2\end{vmatrix},\ \begin{vmatrix}X_1 Y_1\\ X_2 Y_2\end{vmatrix}\right\} \qquad (1)$$

---

[1] We find the vector product $(X_1\mathbf{i}+Y_1\mathbf{j}+Z_1\mathbf{k})\times(X_2\mathbf{i}+Y_2\mathbf{j}+Z_2\mathbf{k})$ using the table in Sec. 113 and the Properties 2, 3, 4, Sec. 112 (cf. Example 3, Sec. 113).

The expressions given between the vertical bars are second-order determinants (Sec. 12).

Practical rule. To obtain the coordinates of the vector $a_1 \times a_2$ form the array

$$\begin{matrix} X_1 Y_1 Z_1 \\ X_2 Y_2 Z_2 \end{matrix} \qquad (2)$$

Covering the first column, we find the first coordinate:

$$\begin{matrix} Y_1 Z_1 \\ Y_2 Z_2 \end{matrix}$$

Covering the second column and *taking the remaining determinant with opposite sign* $\left(-\begin{vmatrix} X_1 Z_1 \\ X_2 Z_2 \end{vmatrix}\right.$ or, what is the same thing, $\left.\begin{vmatrix} Z_1 X_1 \\ Z_2 X_2 \end{vmatrix}\right)$, we find the second coordinate.

Covering the third column (the remaining determinant is again taken with its own sign), we find the third coordinate.

Example 1. Find the vector product of the vectors $a_1\{3, -4, -8\}$ and $a_2\{-5, 2, -1\}$.

Solution. Form the array

$$\begin{matrix} 3 & -4 & -8 \\ -5 & 2 & -1 \end{matrix}$$

Covering the first column, we obtain the first coordinate

$$\begin{vmatrix} -4 & -8 \\ 2 & -1 \end{vmatrix} = (-4)\cdot(-1) - 2\cdot(-8) = 20$$

Covering the second column, we find the determinant

$$\begin{vmatrix} 3 & -8 \\ -5 & -1 \end{vmatrix}$$

Interchanging columns (this reverses the sign), we obtain the second coordinate $\begin{vmatrix} -8 & 3 \\ -1 & -5 \end{vmatrix} = 43$.

Covering the third column, we obtain the third coordinate $\begin{vmatrix} 3 & -4 \\ -5 & 2 \end{vmatrix} = -14$.

Thus, $a_1 \times a_2 = \{20, 43, -14\}$.

Note. To avoid mistakes in the sign when computing the second coordinate, use the following table instead of array (2):

$$\begin{matrix} X_1 & Y_1 & Z_1 & X_1 & Y_1 \\ X_2 & Y_2 & Z_2 & X_2 & Y_2 \end{matrix} \qquad (3)$$

This array is obtained from (2) by adjoining the first two columns. Covering the first column in (3), we take the next two in succession. Then, covering the second column as well, we take the next two in succession. Finally, covering the third column too, we take the last two. The columns do not have to be interchanged in any one of the three determinants obtained.

Example 2. Find the area S of a triangle with specified vertices $A_1(3, 4, -1)$, $A_2(2, 0, 4)$, $A_3(-3, 5, 4)$.

Solution. The desired area is equal to half the area of a parallelogram constructed on the vectors $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_1A_3}$. We find (Sec. 99) $\overrightarrow{A_1A_2}=\{(2-3), (0-4), (4+1)\}=\{-1, -4, 5\}$ and $\overrightarrow{A_1A_3}=\{-6, 1, 5\}$. The area of the parallelogram is equal to the absolute value of the vector product $\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}$, and the vector product is equal to $\{-25, -25, -25\}$. Hence

$$S=\frac{1}{2}\left|\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\right|=\frac{1}{2}\sqrt{(-25)^2+(-25)^2+(-25)^2}=$$
$$=\frac{1}{2}\sqrt{1875}\approx 21.7$$

115. Coplanar Vectors

Three or more vectors are called *coplanar* if, when brought to a common origin, they all lie in one plane.

If at least one of the three vectors is a null vector, all three are still considered coplanar.

The criterion of coplanarity is given in Secs. 116, 120.

116. Scalar Triple Product

The *scalar triple product* of three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (taken in that order) is the scalar product of the vector $\boldsymbol{a}$ by the vector product $\boldsymbol{b}\times\boldsymbol{c}$, i. e. the number $\boldsymbol{a}(\boldsymbol{b}\times\boldsymbol{c})$ or, what is the same thing, $(\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{a}$.

Notation: $\boldsymbol{abc}$.

Criterion of coplanarity. If the system $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is right-handed, then $\mathbf{abc} > 0$; if it is left-handed, then $\mathbf{abc} < 0$. But if the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar (Sec. 115), then $\mathbf{abc}=0$. In other words, the *vanishing of the triple product* $\mathbf{abc}$ *is a criterion of the coplanarity of the vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Geometrical interpretation of a triple product. A triple product $\mathbf{abc}$ of three noncoplanar vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is equal to the volume of a parallelepiped constructed on the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ with the plus sign if the system $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is right-handed and with the minus sign if the system is left-handed.

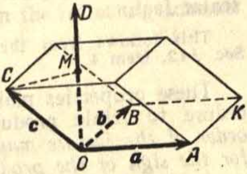


Fig. 160

Explanation. Construct (Figs. 160, 161) the vector

$$\overrightarrow{OD}=\mathbf{a}\times\mathbf{b} \qquad (1)$$

Then the area of the base $OAKB$ is equal to

$$S=|\overrightarrow{OD}| \qquad (2)$$

The altitude H (length of the vector $\overrightarrow{OM}$) with plus or minus sign is (Sec. 92, Item 2) the algebraic projection of the vector $\mathbf{c}$ along the direction $\overrightarrow{OD}$, i. e.

$$H=\pm \operatorname{pr}_{\overrightarrow{OD}} \mathbf{c} \qquad (3)$$

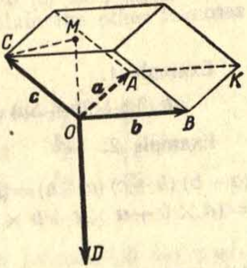


Fig. 161

The plus sign is used when $\overrightarrow{OM}$ and $\overrightarrow{OD}$ are in the same direction (Fig. 160); this is the case for a right-handed system of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The minus sign corresponds to a left-handed system (Fig. 161). From (2) and (3) we get

$$V=SH=\pm |\overrightarrow{OD}| \operatorname{pr}_{\overrightarrow{OD}} \mathbf{c}$$

but $|\overrightarrow{OD}| \operatorname{pr}_{\overrightarrow{OD}} \mathbf{c}$ is the scalar product $\overrightarrow{OD}\cdot\mathbf{c}$ (Sec. 104), i. e. $(\mathbf{a}\times\mathbf{b})\,\mathbf{c}$. Hence

$$V=\pm(\mathbf{a}\times\mathbf{b})\,\mathbf{c}$$

117. Properties of a Scalar Triple Product

1. A triple product does not change in a circular permutation of the factors (Sec. 110); an interchange of two vectors reverses the sign:

$$\mathbf{abc}=\mathbf{bca}=\mathbf{cab}=-(\mathbf{bac})=-(\mathbf{cba})=-(\mathbf{acb})$$

This follows from the geometrical interpretation (Sec. 116) and from Sec. 110.

2. $(\mathbf{a}+\mathbf{b})\,\mathbf{cd}=\mathbf{acd}+\mathbf{bcd}$ (distributive property). It extends to any number of terms.

This follows from the definition of a triple product and from Sec. 112, Item 3.

3. $(m\mathbf{a})\,\mathbf{bc}=m\,(\mathbf{abc})$ (associative property relative to the scalar factor).

This follows from the definition of a triple product and from Sec. 112, Item 4.

These properties make it possible to apply algebraic procedure to triple products, with the sole difference that the *order of the factors may be changed only if allowance is made for the sign of the product* (Item 1).

4. A triple product having at least two equal factors is zero:

$$\mathbf{aab}=0$$

Example 1.

$$\mathbf{ab}\,(3\mathbf{a}+2\mathbf{b}-5\mathbf{c})=3\mathbf{aba}+2\mathbf{abb}-5\mathbf{abc}=-5\mathbf{abc}$$

Example 2.

$$(\mathbf{a}+\mathbf{b})\,(\mathbf{b}+\mathbf{c})\,(\mathbf{c}+\mathbf{a})=(\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c})\,(\mathbf{c}+\mathbf{a})=$$
$$=(\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c})\,(\mathbf{c}+\mathbf{a})=\mathbf{abc}+\mathbf{acc}+\mathbf{aca}+\mathbf{aba}+$$
$$+\mathbf{bcc}+\mathbf{bca}$$

All the terms, except the two extreme ones, are equal to zero. Besides, $\mathbf{bca}=\mathbf{abc}$ (Property 1). Therefore

$$(\mathbf{a}+\mathbf{b})\,(\mathbf{b}+\mathbf{c})\,(\mathbf{c}+\mathbf{a})=2\mathbf{abc}$$

118. Third-Order Determinant [1]

In many cases, in particular when computing triple products, it is convenient to employ notation like

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad (1)$$

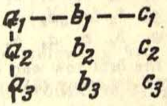
---

[1] Determinants are fully discussed in Secs. 182 to 185.

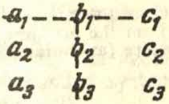
This is an abbreviation of the expression

$$a_1\begin{vmatrix}b_2 & c_2\\ b_3 & c_3\end{vmatrix}-b_1\begin{vmatrix}a_2 & c_2\\ a_3 & c_3\end{vmatrix}+c_1\begin{vmatrix}a_2 & b_2\\ a_3 & b_3\end{vmatrix} \tag{2}$$

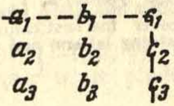
Expression (1) is called a *determinant of the third order.* The determinants of the second order which enter into (2) are constructed as follows. Delete from array (1) the row and column containing a_1, as shown in the following scheme:



The remaining determinant enters into (2) as a factor with a_1 deleted. In similar fashion we obtain the other two determinants of formula (2):



and



Remember that the *middle term in formula* (2) ***has a minus sign!***

Example 1. Evaluate the determinant

$$\begin{vmatrix}-2 & -1 & -3\\ -1 & 4 & 6\\ 1 & 5 & 9\end{vmatrix}$$

We have

$$\begin{vmatrix}-2 & -1 & -3\\ -1 & 4 & 6\\ 1 & 5 & 9\end{vmatrix}=-2\begin{vmatrix}4 & 6\\ 5 & 9\end{vmatrix}+1\begin{vmatrix}-1 & 6\\ 1 & 9\end{vmatrix}-3\begin{vmatrix}-1 & 4\\ 1 & 5\end{vmatrix}=$$

$$=-2\cdot 6+1\cdot(-15)-3\cdot(-9)=0$$

Note 1. Since $\begin{vmatrix}a_2 & c_2\\ a_3 & c_3\end{vmatrix}=-\begin{vmatrix}c_2 & a_2\\ c_3 & a_3\end{vmatrix}$, the third-order determinant may be represented as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad (3)$$

Here, all second-order determinants have a plus sign.

Note 2. Computation by formula (3) may be mechanized in the following manner. Adjoin the first two columns to array (1); this yields the array

$$\begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix} \qquad (4)$$

Take the letter a_1 in the first row and descend diagonally to the right, as shown by the arrow in array (5):

$$\begin{array}{cccccc} a_1 & & b_1 & c_1 & a_1 & b_1 \\ & \searrow & & & & \\ a_2 & \Big| & b_2 & c_2 \Big| & a_2 & b_2 \\ a_3 & \Big| & b_3 & c_3 \Big| & a_3 & b_3 \end{array} \qquad (5)$$

The second-order determinant indicated by the arrow is multiplied by a_1. This yields $a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$.

Then cover the first column, take b_1 from the first row (the first of the remaining letters) and proceed as before [as indicated in array (6)]

$$\begin{array}{ccccc} b_1 & & c_1 & a_1 & b_1 \\ & \searrow & & & \\ b_2 & \Big| & c_2 & a_2 \Big| & b_2 \\ b_3 & \Big| & c_3 & a_3 \Big| & b_3 \end{array} \qquad (6)$$

This yields

$$b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}$$

Finally cover the second column and obtain $c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$.

Example 2. Evaluate the determinant

$$D = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 2 & 5 & 2 \end{vmatrix}$$

Form array (4

$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ -1 & 3 & 4 & -1 & 3 \\ 2 & 5 & 2 & 2 & 5 \end{matrix}$$

which yields

$$D = 1 \cdot \begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix} + 3 \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = -14 + 20 - 33 = -27$$

119. Expressing a Triple Product in Terms of the Coordinates of the Factors

If the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ are defined by their coordinates
$\boldsymbol{a}_1=\{X_1, Y_1, Z_1\}$, $\boldsymbol{a}_2=\{X_2, Y_2, Z_2\}$, $\boldsymbol{a}_3=\{X_3, Y_3, Z_3\}$
then the triple product $\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3$ is computed by the formula

$$\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3=\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} \tag{1}$$

This is a consequence of formulas (1), Sec. 107, and (1), Sec. 114.

Example 1. The triple product $\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3$ of the vectors $\boldsymbol{a}_1\{-2, -1, -3\}$, $\boldsymbol{a}_2\{-1, 4, 6\}$, $\boldsymbol{a}_3\{1, 5, 9\}$ is equal to

$$\begin{vmatrix} -2 & -1 & -3 \\ -1 & 4 & 6 \\ 1 & 5 & 9 \end{vmatrix}=0$$

(cf. Sec. 118, Example 1). Hence (Sec. 116), the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are coplanar.

Example 2. The vectors $\{1, 2, 3\}$, $\{-1, 3, 4\}$, $\{2, 5, 2\}$ form a left-handed system because their triple product (Sec. 118, Example 2)

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 2 & 5 & 2 \end{vmatrix}=-27$$

is negative (see Sec. 116).

120. Coplanarity Criterion in Coordinate Form

A (necessary and sufficient) condition for coplanarity of the vectors $\boldsymbol{a}_1\{X_1, Y_1, Z_1\}$, $\boldsymbol{a}_2\{X_2, Y_2, Z_2\}$, $\boldsymbol{a}_3\{X_3, Y_3, Z_3\}$ is (see Sec. 119, Example 1)

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}=0$$

This follows from Sec. 116.

121. Volume of a Parallelepiped

The volume of a parallelepiped constructed on the vectors

$$a_1\{X_1, Y_1, Z_1\},\quad a_2\{X_2, Y_2, Z_2\},\quad a_3\{X_3, Y_3, Z_3\}$$

is

$$V=\pm\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}$$

where the plus sign is taken when the third-order determinant is positive, and the minus sign when the determinant is negative (cf. Sec. 13).

This is a consequence of Secs. 116, 119.

Example 1. Find the volume of a parallelepiped constructed on the vectors $\{1, 2, 3\}$, $\{-1, 3, 4\}$, $\{2, 5, 2\}$.

Solution. We have

$$V=\pm\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 2 & 5 & 2 \end{vmatrix}=\pm(-27)$$

Since the determinant is negative, we take the minus sign. This yields $V=27$.

Example 2. Find the volume V of a triangular pyramid $ABCD$ with vertices $A(2, -1, 1)$, $B(5, 5, 4)$, $C(3, 2, -1)$, $D(4, 1, 3)$.

Solution. We find (Sec. 99)

$$\overrightarrow{AB}=\{(5-2), (5+1), (4-1)\}=\{3, 6, 3\}$$

In the same manner, $\overrightarrow{AC}=\{1, 3, -2\}$, $\overrightarrow{AD}=\{2, 2, 2\}$. The desired volume is equal to $\frac{1}{6}$ of the volume of a parallelepiped constructed on the edges $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$. Therefore

$$V=\pm\frac{1}{6}\begin{vmatrix} 3 & 6 & 3 \\ 1 & 3 & -2 \\ 2 & 2 & 2 \end{vmatrix}$$

Whence we get $V=3$.

122. Vector Triple Product

A *vector triple product* is an expression of the form

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$$

A vector triple product is a vector that is coplanar with vectors $\boldsymbol{b}$ and $\boldsymbol{c}$; it is expressed in terms of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ as follows:

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\,(\boldsymbol{ac})-\boldsymbol{c}\,(\boldsymbol{ab}) \qquad (1)$$

123. The Equation of a Plane

A. A plane (Fig. 162) which passes through a point $M_0\,(x_0, y_0, z_0)$ and is perpendicular to a vector $\boldsymbol{N}\{A, B, C\}$ is represented by the first-degree equation[1])

$$A\,(x-x_0)+B\,(y-y_0)+C\,(z-z_0)=0 \qquad (1)$$

or

$$Ax+By+Cz+D=0 \qquad (2)$$

where D stands for the quantity

$$-(Ax_0+By_0+Cz_0)$$

The vector $\boldsymbol{N}\{A, B, C\}$ is called the *normal vector* to the plane P.

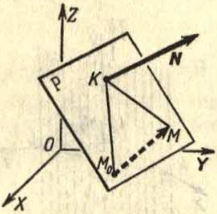


Fig. 162

Note 1. The expression "the plane P is represented by Eq. (1)" means that: (1) the coordinates x, y, z of any point $\boldsymbol{M}$ of plane P satisfy Eq. (1); (2) the coordinates x, y, z of any point exterior to plane P do not satisfy this equation (cf. Sec. 8).

B. Any equation of the first degree $Ax+By+Cz+D=0$ (A, B and C are not all simultaneously zero) represents a plane.

In vector form, Eqs. (1) and (2) are of the form

$$\boldsymbol{N}\,(\boldsymbol{r}-\boldsymbol{r}_0)=0, \qquad (1a)$$

$$\boldsymbol{Nr}+D=0 \qquad (2a)$$

($\boldsymbol{r}_0$ and $\boldsymbol{r}$ are the radius vectors of the points M_0 and M; $D=-\boldsymbol{Nr}_0$).

---

[1]) Eq. (1) is a condition for the perpendicularity of the vectors $N=\{A, B, C\}$ and $M_0M=\{x-x_0, y-y_0, z-z_0\}$. See Secs. 108 and 99.

Example. A plane passing through a point (2, 1, —1) perpendicular to a vector $\{-2, 4, 3\}$ is defined by the equation

$$-2(x-2)+4(y-1)+3(z+1)=0$$

or

$$-2x+4y+3z+3=0$$

Note 2. One and the same plane may be represented by a multiplicity of equations, all the coefficients and the constant term of which are, respectively, proportional (see below, Sec. 125, Note).

124. Special Cases of the Position of a Plane Relative to a Coordinate System

1. The equation $Ax+By+Cz=0$ (constant term $D=0$) represents a plane passing through the origin.

2. The equation $Ax+By+D=0$ (coefficient $C=0$) is a plane parallel to the z-axis OZ, the equation $Ax+Cz+D=0$ is a plane parallel to the y-axis OY, and the equation $By+Cz+{}$ $+D=0$ is a plane parallel to the x-axis OX.

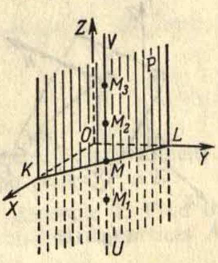


Fig. 163

It is useful to remember that *if the letter z is absent from the equation, the plane is parallel to the z-axis OZ*, etc.

Example. The equation

$$x+y-1=0$$

represents a plane P (Fig. 163) parallel to the z-axis OZ.

Note. In plane analytic geometry, the equation $x+y-1=0$ depicts a *straight* line (KL in Fig. 163). We shall now explain why the same equation in space represents a *plane*.

On the straight line KL take some point M. Since M lies in the xy-plane XOY, $z=0$. In the xy-plane, let the point M have the coordinates $x=\frac{1}{2}$, $y=\frac{1}{2}$ (they satisfy the equation $x+y-1=0$). Then in the three-dimensional system $OXYZ$, the coordinates of the point M will be $x=\frac{1}{2}$, $y=\frac{1}{2}$, $z=0$. These coordinates satisfy the equation $x+y-1=0$ (for greater clarity we shall write it in the form $1x+1y+{}$ $+0\cdot z-1=0$).

Let us now consider the points for which $x=\frac{1}{2}$, $y=\frac{1}{2}$ but $z\neq 0$, for example, the points $M_1\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, $M_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$,

$M_3\left(\frac{1}{2}, \frac{1}{2}, 1\right)$, etc. (see. Fig. 163). Their coordinates also satisfy the equation $x+y+0\cdot z-1=0$. These points fill the "vertical" straight line UV that passes through M. Such vertical straight lines may be constructed for all points of the straight line KL. Together, they will fill the plane P.

The representation of the *straight* line KL in a spatial coordinate system will be given in Sec. 140, Example 4.

3. The equation $Ax+D=0$ $(B=0,\ C=0)$ is a plane parallel both to the y-axis OY and to the z-axis OZ (see Item 2), i. e. it is parallel to the coordinate plane YOZ.

Similarly, the equation $By+D=0$ is a plane parallel to the plane XOZ, and the equation $Cz+D=0$ is a plane parallel to XOY (cf. Sec. 15).

4. The equations $X=0$, $Y=0$, $Z=0$ represent the planes YOZ, XOZ, XOY, respectively.

125. Condition of Parallelism of Planes

If the planes

$$A_1x+B_1y+C_1z+D_1=0 \quad \text{and} \quad A_2x+B_2y+C_2z+D_2=0$$

are parallel, then the normal vectors $N_1\{A_1, B_1, C_1\}$ and $N_2\{A_2, B_2, C_2\}$ are collinear (and conversely). Therefore (Sec. 102) the condition (necessary and sufficient) that the planes be parallel is

$$\frac{A_2}{A_1}=\frac{B_2}{B_1}=\frac{C_2}{C_1}$$

Example 1. The planes

$$2x-3y-4z+11=0 \quad \text{and} \quad -4x+6y+8z+36=0$$

are parallel since $\frac{-4}{2}=\frac{6}{-3}=\frac{8}{-4}$.

Example 2. The planes $2x-3z-12=0$ $(A_1=2,\ B_1=0,\ C_1=-3)$ and $4x+4y-6z+7=0$ $(A_2=4,\ B_2=4,\ C_2=-6)$ are not parallel since $B_1=0$, but $B_2\neq 0$ (Sec. 102, Note).

Note. If not only the coefficients of the coordinates, but also the constant terms are proportional; i. e. if

$$\frac{A_2}{A_1}=\frac{B_2}{B_1}=\frac{C_2}{C_1}=\frac{D_2}{D_1}$$

then the planes coincide. Thus, the equations

$$3x+7y-5z+4=0 \quad \text{and} \quad 6x+14y-10z+8=0$$

represent one and the same plane. Cf. Sec. 18, Note 3.

126. Condition of Perpendicularity of Planes

If the planes

$$A_1x+B_1y+C_1z+D_1=0 \quad \text{and} \quad A_2x+B_2y+C_2z+D_2=0$$

are perpendicular, then their normal vectors $N_1\{A_1, B_1, C_1\}$, $N_2\{A_2, B_2, C_2\}$ are also perpendicular (and conversely). Therefore (Sec. 108), the condition (necessary and sufficient) that the planes be perpendicular is

$$A_1A_2+B_1B_2+C_1C_2=0$$

Example 1. The planes

$$3x-2y-2z+7=0 \quad \text{and} \quad 2x+2y+z+4=0$$

are perpendicular since $3\cdot 2+(-2)\cdot 2+(-2)\cdot 1=0$.

Example 2. The planes

$$3x-2y=0 \quad (A_1=3,\ B_1=-2,\ C_1=0)$$

and

$$z=4 \quad (A_2=0,\ B_2=0,\ C_2=1)$$

are perpendicular.

127. Angle Between Two Planes

The two planes

$$A_1x+B_1y+C_1z+D_1=0 \tag{1}$$

and

$$A_2x+B_2y+C_2z+D_2=0 \tag{2}$$

form four dihedral angles that are pairwise equal. One of them is equal to the angle between the normal vectors $N_1\{A_1, B_1, C_1\}$ and $N_2\{A_2, B_2, C_2\}$. Denoting any one of the dihedral angles by φ, we have

$$\cos\varphi=\pm\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}} \tag{3}$$

Choosing the upper sign, we get $\cos(\widehat{N_1, N_2})$, choosing the lower sign, we get $\cos[180^\circ-(\widehat{N_1, N_2})]$.

Example. The angle between the planes

$$x-y+\sqrt{2}z+2=0 \quad \text{and} \quad x+y+\sqrt{2}z-3=0$$

is determined from the equality

$$\cos\varphi = \pm\frac{1\cdot 1+(-1)\cdot 1+\sqrt{2}\cdot\sqrt{2}}{\sqrt{1+1+(\sqrt{2})^2}\sqrt{1+1+(\sqrt{2})^2}} = \pm\frac{1}{2}$$

We get $\varphi=60°$ or $\varphi=120°$.

If the vector N_1 forms with the x-, y-, z-axes the angles α_1, β_1, γ_1, and the vector N_2, the angles α_2, β_2, γ_2, then

$$\cos\varphi = \pm(\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2) \quad (4)$$

This follows from (3) and formulas (1) to (3), Sec. 101.

128. A Plane Passing Through a Given Point Parallel to a Given Plane

A plane passing through a point $M_1(x_1, y_1, z_1)$ parallel to a plane $Ax+By+Cz+D=0$ is given by the equation

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

This follows from Secs. 123 and 125.

Example. A plane passing through a point $(2, -1, 6)$ parallel to the plane $x+y-2z+5=0$ is given by the equation $(x-2)+(y+1)-2(z-6)=0$, i. e. $x+y-2z+11=0$.

129. A Plane Passing Through Three Points

If the points $M_0(x_0, y_0, z_0)$, $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ do not lie on a straight line, then the plane passing through them (Fig. 164) is given by the equation

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ x_2-x_0 & y_2-y_0 & z_2-z_0 \end{vmatrix} = 0 \quad (1)$$

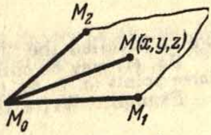


Fig. 164

It expresses coplanarity of the vectors $\overrightarrow{M_0M}$, $\overrightarrow{M_0M_1}$, $\overrightarrow{M_0M_2}$ (see Secs. 120 and 99).

Example. The points $M_0(1, 2, 3)$, $M_1(2, 1, 2)$, $M_2(3, 3, 1)$ do not lie on one straight line since the vectors $\overrightarrow{M_0M_1}\{1, -1, -1\}$ and $\overrightarrow{M_0M_2}\{2, 1, -2\}$ are not collinear. The plane $M_0M_1M_2$ is defined by the equation

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & -1 & -1 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

i. e.

$$x+z-4=0$$

Note. If the points M_0, M_1, M_2 lie on one straight line, then Eq. (1) becomes an identity.

130. Intercepts on the Axes

If the plane $Ax+By+Cz+D=0$ is not parallel to the x-axis (i. e. if $A \neq 0$; Sec. 124), then it intercepts on this axis a segment $a=-\frac{D}{A}$. Similarly, the intercepts on the y-axis and on the z-axis will be $b=-\frac{D}{B}$ (if $B \neq 0$) and $c=-\frac{D}{C}$ (if $C \neq 0$) (cf. Sec. 32).

Example. The plane $3x+5y-4z-3=0$ intercepts on the axes the line segments $a=\frac{3}{3}=1$, $b=\frac{3}{5}$, $c=-\frac{3}{4}$.

131. Intercept Form of the Equation of a Plane

If a plane intercepts on the axes the (nonzero) segments a, b, c, then it may be represented by the equation

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (1)$$

which is called the "intercept form of the equation of a plane".

Eq. (1) may be obtained as an equation of a plane passing through three points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ (see Sec. 129).

Example. Write the equation of the plane

$$3x-6y+2z-12=0$$

in the intercept form.

We find (Sec. 130) $a=4$, $b=-2$, $c=6$. The intercept form of the equation is

$$\frac{x}{4}+\frac{y}{-2}+\frac{z}{6}=1$$

Note 1. A plane passing through the coordinate origin cannot be represented by an intercept-form equation (cf. Sec. 33, Note 1).

Note 2. A plane parallel to the x-axis but not parallel to the other two axes may be represented by the equation $\frac{y}{b}+\frac{z}{c}=1$, where b and c are the y-intercept and z-intercept, respectively. A plane parallel to the x-axis and y-axis may be given by the equation $\frac{z}{c}=1$. Planes parallel to either one or two of the other axes may be represented similarly (cf. Sec. 33, Note 2).

132. A Plane Passing Through Two Points Perpendicular to a Given Plane

A plane P (Fig. 165) which passes through two points $M_0(x_0, y_0, z_0)$ and $M_1(x_1, y_1, z_1)$ and is perpendicular to a plane Q specified by the equation $Ax+By+Cz+D=0$ is represented by the equation

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ A & B & C \end{vmatrix}=0 \qquad (1)$$

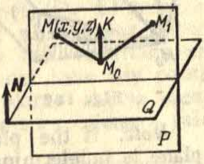


Fig. 165

It expresses (Sec. 120) coplanarity of the vectors $\overrightarrow{M_0M}$, $\overrightarrow{M_0M_1}$ and $N\{A, B, C\} = \overrightarrow{M_0K}$.

Example. A plane passing through the two points $M_0(1, 2, 3)$ and $M_1(2, 1, 1)$ perpendicular to the plane $3x+4y+z-6=0$ is represented by the equation

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2-1 & 1-2 & 1-3 \\ 3 & 4 & 1 \end{vmatrix}=0$$

i. e. $x-y+z-2=0$.

Note. When the straight line M_0M_1 is perpendicular to plane Q, plane P is indeterminate. Accordingly, Eq. (1) becomes an identity.

133. A Plane Passing Through a Given Point Perpendicular to Two Planes

A plane P which passes through the point $M_0(x_0, y_0, z_0)$ and is perpendicular to two (nonparallel) planes Q_1, Q_2:

$$A_1x+B_1y+C_1z+D_1=0, \quad A_2x+B_2y+C_2z+D_2=0$$

is given by the equation

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix}=0 \tag{1}$$

It expresses (Fig. 166) coplanarity of the vectors

$$\overrightarrow{M_0M_1},\ N_1\{A_1,\ B_1,\ C_1\},\ N_2\{A_2,\ B_2,\ C_2\}$$ 1)

Example. A plane that passes through the point (1, 3, 2) and is perpendicular to the planes $x+2y+z-4=0$ and $2x+y+3z+5=0$ is given by the equation

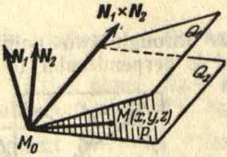


Fig. 166

$$\begin{vmatrix} x-1 & y-3 & z-2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}=0$$

or

$$5x-y-3z+4=0$$

Note. If the planes Q_1 and Q_2 are parallel, then the P plane is indeterminate and Eq. (1) becomes an identity.

134. The Point of Intersection of Three Planes

Three planes may not have a single point in common (if at least two of them are parallel and also if their straight lines of intersection are parallel), may have an infinity of common points (if they all pass through the same straight line) or may have only one common point. In the first case, the system of equations

$$A_1x+B_1y+C_1z+D_1=0,$$
$$A_2x+B_2y+C_2z+D_2=0,$$
$$A_3x+B_3y+C_3z+D_3=0$$

has no solutions, in the second case it has an infinity of solutions, and in the third, only one solution. The investigation is most conveniently carried out by means of determinants (Secs. 183, 190), but elementary algebra will suffice.

---

1) The vector product $N_1\times N_2$ (Fig. 166) serves as the normal vector to the P plane. Thus [Sec. 123, (1a)] the equation of the P plane is $(N_1\times N_2)(r-r_0)=0$, which again yields Eq. (1).

Example 1. The planes

$$7x-3y+z-6=0, \tag{1}$$
$$14x-6y+2z-5=0, \tag{2}$$
$$x+y-5z=0 \tag{3}$$

have no common points since the planes (1) and (2) are parallel (Sec. 125). The system of equations is not consistent [Eqs. (1) and (2) are contradictory].

Example 2. Investigate the following three planes for common points:

$$x+y+z=1, \tag{4}$$
$$x-2y-3z=5, \tag{5}$$
$$2x-y-2z=8 \tag{6}$$

We seek the solution of the system (4)-(6). Eliminating z from (4) and (5), we get $4x+y=8$; eliminating z from (4) and (6), we get $4x+y=10$. These two equations are inconsistent. Hence, the three planes do not have any points in common. Since there are no parallel planes among them, the three straight lines along which the planes intersect pairwise are parallel.

Example 3. Investigate the following planes for common points:

$$x+y+z=1, \quad x-2y-3z=5, \quad 2x-y-2z=6$$

Operating as in Example 2, we both times get $4x+y=8$, which is actually one equation, not two. It has an infinity of solutions. Hence, the three planes have an infinity of common points, i. e. they pass through one straight line.

Example 4. The planes

$$x-y+2=0, \quad x+2y-1=0, \quad x+y-z+2=0$$

have one common point $(-1, 1, 2)$ because the system of equations has the unique solution $x=-1$, $y=1$, $z=2$.

135. The Mutual Positions of a Plane and a Pair of Points

The mutual arrangement of points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and a plane

$$Ax+By+Cz+D=0 \tag{1}$$

may be determined by the following criteria (cf. Sec. 27):

(a) The points M_1 and M_2 lie to one side of the plane (1) when the numbers $Ax_1+By_1+Cz_1+D$ and $Ax_2+By_2+Cz_2+D$ have the same signs.

(b) M_1 and M_2 lie on different sides of the plane (1) when these numbers have opposite signs.

(c) One of the points M_1, M_2 (or both) lies on the plane if one of the numbers (or both) is equal to zero.

Example 1. The points (2, 3, 3) and (1, 2, −1) lie to one side of the plane $6x+3y+2z-6=0$ because the numbers $6\cdot2+3\cdot3+2\cdot3-6=21$ and $6\cdot1+3\cdot2+2(-1)-6=4$ are both positive.

Example 2. The origin (0, 0, 0) and the point (2, 1, 1) lie on different sides of the plane $5x+3y-2z-5=0$ since the numbers $5\cdot0+3\cdot0-2\cdot0-5=-5$ and $5\cdot2+3\cdot1-2\cdot1-5=6$ have opposite signs.

136. The Distance from a Point to a Plane

The distance d from a point $M_1(x_1, y_1, z_1)$ to a plane

$$Ax+By+Cz+D=0 \tag{1}$$

is equal (cf. Sec. 28) to the absolute value of the quantity

$$\delta=\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}} \tag{2}$$

i. e.

$$d=|\delta|=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}} \tag{3}$$

Example. Find the distance from the point (3, 9, 1) to the plane $x-2y+2z-3=0$.

Solution.

$$\delta=\frac{x_1-2y_1+2z_1-3}{\sqrt{1^2+(-2)^2+2^2}}=\frac{1\cdot3-2\cdot9+2\cdot1-3}{3}=-5\frac{1}{3},$$

$$d=|\delta|=5\frac{1}{3}$$

Note 1. The sign of δ indicates the mutual positions of the point M_1 and the origin O relative to the plane (1) (cf. Sec. 28, Note 1).

Note 2. Formula (3) may be derived analytically by reasoning as in Note 2 of Sec. 28. The equation of a straight line which passes through a point M_1 and is perpendicular to the plane (1) is conveniently taken in parametric form (see Secs. 153, 156).

137. The Polar Parameters (Coordinates) of a Plane [1]

The *polar distance* (or *radius vector*) of a plane UVW (Fig. 167) is the length p of the perpendicular OK drawn to the plane from the origin O. The polar distance is positive or zero.

---

[1] Cf. Sec. 29.

If the plane UVW does not pass through the origin, then for the positive direction on the perpendicular OK we take the direction of the vector $\overrightarrow{OK}$. But if UVW goes through the origin, then the positive direction on the perpendicular is taken in arbitrary fashion.

The *polar angles* of the plane UVW are the angles

$$\alpha=\angle XOK, \quad \beta=\angle YOK, \quad \gamma=\angle ZOK$$

between the positive direction of the straight line OK and the coordinate axes (these angles are considered to be positive and not to exceed 180°). The angles α, β, γ are connected (Sec. 101) by the relation

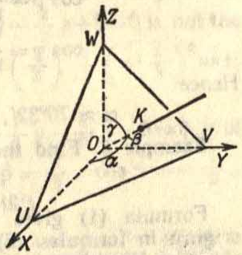


Fig. 167

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

The polar distance p and the polar angles α, β, γ are termed the *polar parameters* (or *polar coordinates*) of the plane UVW

If the plane UVW is given by the equation $Ax+By+Cz+D=0$, then its polar parameters are determined by the formulas

$$p=\frac{|D|}{\sqrt{A^2+B^2+C^2}}, \tag{1}$$

$$\left.\begin{aligned}\cos\alpha&=\mp\frac{A}{\sqrt{A^2+B^2+C^2}},\\ \cos\beta&=\mp\frac{B}{\sqrt{A^2+B^2+C^2}},\\ \cos\gamma&=\mp\frac{C}{\sqrt{A^2+B^2+C^2}}\end{aligned}\right\} \tag{2}$$

where the upper signs hold true when $D>0$, and the lower signs when $D<0$. But if $D=0$, then arbitrarily we take only the upper or only the lower signs.

Example 1. Find the polar parameters of the plane $x-2y+2z-3=0$ ($A=1$, $B=-2$, $C=2$, $D=-3$).

Solution. Formula (1) yields

$$p=\frac{|-3|}{\sqrt{1^2+(-2)^2+2^2}}=\frac{3}{3}=1$$

Formulas (2), where we have to take the lower signs (because $D=-3<0$) yield

$$\cos\alpha=\frac{1}{\sqrt{1^2+(-2)^2+2^2}}=\frac{1}{3},$$

$$\cos\beta=\frac{-2}{\sqrt{1^2+(-2)^2+2^2}}=-\frac{2}{3},$$

$$\cos\gamma=\frac{2}{\sqrt{1^2+(-2)^2+2^2}}=\frac{2}{3}$$

Hence

$$\alpha\approx 70^\circ 32', \quad \beta\approx 131^\circ 49', \quad \gamma\approx 48^\circ 11'$$

Example 2. Find the polar parameters of the plane

$$x-2y+2z=0$$

Formula (1) gives $p=0$ (the plane passes through the origin); in formulas (2) we can take either the upper signs alone or the lower signs alone. In the former case,

$$\cos\alpha=-\frac{1}{3}, \quad \cos\beta=+\frac{2}{3}, \quad \cos\gamma=-\frac{2}{3}$$

hence

$$\alpha\approx 109^\circ 28', \quad \beta\approx 48^\circ 11', \quad \gamma\approx 131^\circ 49'$$

in the latter case,

$$\alpha\approx 70^\circ 32', \quad \beta\approx 131^\circ 49', \quad \gamma\approx 48^\circ 11'$$

138. The Normal Equation of a Plane

A plane with polar distance p (Sec. 137) and polar angles α, β, γ ($\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$; Sec. 101) is given by the equation

$$x\cos\alpha+y\cos\beta+z\cos\gamma-p=0 \qquad (1)$$

This is the *normal form of the equation of a plane*.

Example 1. Set up the normal form of the equation of a plane in which the polar distance is $\frac{1}{\sqrt{3}}$ and all the polar angles are obtuse and equal.

Solution. For $\alpha=\beta=\gamma$ the condition $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ yields $\cos\alpha=\cos\beta=\cos\gamma=\pm\frac{1}{\sqrt{3}}$ and since the angles α, β, γ are obtuse, we have to take the minus sign. The desired equation is $-\frac{1}{\sqrt{3}}x-\frac{1}{\sqrt{3}}y-\frac{1}{\sqrt{3}}z-\frac{1}{\sqrt{3}}=0$.

Note. The same plane can be represented by the equation

$$x+y+z+1=0$$

(both members of the preceding equation have been multiplied by $-\sqrt{3}$), but this is not the normal form of the equation because the coefficients of the coordinates are not cosines of the polar angles (the sum of their squares is not equal to unity) and, what is more, the constant term is positive.

Example 2. The equation $\frac{1}{3}x+\frac{2}{3}y-\frac{2}{3}z+5=0$ is not the normal form, since even though $\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(-\frac{2}{3}\right)^2=1$, the constant term is positive.

Example 3. The equation $-\frac{1}{3}x+\frac{2}{3}y-\frac{2}{3}z-5=0$ is of the normal form; $\cos\alpha=-\frac{1}{3}$, $\cos\beta=\frac{2}{3}$, $\cos\gamma=-\frac{2}{3}$, $p=5$ ($\alpha\approx 109°28'$, $\beta\approx 48°11'$, $\gamma\approx 131°49'$).

Derivation of Eq. (1). The plane under consideration (UVW in Fig. 167) goes through the point K ($p\cos\alpha$, $p\cos\beta$, $p\cos\gamma$) perpendicular to the vector $\overrightarrow{OK}$. Instead of $\overrightarrow{OK}$ we can take the vector $\boldsymbol{a}$ in the same direction with length equal to the scale unit. The coordinates of the vector $\boldsymbol{a}$ are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ (Sec. 101). Applying Eq. (1), Sec. 101, we get the normal form of Eq. (1).

139. Reducing the Equation of a Plane to the Normal Form

To find the normal form of the equation of a plane specified by the equation $Ax+By+Cz+D=0$, it is sufficient to divide both members of the given equation by $\mp\sqrt{A^2+B^2+C^2}$, the upper sign is taken when $D>0$, the lower when $D<0$; if $D=0$, any sign may be taken. This yields the equation

$$\mp\frac{A}{\sqrt{A^2+B^2+C^2}}x\mp\frac{B}{\sqrt{A^2+B^2+C^2}}y\mp\frac{C}{\sqrt{A^2+B^2+C^2}}z-\frac{|D|}{\sqrt{A^2+B^2+C^2}}=0$$

It is in the normal form because the coefficients of x, y, z, by virtue of (2), Sec. 137, are equal to $\cos\alpha$, $\cos\beta$, $\cos\gamma$, respectively, and the constant term is equal to $-p$ by virtue of (1), Sec. 137.

Example 1. Reduce the following equation to the normal form:

$$x-2y+2z-6=0 \tag{1}$$

Divide both members of the equation by $+\sqrt{1^2+(-2)^2+2^2}=3$ (the sign before the radical is plus since the constant term, —6, is negative). We get

$$\frac{1}{3}x-\frac{2}{3}y+\frac{2}{3}z-2=0$$

Hence, $p=2$, $\cos\alpha=\frac{1}{3}$, $\cos\beta=-\frac{2}{3}$, $\cos\gamma=\frac{2}{3}$ ($\alpha\approx70°32'$, $\beta\approx131°49'$, $\gamma\approx48°11'$).

Example 2. Reduce the following equation to the normal form:

$$x-2y+2z+6=0 \tag{2}$$

The constant term is positive. We therefore divide by $-\sqrt{1^2+(-2)^2+2^2}=-3$ and get

$$-\frac{1}{3}x+\frac{2}{3}y-\frac{2}{3}z-2=0$$

Consequently, $p=2$, $\cos\alpha=-\frac{1}{3}$, $\cos\beta=\frac{2}{3}$, $\cos\gamma=-\frac{2}{3}$ ($\alpha\approx109°28'$, $\beta\approx48°11'$, $\gamma\approx131°49'$).

Example 3. Reduce to normal form the equation

$$x-2y+2z=0$$

Since $D=0$ (the plane goes through the origin), we can divide either by $+3$ or by -3. This yields $\frac{1}{3}x-\frac{2}{3}y+\frac{2}{3}z=0$ or $-\frac{1}{3}x+\frac{2}{3}y-\frac{2}{3}z=0$. In both cases, $p=0$. The quantities α, β, γ in the first case are the same as in Example 1, in the second case, the same as in Example 2.

Note. If in the equation $Ax+By+Cz+D=0$ the constant term is negative and $A^2+B^2+C^2=1$, then the equation is in the normal form (Sec. 138, Example 3) and it does not need to be transformed.

140. Equations of a Straight Line in Space

Any straight line UV (Fig. 168) may be represented by a system of two equations:

$$A_1x+B_1y+C_1z+D_1=0, \tag{1}$$
$$A_2x+B_2y+C_2z+D_2=0 \tag{2}$$

which represent (if they are considered *separately*) any two *(distinct)* planes P_1 and P_2 passing through UV. Eqs. (1) and (2) (taken *together*) are termed the *equations of the straight line UV*.

Note. The expression "the straight line UV is represented by the system (1)-(2)" means that (1) the coordinates x, y, z of any point M of the line UV satisfy both equations (1) and (2); (2) the coordinates of any point not lying on UV do not simultaneously satisfy both Eqs. (1), (2), though they may satisfy one of them.

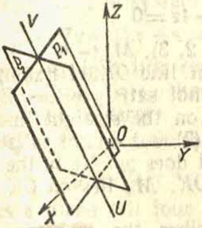


Fig. 168

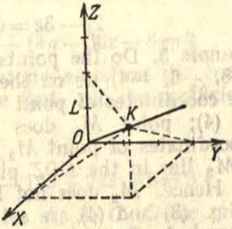


Fig. 169

Example 1. Write the equations of the straight line OK (Fig. 169) that passes through the origin O and the point $K(4, 3, 2)$.

Solution. The straight line OK is the intersection of the planes KOZ and KOX. Taking some point on the z-axis OZ, say $L(0, 0, 1)$, form the equation of the plane KOZ (the plane passing through three points O, K, L; Sec. 129). This gives us

$$\begin{vmatrix} x & y & z \\ 4 & 3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 0, \text{ that is, } 3x - 4y = 0 \qquad (3)$$

In the same way we find the equation

$$2y - 3z = 0 \qquad (4)$$

of the plane KOX. The straight line OK is given by the system of equations (3)-(4).

Indeed, any point M of line OK lies both in the KOZ plane and in the KOX plane; hence, its coordinates satisfy both Eqs. (3) and (4) at the same time. On the other hand, the point N, which does not lie on OK, cannot belong to both planes KOZ and KOX at the same time; hence, its coordinates cannot satisfy both Eqs. (3) and (4) at the same time.

Example 2. The straight line OK of Example 1 may also be given by the system of equations

$$\begin{cases} 3x-4y=0, & (3) \\ 2x-4z=0 & (5) \end{cases}$$

The first describes the KOZ plane, the second, the KOY plane.

The same line OK can be represented by the system

$$2y-3z=0, \quad 2x-4z=0$$

Example 3. Do the points $M_1(2, 2, 3)$, $M_2(-4, -3, -3)$, $M_3(-8, -6, -4)$ lie on the straight line OK of Example 1?

The coordinates of point M_1 do not satisfy either Eq. (3) or Eq. (4); point M_1 does not lie on the straight line UV. The coordinates of point M_2 satisfy (3) but do not satisfy (4); point M_2 lies in the KOZ plane, but does not lie in the KOX plane. Hence, M_2 does not lie on OK. M_3 lies on OK since both Eqs. (3) and (4) are satisfied.

Example 4. Equation $z=0$ describes the xy-plane. The equation $x+y-1=0$ describes the plane P parallel to the z-axis (Sec. 124, Example). The straight line along which the planes XOY and P intersect (KL in Fig. 163) is represented by the system

$$x+y-1=0, \quad z=0$$

141. Condition Under Which Two First-Degree Equations Represent a Straight Line

The system

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, & (1) \\ A_2x+B_2y+C_2z+D_2=0 & (2) \end{cases}$$

represents a straight line if the coefficients A_1, B_1, C_1 are not proportional to the coefficients A_2, B_2, C_2 [in this case the planes (1) and (2) are not parallel (Sec. 125)].

If the coefficients A_1, B_1, C_1 are proportional to the coefficients A_2, B_2, C_2, but the constant terms do not obey the same proportion

$$A_2:A_1=B_2:B_1=C_2:C_1 \neq D_2:D_1$$

then the system is not consistent and does not present any geometric image [planes (1) and (2) are parallel and noncoincident].

If all four quantities A_1, B_1, C_1, D_1 are proportional to the quantities A_2, B_2, C_2, D_2:

$$A_2:A_1=B_2:B_1=C_2:C_1=D_2:D_1$$

then one of the equations (1), (2) is a consequence of the other and the system describes a plane [the planes (1) and (2) arc coincident].

Example 1. The system

$$2x-7y+12z-4=0, \quad 4x-14y+36z-8=0$$

describes a straight line (in the second equation the coefficients A and B are twice those in the first, and the coefficient C is three times as large).

Example 2. The system

$$2x-7y+12z-4=0, \quad 4x-14y+24z-8=0$$

describes a plane (all four quantities A, B, C, D are proportional).

Example 3. The system

$$2x-7y+12z-4=0, \quad 4x-14y+24z-12=0$$

does not represent any geometric image (the quantities A, B, C are proportional and D does not obey that proportion; the system is inconsistent).

142. The Intersection of a Straight Line and a Plane

The straight line L

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, & (1) \\ A_2x+B_2y+C_2z+D_2=0 & (2) \end{cases}$$

and the plane P

$$Ax+By+Cz+D=0 \qquad (3)$$

may not have a single common point (if $L \parallel P$), may have an infinity of common points (if L lies on P) or may only have one common point. The problem reduces[1] to seeking common points of three planes (1), (2), (3), (see Sec. 134).

Example 1. The straight line

$$x+y+z-1=0, \quad x-2y-3z-5=0$$

---

[1] The computations are simplified when the equations of the straight line are taken in parametric form (Sec. 152 and Note in Sec. 153).

does not have common points with the plane

$$2x-y-2z-8=0$$

(they are parallel) (see Example 2, Sec. 134).

Example 2. The straight line

$$x-2y-3z-5=0, \quad 2x-y-2z=6$$

lies in the plane $x+y+z=1$ (see Example 3, Sec. 134).

Example 3. The straight line $x+y-z+2=0$, $x-y+2=0$ intersects the plane $x+2y-1=0$ in the point $(-1, 1, 2)$ (see Example 4, Sec. 134).

Example 4. Determine the coordinates of some point on the straight line L:

$$\begin{cases} 2x-3y-z+3=0, \\ 5x-y+z-8=0 \end{cases}$$

Assign some value, say $x=3$, to the x-coordinate. We then have the system $-3y-z+9=0$, $-y+z+7=0$. Solving it, we find $y=4$, $z=-3$. The point $(3, 4, -3)$ lies on the straight line L (at its intersection with the plane $x=3$ parallel to YOZ). In the same way, taking $x=0$, we find the point $\left(0, -\frac{5}{4}, \frac{27}{4}\right)$ at the intersection of L and the plane YOZ, etc. It is also possible to assign various values to the y- or z-coordinate.

Example 5. Determine the coordinates of some point on the straight line L:

$$\begin{cases} 5x-3y+2z-4=0, \\ 8x-6y+4z-3=0 \end{cases}$$

Unlike the preceding example, arbitrary values cannot be assigned to the x-coordinate. For instance, for $x=0$ we get the inconsistent system $-3y+2z-4=0$, $-6y+4z-3=0$. The straight line L is parallel to the plane ZOY. We can assign arbitrary values to the y- or z-coordinate; for instance, putting $z=0$, we get the point $\left(\frac{5}{2}, \frac{17}{6}, 0\right)$. For x we will always obtain the same value $x=\frac{5}{2}$ so that the straight line L lies in the plane $x=\frac{5}{2}$ parallel to ZOY.

143. The Direction Vector

A. Any (nonzero) vector $a\{l, m, n\}$ lying on a straight line UV (or parallel to it) is called the *direction vector* of that line. The coordinates l, m, n of the direction vector are called the *direction numbers* of the line.

Note. By multiplying the direction numbers l, m, n by one and the same number k (not equal to zero), we get numbers lk, mk, nk, which will also be direction numbers (these are the coordinates of the vector ak, which is collinear with a).

B. For the direction vector of the straight line UV

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, & (1)\\ A_2x+B_2y+C_2z+D_2=0 & (2)\end{cases}$$

we can take the vector product $N_1\times N_2$, where $N_1=\{A_1, B_1, C_1\}$ and $N_2=\{A_2, B_2, C_2\}$ are normal vectors to the planes P_1 and P_2 (Fig. 170) described by Eqs. (1) and (2). Indeed the straight line UV is perpendicular to the normal vectors N_1, N_2.

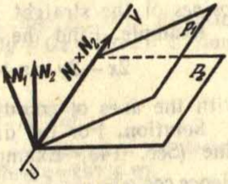


Fig. 170

Example. Find the direction numbers of the straight line

$$2x-2y-z+8=0, \quad x+2y-2z+1=0$$

Solution. We have $N_1=\{2, -2, -1\}$, $N_2=\{1, 2, -2\}$. Taking $a=N_1\times N_2$ for the direction vector of the given straight line, we find

$$a=\left\{\begin{vmatrix}-2 & -1\\ 2 & -2\end{vmatrix}, \begin{vmatrix}-1 & 2\\ -2 & 1\end{vmatrix}, \begin{vmatrix}2 & -2\\ 1 & 2\end{vmatrix}\right\}=\{6, 3, 6\}$$

The direction numbers will be $l=6$, $m=3$, $n=6$.

Note. Multiplying these numbers by $\frac{1}{3}$, we find the direction numbers $l'=2$, $m'=1$, $n'=2$. One can also take the numbers -2, -1, -2 and so forth for the direction numbers.

144. Angles Between a Straight Line and the Coordinate Axes

The angles α, β, γ formed by a straight line L (in one of its two directions) with the coordinate axes are found from the relations

$$\cos\alpha=\frac{l}{\sqrt{l^2+m^2+n^2}},$$

$$\cos\beta = \frac{m}{\sqrt{l^2+m^2+n^2}},$$

$$\cos\gamma = \frac{n}{\sqrt{l^2+m^2+n^2}}$$

where l, m, n are the direction numbers of L.

This is a consequence of Sec. 101.

The quantities $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the *direction cosines* of the straight line L.

Example. Find the angles formed by the straight line

$$2x-2y-z+8=0, \quad x+2y-2z+1=0$$

with the axes of coordinates.

Solution. For the direction numbers of the given straight line (Sec. 143, Example) we can take $l=2$, $m=1$, $n=2$. Hence $\cos\alpha = \frac{2}{\sqrt{2^2+1^2+2^2}} = \frac{2}{3}$, $\cos\beta = \frac{1}{3}$, $\cos\gamma = \frac{2}{3}$; whence $\alpha \approx 48°11'$, $\beta \approx 70°32'$, $\gamma \approx 48°11'$.

145. Angle Between Two Straight Lines

The angle φ between the straight lines L and L' (more exactly, one of the angles between them) is found from the formula

$$\cos\varphi = \frac{ll'+mm'+nn'}{\sqrt{l^2+m^2+n^2}\,\sqrt{l'^2+m'^2+n'^2}} \tag{1}$$

where l, m, n and l', m', n' are the direction numbers of the straight lines L and L', or from the formula

$$\cos\varphi = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' \tag{2}$$

This follows from Sec. 109.

Example. Find the angle between the straight lines

$$\begin{cases} 2x-2y-z+8=0, & 4x+y+3z-21=0, \\ x+2y-2z+1=0, & 2x+2y-3z+15=0 \end{cases}$$

Solution. The direction numbers of the first line (Sec. 143, Example) are $l=2$, $m=1$, $n=2$. If for the direction vector of the second line we take the vector product $\{4, 1, 3\} \times \{2, 2, -3\}$, then the direction numbers will be -9, 18, 6. Multiplying them by $\frac{1}{3}$ (to obtain smaller numbers, Sec. 143, Note), we get $l=-3$, $m=6$, $n=2$. We thus have

$$\cos\varphi=\frac{2\cdot(-3)+1\cdot6+2\cdot2}{\sqrt{2^2+1^2+2^2}\sqrt{(-3)^2+6^2+2^2}}=\frac{4}{21}$$

whence $\varphi\approx79^\circ01'$.

146. Angle Between a Straight Line and a Plane

The angle ψ between the straight line L (with direction numbers l, m, n) and the plane $Ax+By+Cz+D=0$ is found from the formula

$$\sin\psi=\frac{|Al+Bm+Cn|}{\sqrt{A^2+B^2+C^2}\sqrt{l^2+m^2+n^2}}$$

This follows from Sec. 145 (if φ is the angle between the straight line L and the normal vector $\{A, B, C\}$, then $\varphi=90^\circ\pm\psi$).

Example. Find the angle between the straight line

$$3x-2y=24,\quad 3x-z=-4$$

and the plane $6x+15y-10z+31=0$. We have $l=2$, $m=3$, $n=6$ (Sec. 143) and find

$$\sin\varphi=\frac{|6\cdot2+15\cdot3+(-10)\cdot6|}{\sqrt{6^2+15^2+(-10)^2}\sqrt{2^2+3^2+6^2}}=\frac{3}{133}$$

whence $\varphi\approx1^\circ18'$.

147. Conditions of Parallelism and Perpendicularity of a Straight Line and a Plane

The condition for a straight line with direction numbers l, m, n to be *parallel* to a plane $Ax+By+Cz+D=0$ is

$$Al+Bm+Cn=0 \tag{1}$$

It expresses the *perpendicularity* of the straight line and the normal vector $\{A, B, C\}$.

The condition for *perpendicularity* of a straight line and a plane (same notation) is

$$\frac{l}{A}=\frac{m}{B}=\frac{n}{C} \tag{2}$$

This expresses the *parallelism* of a straight line and a normal vector.

148. A Pencil of Planes [1]

The collection of planes passing through one and the same straight line UV is called a *pencil of planes*. The line UV is the *axis* of the pencil.

If we know the equations of two distinct planes P_1 and P_2

$$A_1x+B_1y+C_1z+D_1=0, \tag{1}$$
$$A_2x+B_2y+C_2z+D_2=0 \tag{2}$$

belonging to a pencil (i.e. the equations of the axis of the pencil: see Sec. 140), then every plane of the pencil may be represented by an equation of the form

$$m_1(A_1x+B_1y+C_1z+D_1)+m_2(A_2x+B_2y+C_2z+D_2)=0 \tag{3}$$

Conversely, for any values of m_1, m_2 (not all zero simultaneously) Eq. (3) represents a plane belonging to the pencil with axis UV. [2] In particular, for $m_1=0$ we get the plane P_2 and for $m_2=0$, the plane P_1. Eq. (3) is called the *equation of the pencil of planes*. [3]

When $m_1 \neq 0$, we can divide Eq. (3) by m_1. Denoting $m_2:m_1$ in terms of λ we get the equation

$$A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0 \tag{4}$$

Here, all possible values are given to the single letter λ; but from (4) we cannot obtain the equation of the plane P_2.

Example 1. Let there be given the equations

$$5x-3y=0, \tag{5}$$
$$3z-4x=0 \tag{6}$$

of two planes of a pencil, i.e. the equations of the axis of the pencil. The equation of the pencil is

$$m_1(5x-3y)+m_2(3z-4x)=0 \tag{7}$$

For example, taking $m_1=2$, $m_2=-3$ we will have

$$2(5x-3y)+(-3)(3z-4x)=0 \tag{8}$$

Eq. (8) or

$$22x-6y-9z=0 \tag{8a}$$

represents one of the planes of the pencil.

---

[1] Cf. Sec. 24.

[2] See below: explanation of Example 1.

[3] If planes (1) and (2) are parallel (but not coincident), then Eq. (3) represents (for all possible values of m_1, m_2) all the planes parallel to the two given planes (parallel pencil of planes).

Explanation. On the straight line UV take an arbitrary point $M(x, y, z)$. Its coordinates x, y, z satisfy the Eqs. (5) and (6) and, hence, Eq. (8). This means that plane (8) passes through any point M of the line UV, i.e. it belongs to the pencil.

Example 2. Find the equation of a plane passing through the straight line UV of Example 1 and through the point (1, 0, 0).

Solution. The desired plane is given by an equation of the form (7). This equation must be satisfied for $x=1$, $y=0$, $z=0$. Substituting these values into (7), we get $5m_1-4m_2=0$, or $m_1:m_2=4:5$. We get the equation

$$4(5x-3y)+5(3z-4x)=0$$

or

$$5z-4y=0$$

Example 3. Find the equations of the projection of the straight line L:

$$\left.\begin{array}{r} 2x+3y+4z+5=0, \\ x-6y+3z-7=0 \end{array}\right\} \qquad (9)$$

on the plane P

$$2x+2y+z+15=0 \qquad (10)$$

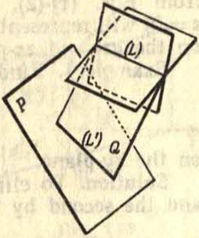


Fig. 171

Solution. The desired projection L' (Fig. 171) is a straight line along which plane P is cut by plane Q (drawn through L perpendicular to P). Plane Q belongs to the pencil with axis L and is given by an equation of the form

$$(2x+3y+4z+5)+\lambda(x-6y+3z-7)=0 \qquad (11)$$

In order to find λ give (11) in the form

$$(2+\lambda)x+(3-6\lambda)y+(4+3\lambda)z+5-7\lambda=0 \qquad (11a)$$

and write the condition of perpendicularity of the planes (10) and (11a):

$$2(2+\lambda)+2(3-6\lambda)+1(4+3\lambda)=0$$

From this we have $\lambda=2$. Putting it into (11a), we obtain the equation of the plane Q. The sought-for projection is given by the equations

$$\left\{\begin{array}{l} 4x-9y+10z-9=0, \\ 2x+2y+z+15=0 \end{array}\right.$$

149. Projections of a Straight Line on the Coordinate Planes

Let a straight line be represented by the equations

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, & (1)\\ A_2x+B_2y+C_2z+D_2=0 & (2)\end{cases}$$

where C_1 and C_2 are not simultaneously zero (case $C_1=C_2=0$ is considered below in Example 3). To find the projection of the straight line on the xy-plane, it suffices to eliminate z from Eqs. (1)-(2). The resulting equation (together with $z=0$) will represent the desired projection.[1] The projections on the yz- and zx-planes are found in similar fashion.

Example 1. Find the projection of the straight line L

$$\begin{cases} 2x+4y-3z-12=0, & (3)\\ x-2y+4z-10=0 & (4)\end{cases}$$

on the xy-plane.

Solution. To eliminate z, multiply the first equation by 4 and the second by 3 and add. This yields

$$4(2x+4y-3z-12)+3(x-2y+4z-10)=0 \qquad (5)$$

or

$$11x+10y-78=0 \qquad (6)$$

This equation together with the equation

$$z=0 \qquad (7)$$

is the projection L' of the straight line L on the xy-plane.

Explanation. Plane (5) passes through the straight line L (Sec. 148). On the other hand, as will be seen from (6) (which does not contain z), this plane (Sec. 124, Item 2) is perpendicular to the xy-plane. Hence, the straight line along which plane (6) intersects plane (7) is the projection of L on the plane (7) (cf. Sec. 148, Example 3).

Example 2. The projection of the straight line L

$$\begin{cases} 3x-5y+4z-12=0, & (8)\\ 2x-5y-4=0 & (9)\end{cases}$$

on the plane $z=0$ is represented (in the plane coordinate system XOY) by Eq. (9). There is no need to eliminate the z-coordinate since it is already absent from Eq. (9). The plane (9) is perpendicular to the xy-plane; it projects the straight line L on XOY.

---

[1] See Example 1, Explanation.

Example 3. Find the projections of the straight line L

$$\begin{cases} 2x-3y=0, & (10)\\ x+y-4=0 & (11)\end{cases}$$

on the coordinate planes.

Solution. In both equations z is absent, so that both planes P_1 and P_2 (Fig. 172) are perpendicular to the xy-plane. The straight line L is perpendicular to XOY and is projected on the xy-plane in the point N with z-coordinate $z_N=0$. From the system (10)-(11) we find $x_N=\frac{12}{5}$, $y_N=\frac{8}{5}$.

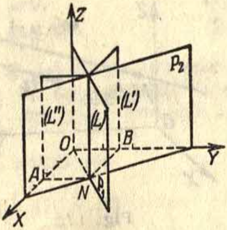


Fig. 172

The equation of the projection L' on the yz-plane may be found in the usual way by eliminating x from (10) and (11). We get $y=\frac{8}{5}$, which is the same equality found above for y_N (from the figure it is evident that the straight line L' is at a distance OB from OZ, equal to $y_N=AN$). The equation of the projection L'' on the xz-plane is $x=\frac{12}{5}$.

150. Symmetric Form of the Equation of a Straight Line

The straight line L passing through a point $M_0(x_0, y_0, z_0)$ and having the direction vector $\boldsymbol{a}\{l, m, n\}$ (Sec. 143) is given by the equations

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n} \qquad (1)$$

which express the collinearity of the vectors $\boldsymbol{a}\{l, m, n\}$ and $\overrightarrow{M_0M}\{x-x_0, y-y_0, z-z_0\}$ (Fig. 173). They are called the *symmetric (standard)* form of the equation of a straight line.

Note 1. Since for the point M_0 we can take any point on L, and the direction vector $\boldsymbol{a}$ may be replaced by a direction vector $k\boldsymbol{a}$ (Sec. 143), an arbitrary value can be assigned separately to each of the quantities x_0, y_0, z_0, l, m, n.

Example 1. Write the symmetric equations of a straight line passing through the points $A(5, -3, 2)$ and $B(3, 1, -2)$ For M_0 we can take the point A, for the vector $\boldsymbol{a}$ we can take $\overrightarrow{AB}=\{-2, 4, -4\}$. The symmetric equations will then be

$$\frac{x-5}{-2}=\frac{y+3}{4}=\frac{z-2}{-4} \tag{2}$$

But if we take B for M_0 and the vector $-\frac{1}{2}\overrightarrow{AB}=\{1, -2, 2\}$ for $\boldsymbol{a}$, then the symmetric equations will be

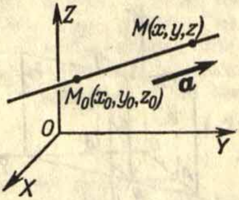


Fig. 173

$$\frac{x-3}{1}=\frac{y-1}{-2}=\frac{z+2}{2} \tag{3}$$

Note 2. Of the three equations

$$\frac{x-5}{-2}=\frac{y+3}{4}, \quad \frac{x-5}{-2}=\frac{z-2}{-4}, \quad \frac{y+3}{4}=\frac{z-2}{-4} \tag{4}$$

contained in (2), *only two* (no matter which) are independent, while the third is a consequence of them; for instance, subtracting the second from the first, we get the third. Each of the Eqs. (4) represents a plane passing through the straight line AB perpendicular to one of the coordinate planes. At the same time, it represents the projection of the straight line AB on the respective coordinate plane (Sec. 149).

Example 2. The symmetric equations of the straight line passing through the points $M_0(5, 0, 1)$, $M_1(5, 6, 5)$ will be

$$\frac{x-5}{0}=\frac{y-0}{6}=\frac{z-1}{4} \tag{5}$$

The expression $\frac{x-5}{0}$ is conventional, signifying (Sec. 102, Note) that $x-5=0$, so that in place of (5) we have to write

$$x=5, \quad \frac{y}{6}=\frac{z-1}{4} \tag{6}$$

The straight line M_0M_1 is perpendicular to the x-axis (since $l=0$).

Example 3. The symmetric equations of the straight line passing through the points $A(2, 4, 3)$ and $B(2, 4, 5)$ will be

$$\frac{x-2}{0}=\frac{y-4}{0}=\frac{z-3}{2}$$

This notation means that $x=2$ and $y=4$.

The quantity z assumes various (any) values for distinct points of the straight line AB. AB is parallel to the z-axis (since $l=m=0$).

151. Reducing the Equations of a Straight Line to Symmetric Form

In order to reduce the straight-line equations

$$A_1x+B_1y+C_1z+D_1=0, \tag{1}$$
$$A_2x+B_2y+C_2z+D_2=0 \tag{2}$$

to symmetric form (Sec. 150), one has to determine the coordinates x_0, y_0, z_0 of some point lying on the straight line (Examples 4 and 5, Sec. 142) and the direction numbers l, m, n (Sec. 143).

Example 1. Reduce the straight-line equations

$$2x-3y-z+3=0, \quad 5x-y+z-8=0$$

to symmetric form.

Solution. As in Sec. 142 (Example 4) we find on the given straight line the point $M_0(3, 4, -3)$, $x_0=3$, $y_0=4$, $z_0=-3$. Computing the direction numbers

$$l=\begin{vmatrix}-3 & -1\\ -1 & 1\end{vmatrix}=-4; \quad m=\begin{vmatrix}-1 & 2\\ 1 & 5\end{vmatrix}=-7,$$
$$n=\begin{vmatrix}2 & -3\\ 5 & -1\end{vmatrix}=13$$

we get the equations in symmetric form

$$\frac{x-3}{-4}=\frac{y-4}{-7}=\frac{z+3}{13}$$

Example 2. Reduce to symmetric form the equations

$$x+2y-3z-2=0, \quad -3x+4y-6z+21=0$$

Assign some value to the y-coordinate or z-coordinate (an arbitrary value cannot be assigned to the x-coordinate; cf. Sec. 142, Example 5); for example, put $y=0$. This gives the point $M_0(5, 0, 1)$. The direction numbers will be $l=0$, $m=15$, $n=10$ or $\left(\text{multiplying by } \frac{1}{5}\right)$ $l=0$, $m=3$, $n=2$.

The symmetric equations are

$$\frac{x-5}{0}=\frac{y}{3}=\frac{z-1}{2}$$

(cf. Sec. 150, Example 2).

Example 3. The same for the straight line

$$x+y-6=0, \quad x-y+2=0 \tag{3}$$

The values x_0 and y_0 are fully determined by equations (3): $x_0=2$, $y_0=4$. To the z_0-coordinate we can assign any value, say $z_0=3$. We then find the direction numbers $l=0$, $m=0$, $n=2$. This yields the symmetric form of the equation (cf. Sec. 150, Example 3):

$$\frac{x-2}{0}=\frac{y-4}{0}=\frac{z-3}{2}$$

152. Parametric Equations of a Straight Line

Each of the ratios $\frac{x-x_0}{l}$, $\frac{y-y_0}{m}$, $\frac{z-z_0}{n}$ (Sec. 150) is equal to the quotient (Sec. 90) obtained by dividing the vector

$$\overrightarrow{M_0M}\left\{x-x_0,\ y-y_0,\ z-z_0\right\}$$

by the (collinear) vector $\boldsymbol{a}\left\{l, m, n\right\}$. Denote this quotient by t. Then

$$\left.\begin{aligned} x&=x_0+lt,\\ y&=y_0+mt,\\ z&=z_0+nt \end{aligned}\right\} \tag{1}$$

These are the *parametric equations of a straight line*. When the quantity t (*parameter*) takes on various values, the point $M(x, y, z)$ moves along a straight line. When $t=0$ it coincides with M_0; positive and negative values of t correspond to points located on the straight line on either side of M_0.

In vector form, the three Eqs. (1) are replaced by one:

$$\boldsymbol{r}=\boldsymbol{r}_0+\boldsymbol{a}t \tag{2}$$

153. The Intersection of a Plane with a Straight Line Represented Parametrically

A common point (if such exists) of the plane P

$$Ax+By+Cz+D=0 \tag{1}$$

and the straight line L

$$x=x_0+lt, \quad y=y_0+mt, \quad z=z_0+nt \tag{2}$$

is found from formulas (2) if into (2) is substituted the value of t as defined from the equation[1]

$$(Al+Bm+Cn)\,t+Ax_0+By_0+Cz_0+D=0 \tag{3}$$

This is obtained if expressions (2) are substituted into (1).

Example 1. Find the point of intersection of the plane

$$2x+3y+3z-8=0$$

with the straight line

$$\frac{x+5}{3}=\frac{y-3}{-1}=\frac{z+3}{2}$$

Solution. In parametric form, the equations of the straight line will be

$$x=-5+3t, \quad y=3-t, \quad z=-3+2t \tag{4}$$

Substituting into the equation $2x+3y+3z-8=0$, we get $9t-18=0$, whence $t=2$. Putting this value into (4) we obtain $x=1$, $y=1$, $z=1$. The desired point is (1, 1, 1).

Example 2. Find the point of intersection of the plane $3x+y-4z-7=0$ with the straight line of Example 1.

Solution. In the same manner we get $0\cdot t-7=0$; this equation has no solution. There is no point of intersection (the straight line is parallel to the plane).

Example 3. Find the point of intersection of the plane $3x+y-4z=0$ with the straight line of Example 1.

Solution. In the same manner, we get $0\cdot t+0=0$; this equation has an infinity of solutions (the straight line lies in the plane).

Note. Taking advantage of the parametric equations (4), we introduced a fourth unknown t and obtained four equations (in place of the three that are given). This complication is compensated for by the greater facility of solving the system.

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[1] In exceptional cases, Eq. (3) may not have any solution (see Example 2 below) or it may have an infinity of solutions (see Example 3 below).

154. The Two-Point Form of the Equations of a Straight Line

A straight line passing through the points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ is given by the equations

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \tag{1}$$

For examples see Sec. 150.

155. The Equation of a Plane Passing Through a Given Point Perpendicular to a Given Straight Line

A plane passing through the point $M_0(x_0, y_0, z_0)$ perpendicular to the straight line

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

has a normal vector $\{l_1, m_1, n_1\}$ and, hence, is represented by the equation

$$l_1(x-x_0)+m_1(y-y_0)+n_1(z-z_0)=0$$

or, in vector form,

$$\mathbf{a}_1(\mathbf{r}-\mathbf{r}_0)=0$$

Example. A plane passing through the point $(-1, -5, 8)$ perpendicular to the straight line $\frac{x}{0}=\frac{y}{2}=\frac{z-3}{5}$ is represented by the equation $2(y+5)+5(z-8)=0$, or

$$2y+5z-30=0$$

156. The Equations of a Straight Line Passing Through a Given Point Perpendicular to a Given Plane

The straight line passing through the point $M_0(x_0, y_0, z_0)$ perpendicular to the plane $Ax+By+Cz+D=0$ has the direction vector $\{A, B, C\}$ and, hence, is given by the symmetric (Sec. 150) equations

$$\frac{x-x_0}{A}=\frac{y-y_0}{B}=\frac{z-z_0}{C} \tag{1}$$

Example. The straight line passing through the origin perpendicular to the plane $3x+5z-5=0$ is given by the symmetric equations $\frac{x}{3}=\frac{y}{0}=\frac{z}{5}$ or the parametric (Sec. 152) equations $x=3t$, $y=0$, $z=5t$.

157. The Equation of a Plane Passing Through a Given Point and a Given Straight Line

A plane passing through the point $M_0(x_0, y_0, z_0)$ and the straight line L

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \tag{1}$$

which does not pass through M_0 is represented by the equation

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ l & m & n \end{vmatrix}=0 \tag{2}$$

or, in vector form,

$$(\mathbf{r}-\mathbf{r}_0)(\mathbf{r}_1-\mathbf{r}_0)\,\mathbf{a}=0 \qquad \text{(2a)}$$

The equation (2), or (2a), expresses coplanarity of the vectors (Fig. 174) $\overrightarrow{M_0M}$, $\overrightarrow{M_0M_1}$ and $\mathbf{a}\{l, m, n\}$.

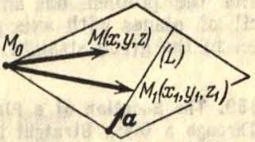


Fig. 174

Example. A plane passing through the point $M_0(5, 2, 3)$ and the straight line

$$\frac{x+1}{z}=\frac{y+1}{1}=\frac{z-5}{3}$$

is given by the equation

$$\begin{vmatrix} x-5 & y-2 & z-3 \\ -6 & -3 & 2 \\ 2 & 1 & 3 \end{vmatrix}=0$$

i.e.

$$x-2y-1=0$$

Note. If straight line (1) passes through the point M_0, then Eq. (2) becomes an identity and the problem has an infinity of solutions (we get a pencil of planes with axis L; Sec. 148).

158. The Equation of a Plane Passing Through a Given Point Parallel to Two Given Straight Lines

A plane passing through a point $M_0(x_0, y_0, z_0)$ parallel to given (mutually nonparallel) straight lines L_1 and L_2 (or to vectors a_1 and a_2) is represented by the equation

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \tag{1}$$

where l_1, m_1, n_1 and l_2, m_2, n_2 are the direction numbers of the given straight lines (or the coordinates of the given vectors). In vector form

$$(r-r_0)\,a_1a_2 = 0 \tag{1a}$$

Eq. (1) or (1a) expresses the coplanarity of the vectors $\overrightarrow{M_0M}$, a_1, a_2 (M is an arbitrary point in the desired plane).

Note. If the straight lines L_1 and L_2 are parallel, i.e. if a_1 and a_2 are collinear, then Eq. (1) becomes an identity, and the problem has an infinity of solutions (we get a pencil of planes with axis passing through the point M_0 parallel to the given straight lines).

159. The Equation of a Plane Passing Through a Given Straight Line and Parallel to Another Given Straight Line

Let L_1 and L_2 be nonparallel straight lines. Then a plane passing through L_1 and parallel to the straight line L_2 is given by the equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \tag{1}$$

where x_1, y_1, z_1 are coordinates of some point M_1 of L_1. Here we have a particular case of Sec. 158 (M_1 playing the role of M_0). The note in Sec. 158 remains valid.

168. The Equation of a Plane Passing Through a Given Straight Line and Perpendicular to a Given Plane

A plane P passing through a given straight line L_1

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \tag{1}$$

and perpendicular to a given plane Q

$$Ax+By+Cz+D=0 \tag{2}$$

(not perpendicular to L_1) is represented by the equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ A & B & C \end{vmatrix}=0 \tag{3}$$

In vector form

$$(\mathbf{r}-\mathbf{r}_1)\,\mathbf{a}_1\mathbf{N}=0 \tag{3a}$$

Explanation. The plane P passes through the straight line L_1 and is parallel to the normal $\mathbf{N}(A, B, C)$ to plane Q (cf. Sec. 150).

Note. If the plane (2) is perpendicular to the straight line (1), Eq. (3) becomes an identity and the problem has an infinity of solutions (see Sec. 158, Note).

The projection of a straight line on any plane. Plane (3) projects the straight line L_1 on the plane Q. Hence the straight line L', which is the projection of L_1 on the plane Q, is given by the system of equations (2)-(3) (cf. Sec. 165).

169. The Equations of a Perpendicular Dropped from a Given Point onto a Given Straight Line

A perpendicular dropped from a point $M_0(x_0, y_0, z_0)$ onto a straight line L_1,

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \tag{1}$$

that does not pass through M_0 is given by the equations

$$\left\{\begin{array}{l} l_1(x-x_0)+m_1(y-y_0)+n_1(z-z_0)=0, \qquad (2)\\ \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ l_1 & m_1 & n_1 \end{vmatrix}=0 \qquad (3) \end{array}\right.$$

or, in vector form, by the equations

$$\begin{cases} \mathbf{a}_1(\mathbf{r}-\mathbf{r}_0)=0, & (2a)\\ (\mathbf{r}-\mathbf{r}_0)(\mathbf{r}_1-\mathbf{r}_0)\,\mathbf{a}_1=0 & (3a)\end{cases}$$

Taken separately, Eq. (2) represents a plane Q (Fig. 175) drawn through M_0 perpendicular to L_1 (Sec. 155), while Eq. (3) represents plane R drawn through the point M_0 and the straight line L_1 (Sec. 157).

Note. If the straight line L_1 passes through the point M_0, Eq. (3) becomes an indentity (Sec. 120) (an infinity of perpendiculars to L can be drawn through a point taken on the straight line L).

Example. Find the equation of the perpendicular dropped from the point (1, 0, 1) onto the straight line

$$x=3z+2, \quad y=2z \tag{1a}$$

Also find the foot of the perpendicular.

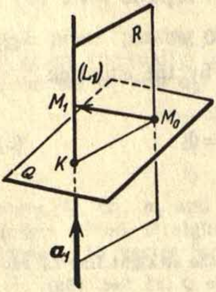


Fig. 175

Solution. Eqs. (1a) may be written in symmetric form (Sec. 151) as

$$\frac{x-2}{3}=\frac{y}{2}=\frac{z}{1} \tag{1b}$$

The desired perpendicular is then given by the equations

$$\begin{cases} 3(x-1)+2(y-0)+1(z-1)=0, & (2b)\\ \begin{vmatrix} x-1 & y & z-1\\ 2-1 & 0 & 0-1\\ 3 & 2 & 1\end{vmatrix}=0 & (3b)\end{cases}$$

or, after simplifications,

$$3x+2y+z-4=0, \tag{2c}$$

$$x-2y+z-2=0 \tag{3c}$$

The coordinates of the foot K of the perpendicular may be found by solving the system of three equations (1b), (2c). Eq. (3c) should be satisfied by itself. We obtain $K\left(\frac{11}{7}, -\frac{2}{7}, -\frac{1}{7}\right)$.

Note. The system of three equations (1b)-(3c) has an infinity of solutions because the plane R passes through the straight line L_1 and does not intersect it.

162. The Length of a Perpendicular Dropped from a Given Point onto a Given Straight Line

Given: point $M_0(x_0, y_0, z_0)$ and straight line L_1 represented by Eq. (1), Sec. 161. It is required to find the distance from M_0 to L_1, i. e. the length of the perpendicular M_0K (Fig. 175) dropped from M_0 onto L_1.

One can first find the foot K of the perpendicular (Sec. 161, Example) and then the length of the segment M_0K. A simpler way is to apply the formula (in the notation of Sec. 161)

$$d=\frac{\sqrt{\begin{vmatrix} y_0-y_1 & z_0-z_1 \\ m_1 & n_1 \end{vmatrix}^2+\begin{vmatrix} z_0-z_1 & x_0-x_1 \\ n_1 & l_1 \end{vmatrix}^2+\begin{vmatrix} x_0-x_1 & y_0-y_1 \\ l_1 & m_1 \end{vmatrix}^2}}{\sqrt{l_1^2+m_1^2+n_1^2}} \tag{1}$$

or, in vector form,

$$d=\frac{\sqrt{[(\mathbf{r}_0-\mathbf{r}_1)\times \mathbf{a}_1]^2}}{\sqrt{\mathbf{a}_1^2}} \tag{1a}$$

The numerator of expression (1a) is (Sec. 111) the area of a parallelogram M_1M_0BA (Fig. 176, where $M_1A=\mathbf{a}_1$) and the denominator is the length of the base M_1A. Hence, the fraction is equal to the altitude M_0K of the parallelogram.

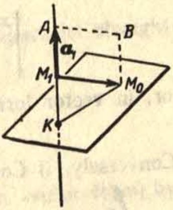


Fig. 176

Example. Find the length of the perpendicular dropped from point $M_0(1, 0, 1)$ onto the straight line $x=3z+2$, $y=2z$.

Solution. In the example in Sec. 161 we found

$$K\left(\frac{11}{7}, -\frac{2}{7}, -\frac{1}{7}\right)$$

Consequently,

$$d=\left|M_0K\right|=$$

$$=\sqrt{\left(\frac{11}{7}-1\right)^2+\left(-\frac{2}{7}\right)^2+\left(-\frac{1}{7}-1\right)^2}=2\sqrt{\frac{3}{7}}$$

Now apply formula (1). According to (1b), Sec. 161, we have $x_1=2$, $y_1=0$, $z_1=0$, $l_1=3$, $m_1=2$, $n_1=1$, so that

$$\begin{vmatrix} y_0-y_1 & z_0-z_1 \\ m_1 & n_1 \end{vmatrix}=\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}=-2,$$

$$\begin{vmatrix} z_0-z_1 & x_0-x_1 \\ n_1 & l_1 \end{vmatrix}=\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}=4,$$

$$\begin{vmatrix} x_0-x_1 & y_0-y_1 \\ l_1 & m_1 \end{vmatrix}=\begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix}=-2$$

We obtain

$$d=\frac{\sqrt{(-2)^2+4^2+(-2)^2}}{\sqrt{3^2+2^2+1^2}}=2\sqrt{\frac{3}{7}}$$

163. The Condition for Two Straight Lines Intersecting or Lying in a Single Plane

If the straight lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \tag{1}$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \tag{2}$$

lie in a single plane, then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \tag{3}$$

or, in vector form,

$$(\mathbf{r}_2-\mathbf{r}_1)\,\mathbf{a}_1\mathbf{a}_2=0 \tag{3a}$$

Conversely, if Condition (3) is fulfilled, then the straight lines lie in a single plane.

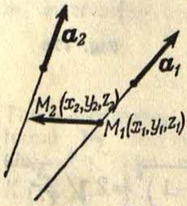


Fig. 177

Explanation. If the straight lines (1) and (2) lie in a single plane, then the straight line M_1M_2 (Fig. 177) also lies in that plane, i.e. the vectors $\overrightarrow{M_1M_2}$, $\mathbf{a}_1$, $\mathbf{a}_2$ are coplanar (and conversely). This is what Eq. (3) expresses (see Sec. 120).

Note. If $\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$ [here, (3) will definitely be satisfied], then the straight lines are parallel, otherwise the straight lines satisfying Condition (3) intersect.

Example. Determine whether the straight lines

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}, \tag{1}$$

$$\frac{x+1}{2}=\frac{y-1}{1}=\frac{z+1}{4} \tag{2}$$

intersect, and if they do, at what point.

Solution. The straight lines (1) and (2) lie in one plane since the determinant (3) equal to $\begin{vmatrix} -1 & 1 & -1 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{vmatrix}$ vanishes. These lines are not parallel (the direction numbers are not proportional). In order to find the point of intersection, we have to solve a system of four equations (1), (2) in three unknowns. As a rule, such a system does not have any solutions, but in the given case [because Condition (3) is fulfilled] there is a solution. Solving a system of any three equations, we get $x=1$, $y=2$, $z=3$. The fourth equation is satisfied. The point of intersection is (1, 2, 3).

164. The Equations of a Line Perpendicular to Two Given Straight Lines

The straight line UV intersecting two nonparallel straight lines (L_1 and L_2 in Fig. 178)

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1},$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

and perpendicular to them is represented (in vector form) by the equations

$$(\boldsymbol{r}-\boldsymbol{r}_1)\,\boldsymbol{a}_1\boldsymbol{a}=0, \tag{1}$$

$$(\boldsymbol{r}-\boldsymbol{r}_2)\,\boldsymbol{a}_2\boldsymbol{a}=0 \tag{2}$$

where $\boldsymbol{a}_1=\{l_1, m_1, n_1\}$, $\boldsymbol{a}_2=\{l_2, m_2, n_2\}$ and $\boldsymbol{a}=\boldsymbol{a}_1\times\boldsymbol{a}_2$.

Taken separately, Eq. (1) is the plane P_1 drawn through the straight line L_1 parallel to the vector $\boldsymbol{a}=\boldsymbol{a}_1\times\boldsymbol{a}_2$ (Sec. 159). Similarly, (2) is plane P_2 drawn through L_2 parallel to $\boldsymbol{a}$.

Point K_1, where UV intersects L_1, is found at the intersection of L_1 with the plane P_2. Similarly, we find, the point K_2 and then the length of the common perpendicular K_1K_2.

Note. If L_1 and L_2 are parallel [then $\boldsymbol{a}=0$ and Eqs. (1), (2) become identities], there is an infinity of straight lines UV. To obtain the equation of one of them, take on L_1 (Fig. 179) an arbitrary point K_1 and form the equation of the straight line passing through K_1 in the direction of the vector $\boldsymbol{a}_1\times\boldsymbol{b}$, where $\boldsymbol{b}=\boldsymbol{a}_1\times(\boldsymbol{r}_2-\boldsymbol{r}_1)$.

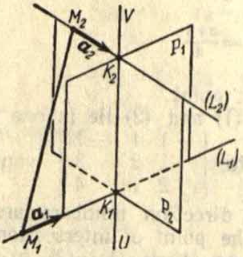


Fig. 178

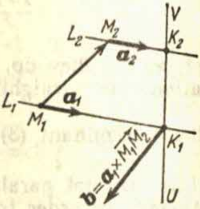


Fig. 179

Example 1. Find the equations of the perpendicular to the straight lines

$$x=2+2t, \qquad y=1+4t, \qquad z=-1-t, \tag{3}$$

$$x=-31+3t', \quad y=6+2t', \quad z=3+6t' \tag{4}$$

Solution. We have $\boldsymbol{a}_1=\{2,\ 4,\ -1\}$, $\boldsymbol{a}_2=\{3,\ 2,\ 6\}$, $\boldsymbol{a}=\boldsymbol{a}_1\times\boldsymbol{a}_2=\{26,\ -15,\ -8\}$.

The desired perpendicular is given by the equations

$$\begin{vmatrix} x-2 & y-1 & z+1 \\ 2 & 4 & -1 \\ 26 & -15 & -8 \end{vmatrix}=0,$$

$$\begin{vmatrix} x+31 & y-6 & z-3 \\ 3 & 2 & 6 \\ 26 & -15 & -8 \end{vmatrix}=0$$

or, after simplifying,

$$\left\{\begin{aligned} &47x+10y+134z+30=0, && (5)\\ &74x+180y-97z+1505=0 && (6)\end{aligned}\right.$$

The point K_1 of intersection of the common perpendicular with the straight line (3) is found from the system (3)-(6). We get $K_1(-2, -7, 1)$. Similarly, we find $K_2(-28, 8, 9)$. The length d of the common perpendicular is

$$d=\sqrt{(-2+28)^2+(-7-8)^2+(1-9)^2}=\sqrt{965}$$

Example 2. Find the equations of the line perpendicular to the straight lines

$$x=2+2t, \quad y=3+2t, \quad z=t, \tag{7}$$

$$x=5+2t', \quad y=4+2t', \quad z=1+t' \tag{8}$$

The lines are parallel: $\boldsymbol{a}_1=\boldsymbol{a}_2=\{2, 2, 1\}$, $\boldsymbol{r}_2-\boldsymbol{r}_1=\{3, 1, 1\}$, $\boldsymbol{b}=\boldsymbol{a}_1\times(\boldsymbol{r}_2-\boldsymbol{r}_1)=\{1, 1, -4\}$. The direction vector of the common perpendicular $\boldsymbol{a}_1\times\boldsymbol{b}=\{-9, 9, 0\}$ or, multiplying by $\frac{1}{9}$, $\{-1, 1, 0\}$. For the initial point we take an arbitrary point $K_1(2+2t; 3+2t; t)$ of the line (7). We obtain the equation of the common perpendicular

$$\frac{x-(2+2t)}{-1}=\frac{y-(3+2t)}{1}=\frac{z-t}{0} \tag{9}$$

where t is an arbitrary number. To find the point K_2 of intersection of the common perpendicular (9) with the straight line (8), substitute expression (8) into Eq. (9). This yields

$$\frac{3+2(t'-t)}{-1}=\frac{1+2(t'-t)}{1}=\frac{1+(t'-t)}{0}$$

Any one of these equations yields $t'=t-1$; substituting into (8), we get $K_2(3+2t, 2+2t, t)$, so that

$$d=|K_1K_2|=$$

$$=\sqrt{[(3+2t)-(2+2t)]^2+[(2+2t)-(3+2t)]^2+[t-t]^2}=\sqrt{2}$$

165. The Shortest Distance Between Two Straight Lines

The shortest distance between the straight lines L_1 and L_2 is the length d of their common perpendicular. It can be found by forming the equations of the common perpendicular (Sec. 164, Examples 1 and 2). A simpler way, however, is to find d directly.

(1) If L_1 and L_2 are not parallel (Fig. 180), then

$$d=\frac{|(\boldsymbol{r}_2-\boldsymbol{r}_1)\,\boldsymbol{a}_1\boldsymbol{a}_2|}{|\boldsymbol{a}_1\times\boldsymbol{a}_2|}=\frac{|\boldsymbol{r}_2-\boldsymbol{r}_1)\,\boldsymbol{a}_1\boldsymbol{a}_2|}{\sqrt{(\boldsymbol{a}_1\times\boldsymbol{a}_2)^2}} \tag{1}$$

($\boldsymbol{r}_1$, $\boldsymbol{r}_2$ are the radius vectors of the points M_1, M_2; $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ are the direction vectors of the straight lines L_1, L_2).

The numerator of the fraction (1) is (Sec. 121) the volume of a parallelepiped constructed on the vectors $\overrightarrow{M_1M_2}$, $\boldsymbol{a}_1$, $\boldsymbol{a}_2$. The denominator is the area of its base (Sec. 111). Consequently, the whole fraction is the altitude $K_1K_2=d$.

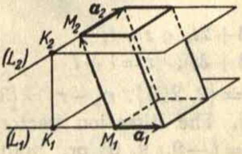


Fig. 180

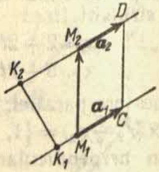


Fig. 181

For intersecting straight lines (the vectors $\overrightarrow{K_1K_2}$, $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ are coplanar), formula (1) yields $d=0$. For parallel straight lines (the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ are collinear) it fails $\left(\text{yielding } \frac{0}{0}\right)$.

(2) If the straight lines L_1, L_2 are parallel (Fig. 181), then

$$d=\frac{|(\boldsymbol{r}_2-\boldsymbol{r}_1)\times\boldsymbol{a}_1|}{|\boldsymbol{a}_1|}=\frac{\sqrt{[(\boldsymbol{r}_2-\boldsymbol{r}_1)\times\boldsymbol{a}_1]^2}}{\sqrt{\boldsymbol{a}_1^2}} \tag{2}$$

(in place of $\boldsymbol{a}_1$ we can take $\boldsymbol{a}_2$).

The numerator of the fraction (2) is the area of the parallelogram M_1M_2DC, the denominator is the length of the base M_1C. The whole fraction is the altitude $K_1K_2=d$.

Example 1. Find the shortest distance between the straight lines of Example 1, Sec. 164 [$\boldsymbol{r}_1=\{2,\ 1,\ -1\}$, $\boldsymbol{r}_2=\{-31,\ 6,\ 3\}$, $\boldsymbol{a}_1=\{2,\ 4,\ -1\}$, $\boldsymbol{a}_2=\{3,\ 2,\ 6\}$].

Solution. The lines are not parallel. We have

$$\boldsymbol{a}_1\times\boldsymbol{a}_2=\left\{\begin{vmatrix}4 & -1\\ 2 & 6\end{vmatrix},\ \begin{vmatrix}-1 & 2\\ 6 & 3\end{vmatrix},\ \begin{vmatrix}2 & 4\\ 3 & 2\end{vmatrix}\right\}=\{26,\ -15,\ -8\},$$

$$(\boldsymbol{r}_2-\boldsymbol{r}_1)\,\boldsymbol{a}_1\boldsymbol{a}_2=-33\cdot 26+5\cdot(-15)+4\cdot(-8)=-965$$

Formula (1) yields

$$d=\frac{965}{\sqrt{26^2+(-15)^2+(-8)^2}}=\frac{965}{\sqrt{965}}=\sqrt{965}$$

Example 2. Find the shortest distance between the straight lines of Example 2, Sec. 164 [$a_1 = a_2 = \{2, 2, 1\}$, $r_2 - r_1 = \{3, 1, 1\}$].

Solution. The lines are parallel, and formula (2) yields

$$d = \frac{\sqrt{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}^2 + \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}^2 + \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix}^2}}{\sqrt{2^2 + 2^2 + 1}} = \sqrt{2}$$

Note. A sign can be prefixed to the shortest distance between straight lines (if they are not perpendicular and not parallel) (see Sec. 165a).

165a. Right-Handed and Left-Handed Pairs of Straight Lines

Definition. A pair of nonperpendicular skew lines L_1, L_2 (Fig. 180) is called *right-handed* if for an observer standing on the extension of some transversal K_1K_2 beyond L_2 the shortest rotation of L_1 to a position parallel to L_2 is performed counterclockwise. Otherwise the pair L_1, L_2 is *left-handed.*

Note 1. A right-handed pair remains right-handed and a left-handed pair left-handed irrespective either of the choice of points K_1, K_2 on the lines L_1, L_2 or of the labelling of the straight lines (the first may be labelled L_2 and the second L_1). Indeed, although the rotation will be reversed, the observer will now be on the continuation of the transversal beyond the straight line L_1, so that for him the direction of the rotation remains unchanged.

Note 2. The concepts of a left-handed and right-handed pair are meaningless with respect to the straight lines L_1, L_2 lying in a single plane and also with respect to perpendicular lines.

Example. If in going in or out, the handle of the corkscrew turns through 60°, then the initial and terminal positions of the axis of the handle form a right-handed pair of straight lines (if for the straight line L_1 we take the axis of the handle in the upper position, then the observer mentioned in the definition must look upwards; otherwise, downwards). In a rotation of the handle through 120° the initial and terminal positions of the axis form a left-handed pair.

Test for right-handedness and left-handedness. Let a_1, a_2 be some (nonzero) vectors collinear with the straight lines

L_1, L_2. If the triple scalar product $\overrightarrow{K_1K_2}\mathbf{a}_1\mathbf{a}_2$ is of the same sign as the scalar product $\mathbf{a}_1\mathbf{a}_2$, then the pair L_1, L_2 is right-handed; if the signs are opposite, then it is left-handed.

When $\overrightarrow{K_1K_2}\mathbf{a}_1\mathbf{a}_2=0$, the lines L_1, L_2 lie in one plane; when $\mathbf{a}_1\mathbf{a}_2=0$, the lines L_1, L_2 are perpendicular. In neither case is the pair L_1, L_2 right-handed or left-handed (see Note 2).

The sign of the shortest distance between two straight lines. To the shortest distance between nonperpendicular intersecting lines we can prefix a sign: positive if the pair is right-handed, and negative if it is left-handed.

Using the letter δ to denote the shortest distance between straight lines (with sign taken into consideration) we have the following formula in place of (1), Sec. 165:

$$\delta=\frac{\mathbf{a}_1\mathbf{a}_2}{|\mathbf{a}_1\mathbf{a}_2|}\;\frac{(\mathbf{r}_2-\mathbf{r}_1)\,\mathbf{a}_1\mathbf{a}_2}{|\mathbf{a}_1\times\mathbf{a}_2|} \tag{1}$$

It also holds true for intersecting (but not perpendicular) lines and then yields $\delta=0$. For perpendicular lines, formula (1) does not hold true because the first factor $\frac{\mathbf{a}_1\mathbf{a}_2}{|\mathbf{a}_1\mathbf{a}_2|}$ becomes indeterminate, $\frac{0}{0}$ (if the straight lines are not perpendicular, then the first factor is equal either to $+1$ or to -1). Neither is (1) valid for parallel lines, since the second factor becomes indeterminate. See Note 2.

166. Transformation of Coordinates

1. **Translation of the origin.** When a system of coordinates $OXYZ$ is replaced by a new system $O'X'Y'Z'$ having axes in the same directions, the old coordinates (x, y, z) of a *point* are expressed in terms of the new coordinates (x', y', z') by the formulas

$$x=a+x', \qquad y=b+y', \qquad z=c+z' \tag{1}$$

where a, b, c are the coordinates of the new origin O' in the old system (cf. Sec. 35).

In this replacement, the coordinates of any *vector* remain unchanged.

2. **Rotation of axes.** When replacing a system $OXYZ$ by a new system $O'X'Y'Z'$ with the same origin, the old coordinates of a point are expressed by new formulas:

$$\left.\begin{aligned}
x&=x'\cos(\widehat{\mathbf{i}', \mathbf{i}})+y'\cos(\widehat{\mathbf{j}', \mathbf{i}})+z'\cos(\widehat{\mathbf{k}', \mathbf{i}}),\\
y&=x'\cos(\widehat{\mathbf{i}', \mathbf{j}})+y'\cos(\widehat{\mathbf{j}', \mathbf{j}})+z'\cos(\widehat{\mathbf{k}', \mathbf{j}}),\\
z&=x'\cos(\widehat{\mathbf{i}', \mathbf{k}})+y'\cos(\widehat{\mathbf{j}', \mathbf{k}})+z'\cos(\widehat{\mathbf{k}', \mathbf{k}})
\end{aligned}\right\} \tag{2}$$

where $\widehat{(\mathbf{i}', \mathbf{i})}$ is the angle between the vectors $\mathbf{i}'$ and $\mathbf{i}$, i.e. between the new and old axis of abscissas, $\widehat{(\mathbf{j}', \mathbf{i})}$ is the angle between the new axis of ordinates and the old axis of abscissas and so forth.[1]

In this substitution, the coordinates of any vector are transformed in accordance with the same formulas.

Note. Of the nine quantities $\cos \widehat{(\mathbf{i}', \mathbf{i})}$, $\cos \widehat{(\mathbf{j}', \mathbf{j})}$, etc., any three may be specified in arbitrary fashion, the other six satisfy the relations

$$\left.\begin{aligned}
&\cos^2 \widehat{(\mathbf{i}, \mathbf{i}')} + \cos^2 \widehat{(\mathbf{i}, \mathbf{j}')} + \cos^2 \widehat{(\mathbf{i}, \mathbf{k}')} = 1,\\
&\cos^2 \widehat{(\mathbf{j}, \mathbf{i}')} + \cos^2 \widehat{(\mathbf{j}, \mathbf{j}')} + \cos^2 \widehat{(\mathbf{j}, \mathbf{k}')} = 1,\\
&\cos^2 \widehat{(\mathbf{k}, \mathbf{i}')} + \cos^2 \widehat{(\mathbf{k}, \mathbf{j}')} + \cos^2 \widehat{(\mathbf{k}, \mathbf{k}')} = 1
\end{aligned}\right\} \quad (3)$$

and

$$\left.\begin{aligned}
&\cos \widehat{(\mathbf{i}, \mathbf{i}')} \cos \widehat{(\mathbf{j}, \mathbf{i}')} + \cos \widehat{(\mathbf{i}, \mathbf{j}')} \cos \widehat{(\mathbf{j}, \mathbf{j}')} + \cos \widehat{(\mathbf{i}, \mathbf{k}')} \cos \widehat{(\mathbf{j}, \mathbf{k}')} = 0,\\
&\cos \widehat{(\mathbf{i}, \mathbf{i}')} \cos \widehat{(\mathbf{k}, \mathbf{i}')} + \cos \widehat{(\mathbf{i}, \mathbf{j}')} \cos \widehat{(\mathbf{k}, \mathbf{j}')} + \cos \widehat{(\mathbf{i}, \mathbf{k}')} \cos \widehat{(\mathbf{k}, \mathbf{k}')} = 0,\\
&\cos \widehat{(\mathbf{j}, \mathbf{i}')} \cos \widehat{(\mathbf{k}, \mathbf{i}')} + \cos \widehat{(\mathbf{j}, \mathbf{j}')} \cos \widehat{(\mathbf{k}, \mathbf{j}')} + \cos \widehat{(\mathbf{j}, \mathbf{k}')} \cos \widehat{(\mathbf{k}, \mathbf{k}')} = 0
\end{aligned}\right\} \quad (4)$$

Relations (3) follow from (4), Sec. 101, and relations (4) follow from (2), Sec. 145

167. The Equation of a Surface

An equation relating the coordinates x, y, z is called the *equation of a surface* S if the following two conditions hold: (1) the coordinates x, y, z of any point of the surface S satisfy this equation, (2) the coordinates x, y, z of any point not lying on the surface S do not satisfy this equation (cf. Sec. 7).

Note. If we change the system of coordinates, then the equation of the surface will change (the new equation will follow from the old equation by means of the formulas for transforming coordinates, Sec. 166).

Example 1. The equation $x+y+z-1=0$ is an equation of a plane surface. Given a properly chosen rectangular coordinate system, the same surface may be represented by any other first-degree equation.

Example 2. The surface of a sphere of radius R with centre at the origin is given by the equation

$$x^2+y^2+z^2=R^2 \qquad (1)$$

because (1) if the point $M(x, y, z)$ lies on this surface, the distance $OM=\sqrt{x^2+y^2+z^2}$ is equal to the radius R and, hence, Eq. (1) is satisfied; (2) if M does not lie on the surface, then $OM \neq R$, and Eq. (1) is not satisfied.

---

[1] Each of the coefficients of the new coordinates is the cosine of the angle between the corresponding new axis and the old axis associated with the coordinate written on the left side.

Example 3. A sphere of radius R with centre at the point $C(a, b, c)$ is given by the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2 \tag{2}$$

An equation relating the coordinates x, y, z is capable of representing other geometric images or none at all (cf. Sec. 58).

Example 4. The equation $x^2+y^2+z^2+1=0$ does not represent any geometric image at all because it has no (real) solutions.

Example 5. The equation $x^2+y^2+z^2=0$ which has a unique real solution $x=0$, $y=0$, $z=0$ represents a point.

Example 6. The equation $(x-y)^2+(z-y)^2=0$ is satisfied only when $x-y=0$ and $z-y=0$ simultaneously; it represents the straight line $x=y=z$.

168. Cylindrical Surfaces Whose Generatrices Are Parallel to One of the Coordinate Axes

A surface generated by the motion of a straight line (generatrix) which is parallel to some fixed line is called a *cylindrical* surface. Any line intersected by the generatrix in any of its positions is called a *directrix*.

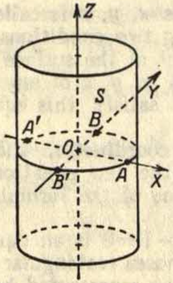


Fig. 182

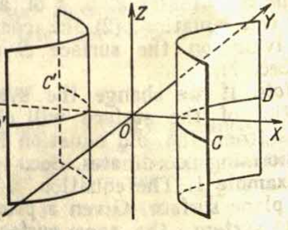


Fig. 183

Any equation which does not have the z-coordinate and represents some line L in the xy-plane represents in space a cylindrical surface whose generatrix is parallel to the z-axis and the line L is the directrix.

Example 1. The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

is, in the xy-plane, an ellipse $ABA'B'$ (Fig. 182) with semiaxes $a=OA$, $b=OB$. In space it is a cylindrical surface S whose generatrices are parallel to the z-axis and whose directrix is the ellipse $ABA'B'$ (*elliptic cylinder*).

Example 2. The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ represents a cylindrical surface (Fig. 183) whose generatrices are parallel to the z-axis and whose directrix is the hyperbola $CDC'D'$ (*hyperbolic cylinder*)

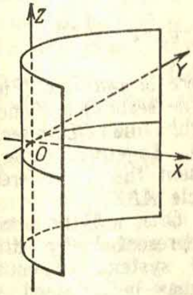


Fig. 184

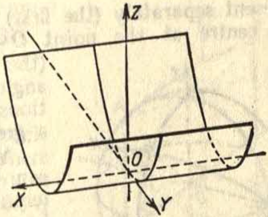


Fig. 185

Example 3. The equation $y^2=2px$ is a *parabolic cylinder* (Fig. 184).

An equation not containing the x (or y) coordinate is a cylindrical surface whose generatrix is parallel to the x-axis (or y-axis).

Example 4. The equation $y^2=2pz$ is a parabolic cylinder located as shown in Fig. 185.

Note. If the directrix is a straight line, then the cylindrical surface is flat. Accordingly, the equation $Ax+By+D=0$ represents a plane in space parallel to the z-axis (cf. Sec. 124, Note).

169. The Equations of a Line

A line may be regarded as the intersection of two surfaces and, accordingly, may be represented by a system of two equations.

Two equations (taken together) relating the coordinates x, y, z are called the *equations of the line* L if the following two conditions are fulfilled: (1) the coordinates of any point M of the line L satisfy both equations; (2) the coordinates of any point not lying on the line L do not satisfy both equations at once (although they may satisfy one of them; cf. Sec. 140).

Example 1. The two equations $y-z=0$, $x-z=0$ represent a straight line as an intersection of two planes (cf. Example 1, Sec. 140).

Example 2. The two equations

$$x^2+y^2+z^2=a^2, \quad y=z$$

represent separately (the first) a sphere of radius a (Fig. 186) with centre at the point O and (the second) a plane LOX (the straight line OL bisects the angle YOZ). Together, these equations represent the circumference of a great circle ALK.

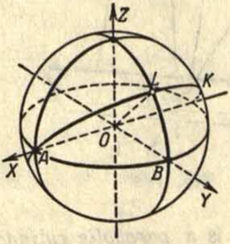


Fig. 186

Note 1. One and the same line can be represented by different (equivalent) systems of equations because it may be obtained as the intersection of various pairs of surfaces.

Note 2. A system of two equations can represent geometric images other than a line or no geometric image at all.

Example 3. The system of equations $x^2+y^2+z^2=25$, $z=5$ represents a point (0, 0, 5) at which the plane $z=5$ touches the sphere $x^2+y^2+z^2=25$.

Example 4. The system of equations $x^2+y^2+z^2=0$, $x+y+z=1$ does not represent any geometric image at all because the first equation is satisfied only by the values $x=0$, $y=0$, but these fail to satisfy the second equation.

170. The Projection of a Line on a Coordinate Plane

1. Let a line L be given by two equations, one of which contains z and the other does not.[1] Then the second represents a "vertical" cylindrical surface and, on the xy-plane, the directrix L_1 of this surface (Sec. 168); the projection of the line L on the xy-plane lies on the L_1 line (covering it entirely or in part).

---

[1] If neither equation contains z, then L is a vertical straight line (or several such lines); it is projected on XOY as a point (cf. Sec. 149, Example 3).

Example 1. The equations

$$z=y+\frac{3}{2}, \quad x^2+y^2=1$$

represent (Fig. 187) the line ABA_1B_1 (ellipse) along which the plane $z=y+\frac{3}{2}$ (plane P in Fig. 187) and the circular cylindrical surface $x^2+y^2=1$ intersect. In the xy-plane, the equation $x^2+y^2=1$ represents the circle $A'B'A'_1B'_1$. The projection of the line ABA_1B_1 coincides with the line $A'B'A'_1B'_1$.

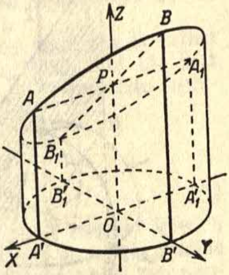


Fig. 187

Example 2. The equations

$$x^2+y^2+z^2=a^2, \quad y=mx$$

represent (Fig. 188) a great circle ("meridian") $APA'P'$ of the sphere O as the intersection of this sphere with the plane $y=mx$ (the plane R in Fig. 188). The equation $y=mx$ represents the straight line UV in the xy-plane. The projection of the meridian $APA'P'$ on the xy-plane lies on UV, but covers only a part of it, the segment AA'.

2. Let both equations representing L contain z; then z has to be eliminated from the given equations in order to find the projection of the line L on the xy-plane.1) The equation obtained by this elimination represents, in the xy-plane, a line L' on which the desired projection lies (covering it completely or partially). Similarly, we find the projections of the line on the xz-plane and yz-plane.

This follows from Item 1.

Example 3. Let us consider a circle (ALK in Fig. 189) represented (cf. Sec. 169, Example 2) by the equations

$$x^2+y^2+z^2=a^2, \tag{1}$$

$$y=z \tag{2}$$

---

1) To eliminate z from the two equations means to find a third equation not containing z and satisfied for all those values of x and y which satisfy the system of the two given equations.

To find its projection on the xy-plane, eliminate z from (1) and (2). This yields the equation

$$x^2+2y^2=a^2 \tag{3}$$

which, in the xy-plane, represents the ellipse $AL'K'$ with semiaxes $OA=a$, $OL'=\frac{a}{\sqrt{2}}$. The projection of the circle ALK covers the ellipse $AL'K'$ entirely.

To find the projection of the circle ALK on the xz-plane, eliminate y from (1) and (2). This yields

$$x^2+2z^2=a^2 \tag{4}$$

which, in the xz-plane, represents an ellipse of the same dimensions as $AL'K'$. The projection of the circle covers this ellipse completely.

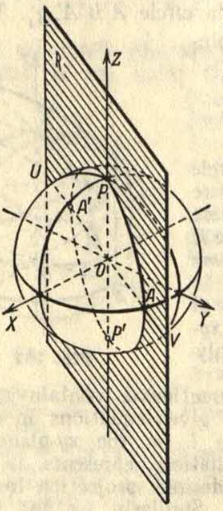


Fig. 188

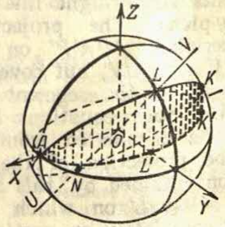


Fig. 189

There is no need to eliminate x in order to find the projection of the circle ALK on the yz-plane because one of the equations ($y=z$) does not contain x anyway. In the yz-plane, the equation $y=z$ represents the entire line UV, but the desired projection only covers a portion of it (the segment NL).

171. Algebraic Surfaces and Their Order

An algebraic equation of the second degree (in three unknowns x, y, z) is any equation of the form

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+Gx+Hy+Kz+L=0$$

where at least one of the six quantities A, B, C, D, E, F is nonzero. Algebraic equations of any degree are defined similarly (cf. Sec. 37).

If a surface S is represented in some rectangular coordinate system by an nth-degree equation, then in any other rectangular system of coordinates it will be represented by an equation of the same degree (cf. Sec. 37).

A surface represented by an nth-degree equation is called an *algebraic surface* of the nth order. Any surface of the first order is a plane. Surfaces of the second order (quadric surfaces) are considered in the following sections.

172. The Sphere

The second-degree equation

$$x^2+y^2+z^2=R^2 \tag{1}$$

represents (Sec. 167, Example 2) a sphere of radius R with centre at the coordinate origin. If the origin does not coincide with the centre of the sphere, then the latter is also represented by a second-degree equation, namely

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2 \tag{2}$$

where a, b, c are coordinates of the centre of the sphere (cf. Sec. 38).

The equation of the second degree

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+Gx+Hy+Kz+L=0 \tag{3}$$

represents a sphere only under the following conditions:

$$A=B=C, \tag{4}$$

$$D=0,\quad E=0,\quad F=0, \tag{5}$$

$$G^2+H^2+K^2-4AL>0 \tag{6}$$

(cf. Sec. 39). Under these conditions we have

$$a=-\frac{G}{2A},\ b=-\frac{H}{2A},\ c=-\frac{K}{2A},\quad R^2=\frac{G^2+H^2+K^2-4AL}{4A^2} \tag{7}$$

Example. The equation

$$x^2+y^2+z^2-2x-4y-4=0$$

$$(A=B=C=1,\quad D=E=F=0,\quad G=-2,\quad H=-4,\quad K=0,\quad L=-4)$$

represents a sphere. Completing the squares in the expressions x^2-2x and y^2-4y and adding the numbers 1^2, 2^2 to the right member to compensate, we obtain the equation

$$(x-1)^2+(y-2)^2+z^2=9$$

or $a=1$, $b=2$, $c=0$, $R=3$.

We find the same using formulas (7).

173. The Ellipsoid

The surface given by the equation[1])

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{1}$$

is called an *ellipsoid*[2]) (Fig. 190). The line of intersection $ABA'B'$ of ellipsoid (1) with the xy-plane is given (Sec. 169) by the system

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad z=0$$

It is equivalent to the system of equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad z=0$$

so that $ABA'B'$ is an ellipse with semiaxes $OA=a$, $OB=b$.

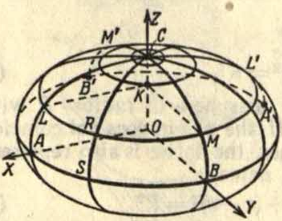


Fig. 190

Sections of ellipsoid (1) by the planes YOZ and XOZ are ellipses $M'CMB$ with semiaxes[3]) $OB=b$, $OC=c$ and $L'CLA$ with semiaxes $OA=a$, $OC=c$.

The section of the ellipsoid by the plane $z=h$ ($LML'M'$ in Fig. 190) is given by the system of equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{h^2}{c^2}, \tag{2}$$

$$z=h \tag{3}$$

---

[1]) Here and in the sequel, the letters a, b, c denote the lengths of certain segments so that the numbers a, b, c are positive.

[2]) The word ellipsoid comes from the Greek and means 'like an ellipse'. The ancient Greek geometers called ellipsoids of revolution (they did not consider any others) *spheroids* (i. e. sphere-like). The term is still used today.

[3]) Earlier (Sec. 41) the letter c was used to denote half the focal length $[c=\sqrt{a^2-b^2}$ so that $c<a]$. Here c has a different meaning and can assume any value.

However, if $|h| > c$, then Eq. (2) does not represent any locus ("imaginary elliptical cylinder"; cf. Sec. 58, Example 5). In this case the plane does not intersect the ellipsoid. For $|h| = c$, Eq. (2) is the axis OZ ($x=0$, $y=0$; cf. Sec. 58, Example 4). This means that the plane $z=c$ has one common point $C(0, 0, c)$ (point of tangency) with the ellipsoid; in the same way, the plane $z=-c$ touches the ellipsoid at the point $C'(0, 0, -c)$ (not indicated in the figure).

But if $|h| < c$, then the desired section is an ellipse with semiaxes

$$KL = a\sqrt{1-\frac{h^2}{c^2}}, \quad KM = b\sqrt{1-\frac{h^2}{c^2}} \tag{4}$$

proportional to a and b.

The dimensions of the sections diminish (all are similar) as one recedes from the xy-plane.

The same holds true for sections parallel to the yz- and zx-planes.

The point O is the centre of symmetry of the ellipsoid (1). The planes XOY, YOZ, XOZ are planes of symmetry, the axes OX, OY, and OZ are axes of symmetry.

General ellipsoid. If all three quantities a, b, c are different (i. e. not one of the ellipses $A'CA$, $B'CB$, ABA' becomes a circle), then the ellipsoid (1) is called *general* (triaxial). The ellipses $A'CA$, $B'CB$, $A'BA$ are called *principal*; their vertices [$A(a, 0, 0)$, $A'(-a, 0, 0)$, $B(0, b, 0)$, $B'(0, -b, 0)$, $C(0, 0, c)$, $C'(0, 0, -c)$] are called the *vertices* of the general ellipsoid. The segments AA', BB', CC' (axes of the principal ellipses) and also their lengths are termed the *axes of the ellipsoid*. If $a > b > c$, then $2a$ is the *major* axis, $2b$ the *mean* axis, and $2c$ the *minor* axis.

Ellipsoid of revolution. If any two of the quantities a, b, c (say a and b) are equal, then the corresponding principal ellipse $A'BA$ and all the sections parallel to it become circles. Any section CRS passing through the z-axis may be obtained by rotating the ellipse CLA about the z-axis, i. e. the ellipsoid is a surface of revolution (ellipses CLA, CRS, CMB, etc. are *meridians*, the circle $A'BA$ is the *equator*). An ellipsoid of this kind is called an *ellipsoid of revolution*. Its equation is of the form

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1 \tag{5}$$

If $a > c$, the ellipsoid of revolution is called *oblate* (Fig. 191*a*), if $a < c$, then *prolate* (Fig. 191*b*). In an ellipsoid of revolution the positions of two of its axes are indeterminate.

If $a=b=c$ the ellipsoid becomes a sphere, and the positions of all three axes become indeterminate.

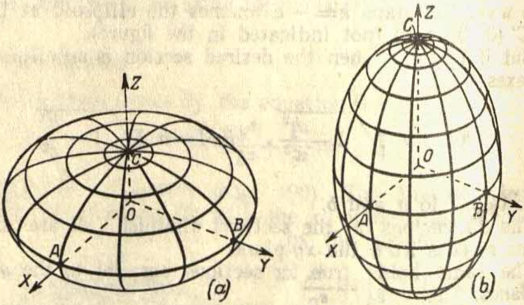


Fig. 191

Note 1. An ellipsoid of revolution may be defined as a surface obtained by uniform compression of a sphere towards its equator (cf. Sec. 40) An oblate ellipsoid of revolution is obtained when the coefficient of compression $k < 1$, prolate when $k > 1$.

A general ellipsoid may be defined as a surface obtained by the uniform compression of an ellipsoid of revolution towards its meridian.

Note 2. The ellipsoid is represented by Eq (1) if its coordinate axes coincide with the axes of the ellipsoid. In other cases, the ellipsoid is described by other equations

Example 1. Determine the surface defined by the equation

$$16x^2+3y^2+16z^2-48=0$$

Solution. The given equation is reduced to the form

$$\frac{x^2}{3}+\frac{y^2}{16}+\frac{z^2}{3}=1.$$

It defines a prolate ellipsoid of revolution with semiaxes $a=c=\sqrt{3}$, $b=4$, and with axis of rotation Oy.

Example 2. Find the surface described by the equation x^2-6x+ $+4y^2+9z^2+36z-99=0$.

Solution. Bring the equation to the form

$$(x-3)^2+4y^2+9(z+2)^2=144$$

Translate the origin to the point $(3, 0, -2)$; then (Sec. 166) we get the equation $x'^2+4y'^2+9z'^2=144$ or

$$\frac{x'^2}{144}+\frac{y'^2}{36}+\frac{z'^2}{16}=1$$

This equation is a general ellipsoid with semiaxes $a=12$, $b=6$, $c=4$; its centre lies in the point $(3, 0, -2)$ and the axes are parallel to the coordinate axes.

174. Hyperboloid of One Sheet

A surface described by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (1)$$

is called a *hyperboloid of one sheet* (Fig. 192).

The term hyperboloid [1] stems from the fact that there are hyperbolas among the sections of this surface. Such, for instance, are sections by the planes $x=0$ (*MNN'M'* in Fig. 192) and $y=0$ (*KLL'K'*). In their planes, these sections are defined by the equations

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad (2)$$

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1 \qquad (3)$$

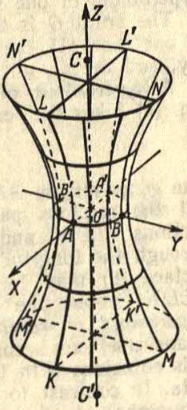


Fig. 192

The words "one sheet" stress the fact that the surface (1), in contrast to a *hyperboloid of two sheets* (see Sec. 175) is not separated into two "sheets", but is a single infinite tube stretching along the z-axis.

The plane

$$z=h \qquad (4)$$

for any value of h (cf. Sec. 173) yields, in a section with the surface (1), the ellipse [2]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{h^2}{c^2} \qquad (5)$$

---

[1] The term means "hyperbola-like".

[2] It is assumed here that $a \neq b$. If $a=b$ the ellipses (5) become circles; see Eq. (6) below.

with semiaxes $a\sqrt{1+\frac{h^2}{c^2}}$, $b\sqrt{1+\frac{h^2}{c^2}}$. All ellipses (5) are similar, their vertices lie on the hyperbolas (2) and (3); the dimensions of the ellipses increase as the section recedes from the *xy*-plane. A section by the *xy*-plane is an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{5'}$$

(*gorge* ellipse $ABA'B'$).

The hyperbolas (2) and (3), and also the ellipse (5′) are called *principal sections*, their vertices $A(a, 0, 0)$, $A'(-a, 0, 0)$, $B(0, b, 0)$, $B'(0, -b, 0)$ are *vertices* of the hyperboloid of one sheet. The segments $AA'=2a$, $BB'=2b$ (real axes of the principal hyperbolas) and, frequently, the straight lines AA', BB' are called *transverse axes*. The segment $CC'=2OC=2c$ laid off on the *z*-axis (the imaginary axis of each of the principal hyperbolas) is called the *longitudinal axis* of the hyperboloid of one sheet.

The point O is the centre of symmetry of the hyperboloid of one sheet (1), the *xy*-, *yz*-, *zx*-planes are planes of symmetry, and the *x*-, *y*-, and *z*-axes are axes of symmetry.

A hyperboloid of revolution of one sheet. If $a=b$, then Eq. (1) takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=1 \tag{6}$$

The gorge ellipse $ABA'B'$ becomes a *gorge circle* of radius a. All the sections parallel to XOY are likewise circles. The sections $KLL'K'$ and $MNN'M'$ (and, in general, all sections through the longitudinal axis) become equal hyperbolas, and surface (6) may be formed by rotation of the hyperbola $KLL'K'$ about the longitudinal axis. Surface (6) is called a *hyperboloid of revolution of one sheet*. The positions of two (transverse) axes become indeterminate, the third (longitudinal) axis coincides with the imaginary axis of the rotating hyperbola. In contrast to the hyperboloid of revolution for $a=b$, a hyperboloid of one sheet (1) for $a \neq b$ is termed *general* (triaxial).

Note. A hyperboloid of revolution of one sheet may be defined as a surface generated by the revolution of a hyperbola about its imaginary axis, a triaxial hyperboloid of one sheet, as the surface obtained by uniform compression of a hyperboloid of revolution of one sheet towards the plane of any one of the meridians.

Example. Determine the type of surface

$$x^2-4y^2-4z^2+16=0$$

Solution. This equation is brought to the form

$$-\frac{x^2}{4^2}+\frac{y^2}{2^2}+\frac{z^2}{2^2}=1$$

It represents a hyperboloid of revolution of one sheet with centre in the point (0, 0, 0) and axis of revolution OX (since the coefficient of x^2 is negative). The radius of the gorge circle $r=2$, the longitudinal semiaxis is equal to 4.

175. Hyperboloid of Two Sheets

The surface described by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1 \qquad (1)$$

is called a *hyperboloid of two sheets* (Fig. 193).

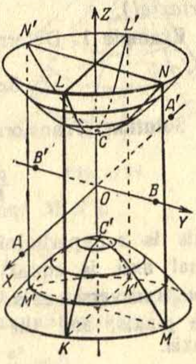


Fig. 193

The sections by the xz- and yz-planes are given by the equations

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}=1, \qquad (2)$$

$$\frac{z^2}{c^2}-\frac{y^2}{b^2}=1 \qquad (3)$$

These are hyperbolas ($KK'L'L$ and $MM'N'N$ in Fig. 193). For each of them the z-axis is a *real* axis (cf. Sec. 174).

The planes $z=h$ do not meet hyperboloid (1) for $|h|<c$ (cf. Sec. 174). For $h=\pm c$, they touch the hyperboloid at the points $C(0, 0, c)$ and $C'(0, 0, -c)$. For $|h|>c$, the sections are ellipses[1]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{h^2}{c^2}-1 \qquad (4)$$

which are similar to one another ($KMK'M'$, $LNL'N'$, and others). Their dimensions increase as they recede from the xy-plane.

Thus, the surface (1) consists of two separate sheets, whence the name: *hyperboloid of two sheets.*

The hyperbolas (2) and (3) are called *principal sections*, their common vertices C and C' are the *vertices* of the hyperboloid of two sheets, their real axis CC' is the *longitudinal axis* of the hyperboloid of two sheets, and the imaginary axes $AA'=2a$ and $BB'=2b$ are called the *transverse axes of symmetry*.

---

[1] See footnote 2 on p. 213.

A hyperboloid of two sheets has a centre O, axes of symmetry OX, OY, OZ and planes of symmetry XOY, YOZ, ZOX. The two sheets of the hyperboloid are symmetric to each other about the xy-plane.

Hyperboloid of revolution of two sheets. Eq. (1), for $a=b$, takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=-1$$

and defines a surface generated by the revolution of a hyperbola about its real axis. It is called a *hyperboloid of revolution of two sheets*. A hyperboloid of two sheets with unequal transverse semiaxes a and b is called *general (triaxial)*.

Example 1. Determine the type of surface

$$3x^2-5y^2-2z^2-30=0$$

Solution. Transform this equation to

$$\frac{y^2}{6}+\frac{z^2}{15}-\frac{x^2}{10}=-1$$

This is a hyperboloid of two sheets (triaxial). The longitudinal axis is equal to $\sqrt{10}$ and coincides with the x-axis; one transverse axis is equal to $\sqrt{6}$ and is directed along the y-axis, and the other is $\sqrt{15}$ and directed along the z-axis.

Example 2. The equation

$$x^2-y^2-z^2=-1$$

is a hyperboloid of *one sheet* (not two sheets). Although we have -1 in the right-hand member, and not $+1$, there are two negative terms in the left-hand member. Representing the equation in the form $y^2+z^2-x^2=1$, we see that the hyperboloid is generated by the revolution of an equilateral hyperbola about its imaginary axis (which coincides with the x-axis).

176. Quadric Conical Surface

A *conical surface* is any surface generated by the motion of a straight line *(generatrix)* passing through a fixed point (*vertex* of the conical surface). Any line (not passing through the vertex) which intersects the generatrix in any of its positions is called the *directrix*.

The surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \qquad (1)$$

which, as shown below, is conical, is called a *quadric conical surface* (Fig. 194).

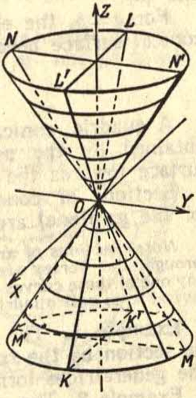


Fig. 194

A section by the *xz*-plane ($y=0$) is given by the equation

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0$$

i. e.

$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=0 \qquad (2)$$

This is a pair of straight lines (KL and $K'L'$) passing through the origin (Sec. 58). The section by the *yz*-plane yields a pair of straight lines (MN and $M'N'$):

$$\left(\frac{y}{b}+\frac{z}{c}\right)\left(\frac{y}{b}-\frac{z}{c}\right)=0 \qquad (3)$$

A section by any other plane $y=kx$ passing through the *z*-axis is given (Sec. 169) by the system of equations

$$y=kx, \quad \frac{x^2}{a^2}+\frac{k^2x^2}{b^2}-\frac{z^2}{c^2}=0 \qquad (4)$$

This too is a pair of straight lines:

$$y=kx, \quad x\sqrt{\frac{1}{a^2}+\frac{k^2}{b^2}}+\frac{z}{c}=0 \qquad (5)$$

and

$$y=kx, \quad x\sqrt{\frac{1}{a^2}+\frac{k^2}{b^2}}-\frac{z}{c}=0 \qquad (6)$$

passing through the origin. Hence, surface (1) is conical and point O is its vertex.

The section of cone (1) by any plane $z=h$ (for $h \neq 0$) is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{h^2}{c^2} \tag{7}$$

It degenerates into a point $O(0, 0, 0)$ for $h=0$. All the ellipses (7) are similar, their vertices lie on the sections (2) and (3).

For $a=b$, the ellipses (7) become circles and the quadric conical surface becomes a circular conical surface:

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=0 \tag{8}$$

A quadric conical surface may be defined as a surface obtained by the uniform compression of a circular conical surface towards the plane of the axial section.

Sections of cone (1) by planes parallel to the xz-plane (or the yz-plane) are hyperbolas.

Note. Sections of any quadric conical surface by planes not passing through the vertex are circles,[1] ellipses, hyperbolas, and parabolas. Any one of these curves may be taken for the directrix. It is therefore advisable to call quadric conical surfaces "elliptical".

Example 1. The equation $x^2+y^2=z^2$ is a circular cone; the section by the xz-plane is a pair of straight lines $x=\pm z$. The generatrices form an angle of 45° with the axis.

Example 2. The equation $-x^2+9y^2+3z^2=0$ is a (noncircular) quadric conical surface. A section by any plane $z=h\,(h \neq 0)$ is the hyperbola $x^2-9y^2=3h^2$; for $h=0$ it becomes a pair of generatrices. The same applies to sections $y=l$. The sections $x=d\,(d \neq 0)$ are ellipses.

177. Elliptic Paraboloid

The surface given by the equation

$$z=\frac{x^2}{2p}+\frac{y^2}{2q} \tag{1}$$

($p>0$, $q>0$) is called an *elliptic paraboloid* (Fig. 195).

Sections by the xz- and yz-planes (principal sections) are parabolas (AOA' BOB'):

$$x^2=2pz, \tag{2}$$

$$y^2=2qz \tag{3}$$

both concave "up".

---

[1] A circular conical surface has one system of parallel circular sections, a noncircular conical surface has two.

The plane $z=0$ touches the paraboloid at the point O, the planes $z=h$ for $h>0$ intersect the paraboloid along similar ellipses

$$\frac{x^2}{2p}+\frac{y^2}{2q}=h \tag{4}$$

with semiaxes $\sqrt{2ph}$, $\sqrt{2qh}$. For $h<0$ these planes do not meet the paraboloid.

The elliptic paraboloid does not have a centre of symmetry; it is symmetric with respect to the xz- and yz-planes and the z-axis. The line OZ is called the *axis* of the elliptic paraboloid, the point O is its *vertex*, and p and q are *parameters*.

For $p=q$, parabolas (2) and (3) become equal, the ellipses (4) turn into circles and the paraboloid (1) becomes a surface generated by the revolution of a parabola about its axis (*paraboloid of revolution*).[1]

The elliptic paraboloid may be defined as a surface generated by uniform compression of a paraboloid of revolution towards one of its meridians.

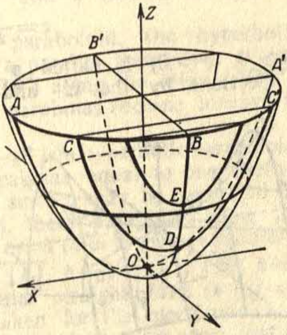


Fig. 195

Example. The surface $z=x^2+y^2$ is a paraboloid of revolution generated by the revolution of the parabola $z=x^2$ about its axis (z-axis). The surface $x=y^2+z^2$ is the same paraboloid situated differently (the axis of revolution coincides with OX).

Note. A section of an elliptic paraboloid by the plane $y=f$ yields the curve $z=\frac{x^2}{2p}+\frac{f^2}{2q}$ (CDC'); this is a parabola equal (Sec. 50) to the parabola AOA' $\left(z=\frac{x^2}{2p}\right)$; its axis is also directed "upwards", and point $D\left(0,\ f,\ \frac{f^2}{2q}\right)$ is the vertex. The coordinates of point D satisfy the equations

---

[1] Parabolic reflectors are in the shape of a paraboloid of revolution (they convert a beam of light emanating from the focus into parallel rays).

$x=0$, $y^2=2qz$, i. e. D lies on the parabola BOB'. Hence, an elliptic paraboloid is a surface generated by the parallel translation of a parabola (AOA') in which its vertex moves along another parabola (BOB'). The planes of the fixed and moving parabolas are perpendicular and the axes are in the same direction.

178. Hyperbolic Paraboloid

The surface defined by the equation

$$z=\frac{x^2}{2p}-\frac{y^2}{2q} \tag{1}$$

($p > 0$, $q > 0$) is called a *hyperbolic paraboloid* (Fig. 196).

Sections by the xz- and yz-planes *(principal sections)* are the parabolas (AOA', BOB')

$$x^2=2pz, \tag{2}$$

$$y^2=-2qz \tag{3}$$

Unlike the principal sections of the elliptic paraboloid (Sec. 177), the parabolas (2) and (3) are concave in *opposite directions* (the parabola AOA' is concave up, the parabola BOB' is concave down). The surface (1) is saddle-shaped.

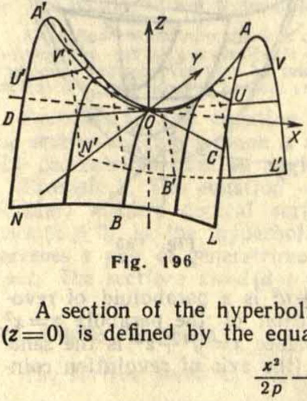


Fig. 196

A section of the hyperbolic paraboloid (1) by the xy-plane ($z=0$) is defined by the equation

$$\frac{x^2}{2p}-\frac{y^2}{2q}=0 \tag{4}$$

This is a *pair of straight lines* [1] OD, OC (Sec. 58, Example 1).

The planes $z=h$, parallel to the xy-plane, intersect the hyperbolic paraboloid along hyperbolas:

$$\frac{x^2}{2p}-\frac{y^2}{2q}=h,\ z=h \tag{5}$$

For $h > 0$, the real axis of these hyperbolas (for example, the hyperbola $UVV'U'$) is parallel to the x-axis; for $h < 0$

---

[1] The hyperbolic paraboloid has an infinity of straight lines; see Sec. 180.

(hyperbola $LNN'L'$), the real axis is parallel to the y-axis. All the hyperbolas (5) lying to one side of the xy-plane are similar; they are pairwise conjugate (Sec. 47) to the hyperbolas (5) lying on the other side of the xy-plane.

The hyperbolic paraboloid does not have a centre; it is symmetric with respect to the xz- and yz-planes and about the z-axis. The straight line OZ is called the *axis* of the hyperbolic paraboloid, the point O is its *vertex*, and p and q are *parameters*.

Note 1. The hyperbolic paraboloid is not a surface of revolution for any value of p and q (unlike the quadric surfaces discussed above).

Note 2. Like the elliptic paraboloid, the hyperbolic paraboloid may be formed by a parallel translation of one of the principal sections (say BOB') along the other (AOA'). But then the fixed and moving parabolas become concave in opposite directions.

Example. The surface $z=x^2-y^2$ is a hyperbolic paraboloid; both principal sections are parabolas equal to one another but in opposite directions. The surface may be generated by a parallel translation of one of these parabolas along the other. The section by the plane $z=h$ $(h \neq 0)$ is an equilateral hyperbola with semiaxes $a=\sqrt{|h|}$, $b=\sqrt{|h|}$. For $h=0$ it becomes a pair of perpendicular straight lines ($x+y=0$, $x-y=0$). If these lines are taken for the coordinate axes (OX', OY'), then the hyperbolic paraboloid under consideration will be represented (Sec. 36) by the equation $z=2x'y'$.

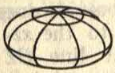
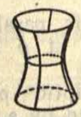
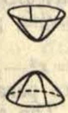
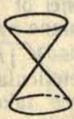
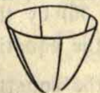
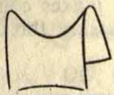
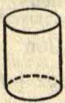
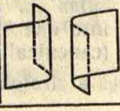
Generally speaking, the equation $z=\frac{xy}{a}$ defines the same hyperbolic paraboloid as the equation $z=\frac{x^2}{2a}-\frac{y^2}{2a}$; only in the former case, the x- and y-axes coincide with the rectilinear generatrices (Sec. 180) passing through the vertex.

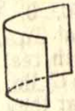
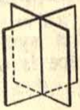
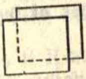
179. Quadric Surfaces Classified

Any second-degree equation

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+Gx+Hy+Kz+L=0$$

can, with the aid of formulas for transforming coordinates (Sec. 166), be converted into one of the 17 equations given below called *standard* (canonical). Then, the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ (No. 14) defines a straight line ($x=0$, $y=0$) and

No.	Standard Equation	Drawing (Schematic)	Type of Surface	Section
1	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$		Ellipsoid (in particular, ellipsoid of revolution and sphere)	173
2	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$		Hyperboloid of one sheet	174
3	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$		Hyperboloid of two sheets	175
4	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$		Quadric conical surface	176
5	$z=\frac{x^2}{2p}+\frac{y^2}{2q}$		Elliptic paraboloid	177
6	$z=\frac{x^2}{2p}-\frac{y^2}{2q}$		Hyperbolic paraboloid	178
7	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$		Elliptic cylinder	168
8	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$		Hyperbolic cylinder	168

No.	Standard Equation	Drawing (Schematic)	Type of Surface	Section
9	$y^2=2px$		Parabolic cylinder	168
10	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$		Pair of intersecting planes	
11	$\frac{x^2}{a^2}=1$		Pair of parallel planes	
12	$x^2=0$		Pair of coincident planes	
13	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$		Imaginary quadric conical surface with real vertex (0 0. 0)	
14	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$		Pair of imaginary planes (intersecting along real straight line)	
15	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$		Imaginary ellipsoid	
16	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$		Imaginary elliptic cylinder	
17	$\frac{x^2}{a^2}=-1$		Pair of imaginary parallel planes	

not a surface. However, we say that it defines a *pair of imaginary surfaces* (intersecting along a real straight line) (cf. Sec. 58, Example 4). The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$ (No. 13) defines only one point (0, 0, 0). However, (by similarity with Eq. No. 4) we say that Eq. No. 13 defines an *imaginary quadric conical surface* (with real vertex).

Equations Nos. 15, 16, 17 do not represent any geometric image. However, we say that they correspond to an *imaginary ellipsoid* (cf. No. 1), an *imaginary elliptic cylinder* (cf. No. 7) and a *pair of imaginary parallel planes* (cf. No. 11), respectively.

Taking advantage of this symbolic terminology, we can say that any quadric surface is one of the 17 surfaces given in the classification.

180. Straight-Line Generatrices of Quadric Surfaces

A surface is called *ruled* if it can be generated by the motion of a straight line *(generatrix)*. Of the quadric surfaces, the cylinder and quadric conical surface and also the hyperboloid of one sheet and the hyperbolic paraboloid are ruled surfaces.

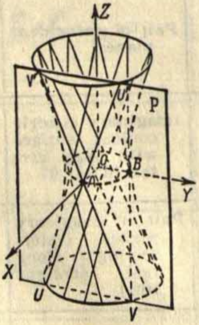


Fig. 197

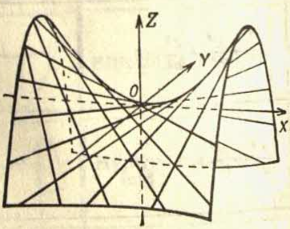


Fig. 198

Both in the hyperboloid of one sheet (Fig. 197) and the hyperbolic paraboloid (Fig. 198), two straight-line generatrices pass through each point. In Fig. 197, through point *A* pass the generatrices *UU'* and *VV'* through point *V*, the generatrices *VA* and *VB*.

There are no straight-line (real) generatrices in the case of the ellipsoid, hyperboloid of two sheets and elliptic paraboloid.

Example. A section of the hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \tag{1}$$

by the plane $x=a$ (plane P in Fig. 197) is defined by the equation $\frac{a^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, i.e.

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \tag{2}$$

This is a pair of straight lines (UU' and VV'). They pass through the vertex $A(a, 0, 0)$ of the gorge ellipse. In exactly the same way, through the vertex $B(0, b, 0)$ pass a pair of straight-line generatrices

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0, \quad y=b \tag{3}$$

A hyperboloid of revolution of one sheet ($a=b$) may be generated[1] by revolution of the straight line UU' (or VV') about the z-axis.

Note. The ruled-surface nature of a hyperboloid of one sheet was utilized by engineer V. Shukhov in the construction of what is called the "Shukhov Tower" of Moscow which for years was used as the Moscow radio and television tower. It was constructed out of steel strips arranged along rectilinear generatrices of a hyperboloid of one sheet. The strips were riveted together at the points of intersection of the two systems of generatrices. Shukhov's structure possesses high strength, though a relatively small amount of material was used in the construction.

181. Surfaces of Revolution

Let L be a line lying in the xz-plane. The equation of a surface generated by rotation of L about the z-axis is obtained from the equation of the line L by replacing x by $\sqrt{x^2+y^2}$.

Example 1. Let a straight line $z=2x$ lying in the plane $y=0$ (straight line PP' in Fig. 199) be rotated about OZ. Then the equation of the conical surface generated by rota-

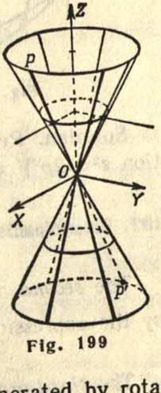


Fig. 199

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[1] If two matches not lying in the same plane are pierced with a pin, and if, taking the end of one of the matches, we rapidly revolve the whole model about it, the other match will clearly sweep out a hyperboloid of one sheet.

tion of the straight line PP' is of the form $z=2\sqrt{x^2+y^2}$, or $x^2+y^2-\frac{z^2}{4}=0$ (cf. Sec. 176).

Similar rules hold when L lies in another coordinate plane and the axis of revolution is some other coordinate axis.

Example 2. Find the equation of a surface generated by rotation of the parabola $y^2=2px$ (LOL' in Fig. 200) about the x-axis.

Solution. Replacing y by $\sqrt{y^2+z^2}$, i.e. y^2 by y^2+z^2, we get $y^2+z^2=2px$ (a paraboloid of revolution about the x-axis).

Example 3. Find the equation of a surface generated by rotation of the parabola $z^2=2px$ (KOK' in Fig. 201) about the z-axis.

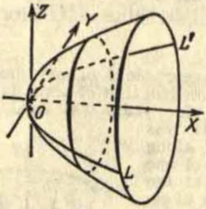


Fig. 200

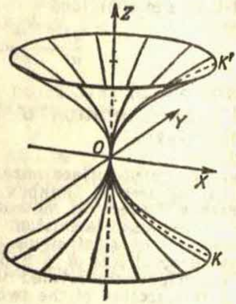


Fig. 201

Solution. Replacing x by $\sqrt{x^2+y^2}$, we obtain the equation $z^2=2p\sqrt{x^2+y^2}$ or $z^4=4p^2(x^2+y^2)$ (a quartic surface).

182. Determinants of Second and Third Order

The *second-order determinant* $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is (Sec. 12) given by the expression

$$a_1b_2-a_2b_1$$

The *third-order determinant*

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{1}$$

is (Sec. 118) given by the expression

$$a_1b_2c_3-a_1b_3c_2+b_1c_2a_3-b_1c_3a_2+c_1a_2b_3-c_1a_3b_2 \tag{2}$$

or, what is the same,

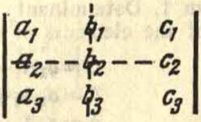
$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \tag{3}$$

The letters a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 are called *elements of the determinant*.

Minors. The determinants $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ of formula (3) are called *minors* (from the Latin, less) of the elements a_1, b_1, c_1.

A minor of any element is the determinant obtained from the given determinant by deleting the row and column in which the element stands.

Examples. The minor of the element b_2 of determinant (1) is the determinant $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$:



The minor of element b_3 is $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, the minor of element c_3 is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Note. In the second-order determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, element b_2 is the minor of element a_1; it may be considered a "first-order determinant". Element b_2 is obtained from a second-order determinant by striking out the upper row and the left column. Similarly, element b_1 is the minor of element a_2, etc.

Cofactor. In formula (3) elements a_1, b_1, c_1 are multiplied by $+\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, $+\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$. These expressions are called the *cofactors* of the elements a_1, b_1, c_1.

Generally, the *cofactor* of an element is its minor with its sign or the opposite sign prefixed in accordance with the following rule:

If the sum of the position numbers of the column and the row in which the element stands is an even number, then the minor has its own sign, if odd, then the sign is reversed.

The cofactors of the elements a_1, b_1, c_1, a_2 and so forth will be denoted, respectively, by A_1, B_1, C_1, A_2, etc.

Example 1. Element b_1 of determinant (1) lies at the intersection of the first row and the second column. Since $1+2=3$ is an odd number, $B_1=-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$.

Example. Find the cofactor of the element c_2.

Solution. Striking out the second row and the third column, we find the minor $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$ of the element c_2. The number of the row of this element is 2, the number of the column, 3. The sum $2+3$ is an odd number. Therefore $C_2=-\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

Example 3. In determinant (1) the cofactor B_2 of element b_2 is $+\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ ($2+2$ is an even number).

Theorem 1. Determinant (1) is equal to the sum of the products of the elements of some row by their cofactors, i.e.

$$\Delta=a_1A_1+b_1B_1+c_1C_1, \tag{4}$$

$$\Delta=a_2A_2+b_2B_2+c_2C_2, \tag{5}$$

$$\Delta=a_3A_3+b_3B_3+c_3C_3 \tag{6}$$

Formula (4) is identical to (3), formulas (5) and (6) are verified by direct computation.

Theorem 2. Determinant (1) is equal to the sum of the products of the elements of some column by their cofactors, i.e.

$$\Delta=a_1A_1+a_2A_2+a_3A_3, \tag{7}$$

$$\Delta=b_1B_1+b_2B_2+b_3B_3, \tag{8}$$

$$\Delta=c_1C_1+c_2C_2+c_3C_3 \tag{9}$$

These two theorems facilitate computing a determinant that has zeros as some of the elements.

Example 4. To evaluate the determinant

$$\Delta=\begin{vmatrix} 2 & 5 & -2 \\ 3 & 8 & 0 \\ 1 & 3 & 5 \end{vmatrix}$$

it is convenient to use (5) or (9).

Formula (5) yields

$$\Delta=-3\begin{vmatrix} 5 & -2 \\ 3 & 5 \end{vmatrix}+8\begin{vmatrix} 2 & -2 \\ 1 & 5 \end{vmatrix}=-3\cdot 31+8\cdot 12=3$$

Formula (9) yields

$$\Delta=-2\begin{vmatrix}3 & 8\\ 1 & 3\end{vmatrix}+5\begin{vmatrix}2 & 5\\ 3 & 8\end{vmatrix}=-2\cdot 1+5\cdot 1=3$$

Example 5. In evaluating the determinant

$$\Delta=\begin{vmatrix}4 & -3 & 2\\ 6 & 11 & 1\\ 0 & 3 & 0\end{vmatrix}$$

it is best to use (6):

$$\Delta=-3\begin{vmatrix}4 & 2\\ 6 & 1\end{vmatrix}=-3\cdot -8=24$$

183. Determinants of Higher Order

The *fourth-order determinant*

$$\Delta=\begin{vmatrix}a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\\ a_4 & b_4 & c_4 & d_4\end{vmatrix} \tag{1}$$

is the expression

$$\Delta=a_1A_1+b_1B_1+c_1C_1+d_1D_1 \tag{2}$$

where A_1, B_1, C_1, D_1 are cofactors (Sec. 182) of the elements a_1 b_1, c_1, d_1, i.e.

$$\left.\begin{aligned} A_1&=\begin{vmatrix}b_2 & c_2 & d_2\\ b_3 & c_3 & d_3\\ b_4 & c_4 & d_4\end{vmatrix} \qquad B_1=-\begin{vmatrix}a_2 & c_2 & d_2\\ a_3 & c_3 & d_3\\ a_4 & c_4 & d_4\end{vmatrix}\\ C_1&=\begin{vmatrix}a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\\ a_4 & b_4 & d_4\end{vmatrix} \qquad D_1=-\begin{vmatrix}a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\\ a_4 & b_4 & c_4\end{vmatrix}\end{aligned}\right\} \tag{3}$$

Example 1. Evaluate the determinant

$$\Delta=\begin{vmatrix}6 & 3 & 0 & 3\\ 4 & 4 & 2 & 1\\ 0 & 4 & 4 & 2\\ 7 & 7 & 8 & 5\end{vmatrix}$$

Solution.

$$A_1=\begin{vmatrix}4 & 2 & 1\\ 4 & 4 & 2\\ 7 & 8 & 5\end{vmatrix}=8, \qquad B_1=-\begin{vmatrix}4 & 2 & 1\\ 0 & 4 & 2\\ 7 & 8 & 5\end{vmatrix}=-16,$$

$$D_1=-\begin{vmatrix}4 & 4 & 2\\ 0 & 4 & 4\\ 7 & 7 & 8\end{vmatrix}=-72$$

(since $c_1=0$, it follows that C_1 does not need to be computed):

$$\Delta=6\cdot8+3\,(-16)+3\,(-72)=-216$$

Theorems 1 and 2 (Sec. 182) hold true for fourth-order determinants. These two theorems are united by the following theorem.

Theorem. A determinant is equal to the sum of the products of the elements of any row (or any column) by their cofactors, i.e.

$$\left.\begin{aligned}
&\Delta=a_1A_1+b_1B_1+c_1C_1+d_1D_1,\\
&\Delta=a_2A_2+b_2B_2+c_2C_2+d_2D_2,\\
&\dots\dots\dots\dots\dots\\
&\Delta=a_1A_1+a_2A_2+a_3A_3+a_4A_4,\\
&\Delta=b_1B_1+b_2B_2+b_3B_3+b_4B_4,\\
&\dots\dots\dots\dots\dots
\end{aligned}\right\}\tag{4}$$

The first of formulas (4) coincides with formula (2) taken as the definition. The remaining may be verified by direct computation, though this is a cumbersome procedure. There are shorter ways.

Example 2. Evaluate the determinant of Example 1 by expanding it in terms of the elements of the second column. We have

$$\Delta=3B_1+4B_2+4B_3+7B_4$$

where

$$B_1=-\begin{vmatrix}4&2&1\\0&4&2\\7&8&5\end{vmatrix}=-16,\qquad B_2=\begin{vmatrix}6&0&3\\0&4&2\\7&8&5\end{vmatrix}=-60,$$

$$B_3=-\begin{vmatrix}6&0&3\\4&2&1\\7&8&5\end{vmatrix}=-66,\qquad B_4=\begin{vmatrix}6&0&3\\4&2&1\\0&4&2\end{vmatrix}=48$$

so that $\Delta=3\cdot(-16)+4\cdot(-60)+4\cdot(-66)+7\cdot48=-216$.

Example 3. Evaluate the same determinant by expanding it in terms of the elements of the third row:

$$\Delta=0\cdot A_3+4B_3+4C_3+2D_3=$$

$$=-4\begin{vmatrix}6&0&3\\4&2&1\\7&8&5\end{vmatrix}+4\begin{vmatrix}6&3&3\\4&4&1\\7&7&5\end{vmatrix}-2\begin{vmatrix}6&3&0\\4&4&2\\7&7&8\end{vmatrix}=-216$$

The *fifth-order determinant*

$$\Delta=\begin{vmatrix}a_1&b_1&c_1&d_1&e_1\\a_2&b_2&c_2&d_2&e_2\\a_3&b_3&c_3&d_3&e_3\\a_4&b_4&c_4&d_4&e_4\\a_5&b_5&c_5&d_5&e_5\end{vmatrix}\tag{5}$$

is the expression

$$\Delta=a_1A_1+b_1B_1+c_1C_1+d_1D_1+e_1E_1\tag{6}$$

where A_1, B_1, C_1, D_1, E_1 are cofactors of the elements a_1, b_1, c_1, d_1, e_1, these cofactors are themselves fourth-order determinants.

Similarly, we define a sixth-order determinant in terms of a determinant of the fifth order, etc.

The theorem of this section holds true for determinants of any order.

184. Properties of Determinants

1. The magnitude of a determinant does not change if each of the rows is substituted by a column of the same position number.

Example 1.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example 2.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. If any two rows or any two columns are interchanged, the absolute value of a determinant remains unaltered, while the sign is reversed.

Example 3.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

(second and third rows interchanged, cf. Sec. 117, Item 1)

Example 4.

$$\begin{vmatrix} 2 & 1 & 5 \\ 3 & 6 & 0 \\ -4 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 5 & 1 & 2 \\ & 6 & 3 \\ 1 & 2 & -4 \end{vmatrix}$$

(first and third columns interchanged)

3. A determinant, the elements of one row (or column) of which are respectively proportional to the elements of the other row (column), is zero. In particular, a determinant with two identical rows (columns) is equal to zero.

Example 5.

$$\begin{vmatrix} 2 & 2 & 2 \\ -5 & -3 & -3 \\ 0 & -1 & -1 \end{vmatrix} = 0$$

(second and third columns are the same)

Example 6.

$$\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ 3a & 3a' & 3a'' \end{vmatrix} = 0$$

(elements of third row are proportional to elements of first row; cf. Sec. 117, Items 1, 3, 4)

4. A common factor of all the elements of one row (or of one column) may be taken outside the sign of the determinant.

Example 7.

$$\begin{vmatrix} ma & ma' & ma'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} = m \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} \quad \text{(cf. Sec. 117, Item 3)}$$

5. If every element of some column (row) is the sum of two terms, then the determinant is equal to the sum of two determinants: one containing only the first term in place of each sum, the other only the second term (the remaining elements of both determinants are the same as in the given determinant).

Example 8.

$$\begin{vmatrix} a_1 & b_1+c_1 & d_1 \\ a_2 & b_2+c_2 & d_2 \\ a_3 & b_3+c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$$

(cf. Sec. 117, Item 2).

6. If to all the elements of some column we add terms proportional to the corresponding elements of another column, then the new determinant is equal to the old one. The same holds true for rows.

This follows from Items 5 and 3.

Example 9. The determinant $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -3 \\ 5 & 0 & 2 \end{vmatrix}$ is equal to 12. Let us add the elements of the second row to the elements of the first row. We get $\begin{vmatrix} 6 & 0 & 0 \\ 4 & 1 & -3 \\ 5 & 0 & 2 \end{vmatrix}$. This determinant is also equal to 12, but is evaluated in simpler fashion (two terms are zero in the expansion in terms of elements of the first row).

Example 10. To evaluate the determinant

$$\begin{vmatrix} 4 & 2 & 3 \\ -1 & 3 & 5 \\ 6 & 3 & -1 \end{vmatrix}$$

add the elements of the second column multiplied by -2 to the elements of the first column. This yields $\begin{vmatrix} 0 & 2 & 3 \\ -7 & 3 & 5 \\ 0 & 3 & -1 \end{vmatrix}$.

This determinant is readily evaluated by expanding the first column in terms of its elements [Sec. 182, Formula (7)]. We have

$$7\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -77$$

Example 11. To evaluate the determinant

$$\begin{vmatrix} 6 & 3 & 0 & 3 \\ 4 & 4 & 2 & 1 \\ 0 & 4 & 4 & 2 \\ 7 & 7 & 8 & 5 \end{vmatrix}$$

subtract the elements of the third column from the elements of the second column. This yields

$$\begin{vmatrix} 6 & 3 & 0 & 3 \\ 4 & 2 & 2 & 1 \\ 0 & 0 & 4 & 2 \\ 7 & -1 & 8 & 5 \end{vmatrix}$$

Now subtract the elements of the fourth column multiplied by 2 from the elements of the third column. This gives

$$\begin{vmatrix} 6 & 3 & -6 & 3 \\ 4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 7 & -1 & -2 & 5 \end{vmatrix}$$

Expanding in terms of the elements of the third row, we get (as in Example 1, Sec. 183):

$$-2\begin{vmatrix} 6 & 3 & -6 \\ 4 & 2 & 0 \\ 7 & -1 & -2 \end{vmatrix} = -216$$

185. A Practical Technique for Computing Determinants

The device explained below is particularly convenient when the elements of the determinant are integers.

Pick a row (or column) in terms of the elements of which we shall carry out the expansion. It is desirable to have zero. The device is calculated to create fresh zeros in the chosen row. To do this, use Property 6, Sec. 184.

Example 1. Evaluate the determinant

$$\Delta=\begin{vmatrix}2 & 5 & 3\\ 0 & 6 & 2\\ 7 & 3 & -1\end{vmatrix}$$

Expand it in terms of the elements of the second row (it has a zero). We then establish another zero (in place of Element 6). To do this, subtract tripled elements of the third column from the elements of the second column. This yields

$$\Delta=\begin{vmatrix}2 & -4 & 3\\ 0 & 0 & 2\\ 7 & 6 & -1\end{vmatrix}=-2\begin{vmatrix}2 & -4\\ 7 & 6\end{vmatrix}=-80$$

Now expand in terms of the elements of the first column where there is one zero. In place of Element 7 we create another zero by subtracting from elements of the third row elements of the first row multiplied by $\frac{7}{2}$, which gives

$$\Delta=\begin{vmatrix}2 & 5 & 3\\ 0 & 6 & 2\\ 0 & -\frac{29}{2} & -\frac{23}{2}\end{vmatrix}=-\frac{1}{2}\begin{vmatrix}2 & 5 & 3\\ 0 & 6 & 2\\ 0 & 29 & 23\end{vmatrix}=$$

$$=-\frac{1}{2}\cdot 2\begin{vmatrix}6 & 2\\ 29 & 23\end{vmatrix}=-80$$

Note. One could foresee that the first way would be more convenient since in the second row Element 6 is a multiple of Element 2, whereas in the first column Element 7 is not a multiple of Element 2. It is desirable for all elements in a chosen row (or column) to be multiples of one element. If one of the elements is equal to 1 or −1, then we should take the row or column with that element.

Example 2. Evaluate the determinant

$$\Delta=\begin{vmatrix}-1 & -2 & 4 & 1\\ 2 & 3 & 0 & 6\\ 2 & -2 & 1 & 4\\ 3 & 1 & -2 & -1\end{vmatrix}$$

We choose the third column (it has a zero and a one). To create a zero in place of Element 4, subtract quadrupled elements of the third row (which has Element 1 of the chosen column) from elements of the first row. The first row will become

$$-9 \quad 6 \quad 0 \quad -15$$

In order to turn element -2 into a zero in the third column, add doubled elements of the third row to elements of the fourth. This fourth row will then take the form

$$7 \quad -3 \quad 0 \quad 7$$

Now, expanding in terms of elements of the third column, we have

$$\Delta=\begin{vmatrix} -9 & 6 & 0 & -15 \\ 2 & 3 & 0 & 6 \\ 2 & -2 & 1 & 4 \\ 7 & -3 & 0 & 7 \end{vmatrix}=1\begin{vmatrix} -9 & 6 & -15 \\ 2 & 3 & 6 \\ 7 & -3 & 7 \end{vmatrix}$$

In the third-order determinant all elements of the second column are multiples of Element -3. Therefore, we add elements of the third row (which contains Element -3) to the elements of the second, and then (first doubling them) add them to the elements of the first row. This yields

$$\Delta=\begin{vmatrix} 5 & 0 & -1 \\ 9 & 0 & 13 \\ 7 & -3 & 7 \end{vmatrix}=-\begin{vmatrix} 5 & -1 \\ 9 & 13 \end{vmatrix}\cdot(-3)=222$$

Example 3. Evaluate the determinant

$$\Delta=\begin{vmatrix} 3 & 7 & -2 & 4 \\ -3 & -2 & 6 & -4 \\ 5 & 5 & -3 & 2 \\ 2 & 6 & -5 & 3 \end{vmatrix}$$

There are no zeros, but in the second row it is easy to make two zeros by adding the elements of the first row to the elements of the second row. This yields

$$\Delta=\begin{vmatrix} 3 & 7 & -2 & 4 \\ 0 & 5 & 4 & 0 \\ 5 & 5 & -3 & 2 \\ 2 & 6 & -5 & 3 \end{vmatrix}$$

Another zero can be created in the second row by subtracting the elements of the second row $\left(\text{multiplied by } \frac{4}{5}\right)$ from the elements of the third column. It is more convenient to produce a one in the second row by subtracting the elements of the third column from the elements of the second. This gives

$$\Delta=\begin{vmatrix} 3 & 9 & -2 & 4 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & -3 & 2 \\ 2 & 11 & -5 & 3 \end{vmatrix}=\begin{vmatrix} 3 & 9 & -38 & 4 \\ 0 & 1 & 0 & 0 \\ 5 & 8 & -35 & 2 \\ 2 & 11 & -49 & 3 \end{vmatrix}$$

(we subtracted quadrupled elements of the second column from the elements of the third column). We now have

$$\Delta=\begin{vmatrix} 3 & -38 & 4 \\ 5 & -35 & 2 \\ 2 & -49 & 3 \end{vmatrix}=-303,$$

186. Using Determinants to Investigate and Solve Systems of Equations

Determinants were first introduced to solve systems of equations of the first degree. In 1750, the Swiss mathematician G. Cramer gave general formulas expressing the unknowns in terms of determinants composed of the coefficients of the system. About a hundred years later the theory of determinants was taken far beyond the limits of algebra into all divisions of mathematics.

In the sections which follow we give basic information on investigating and solving systems of first-degree equations. Geometrical facts are invoked for greater pictorialness.

187. Two Equations in Two Unknowns

Consider the system of equations

$$a_1x+b_1y=h_1, \tag{1}$$

$$a_2x+b_2y=h_2 \tag{2}$$

(each of which defines a straight line in the *xy*-plane; cf. Sec. 19).

Introduce the *notation*

$$\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \textit{(determinant of the system)} \tag{3}$$

$$\Delta_x=\begin{vmatrix} h_1 & b_1 \\ h_2 & b_2 \end{vmatrix}, \quad \Delta_y=\begin{vmatrix} a_1 & h_1 \\ a_2 & h_2 \end{vmatrix} \tag{4}$$

The determinant Δ_x is obtained from Δ by replacing the elements of the first column by the constant terms of the system; Δ_y is obtained in similar fashion.

Three cases are possible.

Case 1. The determinant of the system is nonzero: $\Delta \neq 0$. Then the system has a unique solution:

$$x=\frac{\Delta_x}{\Delta}, \quad y=\frac{\Delta_y}{\Delta} \tag{5}$$

[the straight lines (1) and (2) intersect, formulas (5) yield the coordinates of the point of intersection].

Case 2. The determinant is equal to zero: $\Delta=0$ (i. e. the coefficients of the unknowns are proportional). Let one of the determinants Δ_x, Δ_y be different from zero (i. e. the constant terms are not proportional to the coefficients of the unknowns).

In this case the system does not have any solutions [the straight lines (1) and (2) are parallel but not coincident].

Case 3. $\Delta=0$, $\Delta_x=0$, $\Delta_y=0$ (both the coefficients and the constant terms are proportional).

Then one of the equations (1), (2) is a consequence of the other and the system reduces to a single equation in two unknowns and has an infinity of solutions [the straight lines (1) and (2) coincide].

Example 1.

$$2x+3y=8, \quad 7x-5y=-3$$

Here

$$\Delta=\begin{vmatrix} 2 & 3 \\ 7 & -5 \end{vmatrix}=-31, \quad \Delta_x=\begin{vmatrix} 8 & 3 \\ -3 & -5 \end{vmatrix}=-31,$$

$$\Delta_y=\begin{vmatrix} 2 & 8 \\ 7 & -3 \end{vmatrix}=-62$$

The system has a unique solution:

$$x=\frac{\Delta_x}{\Delta}=1, \quad y=\frac{\Delta_y}{\Delta}=2$$

Example 2.

$$2x+3y=8, \quad 4x+6y=10$$

Here $\Delta=\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}=0$ and $\Delta_x=\begin{vmatrix} 8 & 3 \\ 10 & 6 \end{vmatrix}=18 \neq 0.$

The coefficients are proportional but the constant terms do not obey the same proportion. The system has no solutions.

Example 3.

$$2x+3y=8, \quad 4x+6y=16$$

Here

$$\Delta=\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}=0, \quad \Delta_x=\begin{vmatrix} 8 & 3 \\ 16 & 6 \end{vmatrix}=0, \quad \Delta_y=\begin{vmatrix} 2 & 8 \\ 4 & 16 \end{vmatrix}=0$$

One of the equations is a consequence of the other (for example, the second is obtained from the first by multiplying by 2). The system reduces to a single equation and has an infinity of solutions contained in the formula

$$y=-\frac{2}{3}x+\frac{8}{3} \quad \left(\text{or } x=-\frac{3}{2}y+4\right)$$

188. Two Equations in Three Unknowns

Consider the system of equations

$$a_1x+b_1y+c_1z=h_1, \tag{1}$$
$$a_2x+b_2y+c_2z=h_2 \tag{2}$$

(each of which defines a plane in space; cf. Sec. 141).

Three cases are possible.

Case 1. Of the following three determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} \tag{3}$$

at least one is nonzero, i. e. the coefficients of the unknowns are not proportional. Then the system has an infinity of solutions, and any value can be assigned to *one* of the unknowns. For instance, if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then to the unknown z we can assign any value; the unknowns x and y are determined in unique fashion (Sec. 187, Item 1) from the system

$$a_1x+b_1y=h_1-c_1z,$$
$$a_2x+b_2y=h_2-c_2z$$

[the planes (1) and (2) are not parallel, the system defines a straight line, the quantities (3) are direction numbers, (Sec. 143)].

Case 2. All determinants (3) are equal to zero, but one of the determinants

$$\begin{vmatrix} a_1 & h_1 \\ a_2 & h_2 \end{vmatrix}, \quad \begin{vmatrix} b_1 & h_1 \\ b_2 & h_2 \end{vmatrix}, \quad \begin{vmatrix} c_1 & h_1 \\ c_2 & h_2 \end{vmatrix} \tag{4}$$

is nonzero; i.e. the coefficients of the unknowns are proportional but the constant terms do not obey that proportion. In this case the system has no solutions [planes (1) and (2) are parallel but not coincident].

Case 3. All the determinants (3) and (4) are equal to zero, i.e. the coefficients and the constant terms are proportional. The system then reduces to a single equation and has an infinity of solutions; we can assign any values whatsoever to *two* of the unknowns. For example, if $c_1 \neq 0$, then the unknowns x, y can be given any values [the planes (1) and (2) coincide].

Example 1. Solve the system of equations

$$x-2y-z=15, \quad 2x-4y+2z=2$$

Here

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} = 0, \quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ -4 & 2 \end{vmatrix} = -8,$$

$$\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} = -4$$

Here, there are determinants which are not equal to zero. This means that the system has an infinity of solutions. We can assign any value to the unknown x alone or to the unknown y alone, since $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \neq 0$ and $\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} \neq 0$. *We cannot assign an arbitrary value to the unknown z* (cf. Sec. 142, Example 5).

We solve the system for y and z and get

$$-2y - z = 15 - x, \quad -4y + 2z = 2 - 2x$$

Whence

$$y = \frac{\begin{vmatrix} 15-x & -1 \\ 2-2x & 2 \end{vmatrix}}{-8} = -4 + \frac{1}{2}x, \quad z = \frac{\begin{vmatrix} -2 & 15-x \\ -4 & 2-2x \end{vmatrix}}{-8} = -7$$

(The system defines a straight line perpendicular to the z-axis.)

Example 2. The system

$$7x - 4y + z = 5, \quad 21x - 12y + 3z = 12$$

does not have any solutions since all the determinants (3) are zero (the coefficients of the unknowns are proportional) and the determinant $\begin{vmatrix} a_1 & h_1 \\ a_2 & h_2 \end{vmatrix} = \begin{vmatrix} 7 & 5 \\ 21 & 12 \end{vmatrix}$ is nonzero (the constant terms are not proportional to the coefficients).

(The planes are parallel but not coincident.)

Example 3. Solve the system

$$7x - 4y + z = 5, \quad 21x - 12y + 3z = 15$$

Here, both the coefficients and the constant terms are proportional. The system reduces to a single equation. To any pair of unknowns (x and y, say) can be assigned arbitrary values (then $z = 5 - 7x + 4y$).

(The planes are coincident.)

189. A Homogeneous System of Two Equations in Three Unknowns

A system of first-degree equations is called *homogeneous* if the constant term in each equation is zero.

Consider the homogeneous system

$$a_1x+b_1y+c_1z=0, \qquad (1)$$

$$a_2x+b_2y+c_2z=0 \qquad (2)$$

This is a particular case of the system of Sec. 188. The peculiarity is that *Case 2 cannot occur* [the determinants (4), Sec. 188, are always zero]. The system (1)-(2) will always have an infinity of solutions.

[The planes (1) and (2) pass through the coordinate origin and, consequently, either intersect or coincide.]

Case 1. The coefficients are not proportional, i.e. at least one of the three determinants (3), Sec. 188, is nonzero. Then the solution may be written in symmetric form:

$$x=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} t, \quad y=\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} t, \quad z=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t \qquad (3)$$

(the parameter t is an arbitrary number; cf. Sec. 152). [The parametric equations (3) define the straight line of intersection of the planes (1) and (2).]

Case 2. The coefficients are proportional, i.e. all the determinants $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ are zero.

The system reduces to a single equation (the planes are coincident).

Example 1. Solve the system

$$2x-5y+8z=0, \quad x+4y-3z=0$$

Here

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}=\begin{vmatrix} -5 & 8 \\ 4 & -3 \end{vmatrix}=-17, \quad \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}=\begin{vmatrix} 8 & 2 \\ -3 & 1 \end{vmatrix}=14,$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=\begin{vmatrix} 2 & -5 \\ 1 & 4 \end{vmatrix}=13$$

According to (3) we have

$$x=-17t, \quad y=14t, \quad z=13t$$

In this example, an arbitrary value may be assigned to any one of the unknowns. For example, putting $z=39$, we find $t=3$, hence, $x=-51$, $y=42$.

Example 2. Solve the system

$$x-2y-z=0,\quad 2x-4y+2z=0$$

Here

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ -4 & 2 \end{vmatrix} = -8,\quad \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} = -4,\quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

Hence

$$x=-8t,\quad y=-4t,\quad z=0$$

An arbitrary value may be assigned to one of the unknowns x or y, *but not to the unknown* z, which can only be equal to zero (the straight line lies in the xy-plane).

Example 3. The system

$$7x-4y+z=0,\quad 21x-12y+3z=0$$

reduces to a single equation. Arbitrary values may be assigned to *any pair* of unknowns.

190. Three Equations in Three Unknowns

We consider the system

$$a_1x+b_1y+c_1z=h_1, \tag{1}$$
$$a_2x+b_2y+c_2z=h_2, \tag{2}$$
$$a_3x+b_3y+c_3z=h_3 \tag{3}$$

We introduce the notation

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ (determinant of system)}, \tag{4}$$

$$\Delta_x = \begin{vmatrix} h_1 & b_1 & c_1 \\ h_2 & b_2 & c_2 \\ h_3 & b_3 & c_3 \end{vmatrix},\quad \Delta_y = \begin{vmatrix} a_1 & h_1 & c_1 \\ a_2 & h_2 & c_2 \\ a_3 & h_3 & c_3 \end{vmatrix},\quad \Delta_z = \begin{vmatrix} a_1 & b_1 & h_1 \\ a_2 & b_2 & h_2 \\ a_3 & b_3 & h_3 \end{vmatrix} \tag{5}$$

The determinant Δ_x is obtained from Δ by replacing the elements of the first column by the constant terms. In similar fashion, we obtain Δ_y and Δ_z.

If it turned out that in the determinant Δ the appropriate elements of any two rows, say the first and the second, were proportional, the Eqs. (1) and (2) would either be inconsistent (Sec. 188, Item 2), or would reduce to a single equation (Sec. 188, Item 3). In the first case the given system does not have any solutions, in the second case, in place of the given system we obtain a system of two equations (1) and (3) (which in turn can be reduced to a single equation). Since all of this has already been considered in Sec. 188, we can confine ourselves to the assumption that the determinant Δ *has no pair of rows with proportional elements* [there is no pair of parallel planes among the three planes (1), (2), (3)].

Three cases are possible in this assumption.

Case 1. The determinant of the system is not equal to zero:

$$\Delta \neq 0$$

The system has a unique solution:

$$x=\frac{\Delta_x}{\Delta}, \quad y=\frac{\Delta_y}{\Delta}, \quad z=\frac{\Delta_z}{\Delta} \tag{6}$$

(The three planes intersect in one point.)

Case 2. The determinant of the system is equal to zero: $\Delta=0$; and one of the determinants Δ_x, Δ_y, Δ_z is nonzero, then the other two are nonzero:[1]

$$\Delta_x \neq 0, \quad \Delta_y \neq 0, \quad \Delta_z \neq 0$$

In this case the system has no solutions.

[The equality $\Delta=0$ signifies that the normal vectors to the planes (1), (2), (3) are coplanar, hence, all three planes are parallel to a single straight line. In the case at hand, the three planes form a prismatic surface (Fig. 202).]

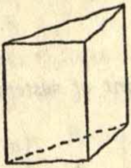


Fig. 202

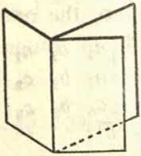


Fig. 203

Case 3. $\Delta=0$, $\Delta_x=0$, $\Delta_y=0$, $\Delta_z=0$. In this case, one of the three equations (no matter which) is a consequence of the other two. The system reduces to two equations in three unknowns and has an infinity of solutions (Sec. 188, Case 1; Cases 2 and 3 cannot be represented due to the foregoing assumption).

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[1] If the corresponding elements in two rows of the determinant Δ are proportional (we did not consider this case), then it may happen that of the three determinants Δ_x, Δ_y, Δ_z only one or only two are equal to zero.

(As in the preceding case, the three planes are parallel to one straight line, but this time they form a pencil; Fig. 203.)

Example 1. Solve the system

$$3x+4y+2z=5, \quad 5x-6y-4z=-3, \quad -4x+5y+3z=1$$

Here

$$\Delta=\begin{vmatrix} 3 & 4 & 2 \\ 5 & -6 & -4 \\ -4 & 5 & 3 \end{vmatrix}=12, \quad \Delta_x=\begin{vmatrix} 5 & 4 & 2 \\ -3 & -6 & -4 \\ 1 & 5 & 3 \end{vmatrix}=12,$$

$$\Delta_y=\begin{vmatrix} 3 & 5 & 2 \\ 5 & -3 & -4 \\ -4 & 1 & 3 \end{vmatrix}=-24, \quad \Delta_z=\begin{vmatrix} 3 & 4 & 5 \\ 5 & -6 & -3 \\ -4 & 5 & 1 \end{vmatrix}=60$$

The system has a unique solution:

$$x=\frac{\Delta_x}{\Delta}=1, \quad y=\frac{\Delta_y}{\Delta}=-2, \quad z=\frac{\Delta_z}{\Delta}=5$$

Example 2. Solve the system of equations

$$x+y+z=5, \quad x-y+z=1, \quad x+z=2$$

Here

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix}=0$$

and

$$\Delta_x=\begin{vmatrix} 5 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{vmatrix}=-2$$

(the determinants Δ_y and Δ_z need not be computed.[1]) The system has no solutions. This is evident by inspection: combining the first two equations termwise, we get $2x+2z=6$, i. e. $x+z=3$, which contradicts the third equation.

Example 3. Solve the system

$$x+y+z=5, \quad x-y+z=1, \quad x+z=3$$

---

[1] The rows of the determinant Δ are not proportional pairwise; see footnote on p. 242.

Here

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 0 & 1\end{vmatrix}=0$$

and

$$\Delta_x=\begin{vmatrix}5 & 1 & 1\\ 1 & -1 & 1\\ 3 & 0 & 1\end{vmatrix}=0$$

The determinants Δ_y and Δ_z are definitely equal to zero.1)

The given system of equations reduces to a system of two equations (any two of the three given equations; the third is a consequence) and has an infinity of solutions. An arbitrary value can be assigned to the unknown x alone or to the unknown z alone (but not to y; see Sec. 188, Item 1).

Let us take the first and third equations and solve them for x and z. We then get

$$x+y=5-z, \quad x=3-z$$

Whence

$$x=3-z, \quad y=2$$

Note. If a system of three equations in three unknowns is homogeneous ($h_1=h_2=h_3=0$), then the second case is impossible. In the first case, the only solution will be $x=0$, $y=0$, $z=0$ (the planes intersect at the origin). Taking (in the third case) any two equations of the system, say (1) and (2), we find all the solutions of the given system from the formulas (3), Sec. 189 (the three planes form a pencil, the axis of which passes through the origin of coordinates).

Example 4. Solve the system

$$x+y+z=0, \quad 3x-y+2z=0, \quad x-3y=0$$

Here

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ 3 & -1 & 2\\ 1 & -3 & 0\end{vmatrix}=0$$

One of the equations is a consequence of the other two. An arbitrary value may be assigned to one of the unknowns (no matter which). Taking the first and third equations, we find [from formulas (3), Sec. 189]

$$x=\begin{vmatrix}1 & 1\\ -3 & 0\end{vmatrix}t=3t, \quad y=\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}t=t, \quad z=\begin{vmatrix}1 & 1\\ 1 & -3\end{vmatrix}t=-4t$$

---

1) The rows of determinant Δ are not proportional pairwise; see footnote on p. 242

190a. A System of n Equations in n Unknowns

An exhaustive list of possible cases is too involved. We therefore confine ourselves to the following.

Let there be given the following system of n equations in n unknowns:

$$\left.\begin{aligned} a_1x+b_1y+c_1z+\ldots+f_1u&=h_1,\\ a_2x+b_2y+c_2z+\ldots+f_2u&=h_2,\\ \cdots\cdots\cdots&\cdots\\ a_nx+b_ny+c_nz+\ldots+f_nu&=h_n \end{aligned}\right\} \tag{1}$$

1. If the nth-order determinant

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 & \ldots & f_1\\ a_2 & b_2 & c_2 & \ldots & f_2\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ a_n & b_n & c_n & \ldots & f_n \end{vmatrix} \quad \textit{(determinant of the system)} \tag{2}$$

is not equal to zero, then the system has a unique solution

$$x=\frac{\Delta_x}{\Delta}\quad y=\frac{\Delta_y}{\Delta},\ \ldots,\ u=\frac{\Delta_u}{\Delta} \tag{3}$$

where Δ_x is the determinant obtained from Δ by replacing the elements $a_1, a_2, \ldots, a_n$ by the corresponding constant terms $h_1, h_2, \ldots, h_n$; similarly, we get the determinants $\Delta_y, \Delta_z, \ldots, \Delta_u$.

2. If $\Delta=0$ and there are nonzero determinants among $\Delta_x, \Delta_y, \ldots, \Delta_u$, then the system has no solution.

3. Now let $\Delta=\Delta_x=\Delta_y=\ldots=\Delta_u=0$ and let one of the minors of the $(n-1)$th order of the determinant Δ (say, the minor obtained by striking out the second row and the third column) be nonzero. Then the system reduces to $n-1$ equations; one of the equations (the second one in accord with the number of the row) is a consequence of the others. To one of the unknowns (the unknown z in accordance with the number of the column) we can assign an arbitrary value. The remaining $n-1$ unknowns are determined in unique fashion from the system of $n-1$ unknowns.

Note. When all the $(n-1)$th-order determinants, which are minors of the determinant Δ, are equal to zero, the system may not have any solutions and may be reduced to $n-2$ equations or to a lesser number of equations.

Example 1. Solve the system

$$\begin{aligned} 3x+7y-2z+4u&=3,\\ -3x-2y+6z-4u&=11,\\ 5x+5y-3z+2u&=6,\\ 2x+6y-5z+3u&=0 \end{aligned}$$

The determinant Δ of the system (see Sec. 185, Example 3) is equal to -303. Using the techniques explained in Sec. 185, we find

$$\Delta_x=-303,\quad \Delta_y=-606,\quad \Delta_z=-303,\quad \Delta_u=909$$

According to formulas (3) we have

$$x=1,\quad y=2,\quad z=1,\quad u=-3$$

Example 2. Solve the system

$$\begin{aligned} x-\ y+2z-\ u&=1,\\ x+\ y+\ z+\ u&=4,\\ 2x+\ 3y\quad\ -5u&=0,\\ 5x+2y+5z-6u&=0 \end{aligned}$$

Here

$$\Delta=\begin{vmatrix} 1 & -1 & 2 & -1\\ 1 & 1 & 1 & 1\\ 2 & 3 & 0 & -5\\ 5 & 2 & 5 & -6 \end{vmatrix}=0$$

Yet

$$\Delta_x=\begin{vmatrix} 1 & -1 & 2 & -1\\ 4 & 1 & 1 & 1\\ 0 & 3 & 0 & -5\\ 0 & 2 & 5 & -6 \end{vmatrix}=\begin{vmatrix} 1 & -1 & 2 & -1\\ 0 & 5 & 7 & 5\\ 0 & 3 & 0 & -5\\ 0 & 2 & 5 & -6 \end{vmatrix}=144\neq 0$$

And so the system has no solutions (if the first equation is termwise multiplied by 2 and the resulting equation is termwise combined with the second and third, we get $5x+2y+5z-6u=6$, but this contradicts the fourth equation)

Example 3. Solve the system

$$\begin{aligned} x-\ y+2z-\ u&=1,\\ x+\ y+\ z+\ u&=4,\\ 2x+3y\quad\ -5u&=0,\\ 5x+2y+5z-6u&=6 \end{aligned}$$

Here

$$\Delta=\Delta_x=\Delta_y=\Delta_z=\Delta_u=0.$$

Striking out the fourth row and the fourth column, we get the minor

$$\begin{vmatrix} 1 & -1 & 2\\ 1 & 1 & 1\\ 2 & 3 & 0 \end{vmatrix}=-3\neq 0$$

The system reduces to three equations

$$\left.\begin{aligned} x-\ y+2z-\ u&=1,\\ x+\ y+\ z+\ u&=4,\\ 2x+3y\quad\ -5u&=0 \end{aligned}\right\}\qquad(4)$$

The fourth equation is a consequence of these three (cf. Example 2). To the unknown u we can assign any value. From (4) we find

$$x=\frac{-24u+21}{-3},\quad y=\frac{11u-14}{-3},\quad z=\frac{16u-19}{-3}$$

FUNDAMENTALS OF MATHEMATICAL ANALYSIS

191. Introductory Remarks

Mathematical analysis comprises a system of disciplines united by the following characteristic features.

Their subject matter embraces quantitative relationships of the surrounding world (in contrast to geometric disciplines which treat of spatial properties). These relationships are expressed by means of *numerical* quantities as in arithmetic. But whereas in arithmetic (and in algebra) one deals mainly with constant quantities (which characterize *states*), in analysis one deals with variable quantities (which describe *processes*, Sec. 195). The underlying concepts involved in the study of relationships between variable quantities are those of the *function* (Sec. 196) and the *limit* (Secs. 203-206).

In this book we consider the following divisions of analysis: differential calculus, integral calculus, the theory of series and the theory of differential equations. The subject matter of each is discussed in its proper place.

In embryo, the methods of mathematical analysis are found in the works of the ancient Greek scholars (e. g. Archimedes). The systematic development of these methods began in the 17th century. On the borderline of the 17th and 18th centuries, Newton[1] and Leibniz[2] completed, in the main, the construction of the differential and integral calculus, and also laid the foundation of the theory of series and differential equations. In the 18th century, Euler elaborated the latter two divisions and laid the foundation for other disciplines of mathematical analysis.

By the end of the 18th century an enormous quantity of factual material had been accumulated, but it was still lacking in logical development. This drawback was overcome through the efforts of such prominent mathematicians, scientists of the 19th century, as Cauchy in France, Lobachevsky in Russia, Abel in Norway, Riemann in Germany, and others.

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[1] Isaac Newton (1642-1727), great English mathematician and physicist; discovered the law of universal gravitation, formulated the basic laws of mechanics and applied them to a study of the motion of terrestrial and celestial bodies; investigated the laws of optics experimentally and theoretically.

[2] Gotfried Wilhelm Leibniz (1646-1716), celebrated German philosopher and mathematician.

192. Rational Numbers

The concept of number grew out of the counting of objects. Counting resulted in the numbers 1, 2, 3, and so on, which are now termed *natural* numbers. Later came the concept of a fractional number, which grew out of the measurements of continuous quantities (lengths, weights, etc.). The *negative* numbers (and *zero*) came into mathematics[1] with the development of algebra.

The integers (i. e. natural numbers 1, 2, 3, etc. and the negative numbers —1, —2, —3, etc. and zero) and fractions are called *rational* numbers (in contrast to *irrational* numbers; Sec. 193). Any rational number may be written in the form $\frac{p}{q}$ (where p and q are integers).

193. Real Numbers

In practical affairs, measurements are carried out by means of instruments. The result of a measurement is expressed as some rational number (say, the thickness of a metallic filament measured by a micrometer may be expressed in millimetres as the number 0.023). Every instrument is limited in accuracy and so in everyday activities the range of rational numbers is quite sufficient, even redundant. But in mathematical *theory*, where measurements are assumed to be absolutely exact, the rational numbers do not suffice. Thus, no rational number is capable of precisely expressing the length of the diagonal of a square if its side is taken as the unit of measure; neither can a rational number express exactly the sine of a 60° angle, the cosine of a 22° angle, the tangent of a 17° angle, the ratio of the circumference to the diameter of a circle, etc. Speaking generally, it is impossible to express the ratio of incommensurable segments exactly by means of a rational number.

In order to express exactly the ratio of incommensurable segments, it is necessary to introduce new numbers called *irrational* numbers.[2] An irrational number expresses the

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[1] In China about 2000 years ago and in India about 1500 years ago. In Europe, negative numbers were recognized only in the 17th century.

[2] A ratio of commensurable line-segments can be expressed by a ratio of integers; this is not possible in the case of incommensurable line-segments. Originally, the ancient Greek mathematicians considered only the ratios of integers, and so when incommensurable quantities were discovered, they received the name irrational, which means

length of a segment that is not commensurable with the unit of measure (scale unit). Together, the rational and irrational numbers are called *real* numbers (in contrast to *imaginary* numbers; see Note 2). Real numbers suffice to express with exactitude the length of any line-segment.

An irrational number cannot be exactly equal to any rational number, but it is possible to find, for every irrational number, a rational number (say, a decimal) which is approximately equal to the irrational number (too large or too small) and which can be made to approach it to any arbitrary degree of accuracy.

Example. For the number log 3 (which is irrational) we can find approximate values 0.4771 (with defect) and 0.4772 (with excess); they differ by only 0.0001 so that the error in each does not, in absolute value, exceed 0.0001. If it is required that the error not exceed 0.00001, we can find the values 0.47712 (with defect) and 0.47713 (with excess). For the evaluation of logarithms see Sec. 272 and also Sec. 242.

Note 1. Rational numbers are also often expressed in approximate fashion. For example, in place of the fraction 1/3 one often takes the values 0.33, 0.333, etc. (with defect), or 0.34, 0.334, etc. (with excess), depending on the accuracy required.

Note 2. An *imaginary* number has the notation bi, where b is a real number and i is the "imaginary unit" defined by the equality $i^2=-1$ (there is no real number that can satisfy this equality). An expression of the form $a+bi$ is called a *complex* number. Complex numbers were introduced into algebra in the middle of the 16th century in connection with the solution of cubic equations. They have been used in analysis since the end of the 17th century.

In this book, all numbers are assumed to be real unless otherwise stated.

194. The Number Line

On the straight line $X'X$ (Fig. 204) choose an origin O, a scale unit OA and a positive direction (say, from X' to X). Then every real number x will be associated with a definite point M, the abscissa of which is equal to x.

"having no relation" (translated from the Greek "alogos"). Later (in the 4th century B. C.) the Greek mathematicians (Eudoxus and then Euclid) began to consider the ratios of incommensurable quantities as well. When new numbers were introduced to describe these relations, they too were called irrational.

In analysis, numbers are depicted in this way (for greater pictorialness) by points. The straight line $X'X$ on which the points are specified is called the *number line* (*number scale*, or *number axis*).

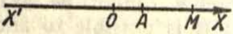

Fig. 204

195. Variable and Constant Quantities

A variable quantity is a quantity which can take on *different* values within the framework of a given problem. In contrast, a constant quantity is one which, within the framework of the given problem, has *one and only one* value. One and the same quantity can be a constant in one problem and a variable in another.

Example 1. In most physical problems, the boiling point T of water is a constant quantity. But when one has to take into account changing atmospheric pressure, T becomes a variable quantity.

Example 2. In the equation of the parabola $y^2=2px$, the coordinates x, y are variables. The parameter p is a constant if we consider only one parabola. But if we consider a set of parabolas with a common x-axis and a common vertex O, then the parameter p is a variable quantity.

Variables are ordinarily denoted by the last letters of the alphabet (x, y, z, u, v, w); constants by the first letters: a, b, c, . . .

196. Function

Definition 1. A quantity y is called a *function* of a variable quantity x if with every value assumed by x we can associate one or several definite values of y. Here, the variable x is called the *argument*.

We can put it otherwise: the quantity y *depends* on the quantity x; accordingly, the argument is called the *independent* variable and the function is termed the *dependent* variable.

Example 1. Let T be the boiling point of water and p, atmospheric pressure. Observations have shown that to every value that p can assume there always corresponds one and the same value of T. Hence, T is a function of the argument p.

The relationship between T and p enables one to determine the pressure (without a barometer) from the boiling poin. of water using a table (a portion is given below):

Table 1

T, °C	70	75	80	85	90	95	100
p, mm	234	289	355	434	526	634	760

In turn, p is a function of the argument T; the dependence of p on T enables one, by observing the pressure, to determine the temperature of the boiling point of water (without a thermometer) using the same table. However, it is more convenient to use a table like the following:

Table 2

p, mm	300	350	400	450	500	550	600	650	700
T, °C	75.8	79.6	83.0	85.9	88.7	91.2	93.5	95.7	97.7

Here, the argument p increases in equal jumps (like the argument T in Table 1).

Note 1. Table 1 may be supplemented by other values of the argument T, say, 65°, 73°, 104°. But there are values which the boiling-point temperature cannot assume. For example, it cannot be less than absolute zero (—273 °C). And, of course, there is no value of p that corresponds to the impossible value $T = -300$ °C. That is why Definition 1 reads: "with every value assumed by x . . ." (and not "every value of x . . .").

Example 2. A body is thrown upwards; s is the height above the earth, t is the time elapsed from the launching.

The quantity s is a function of the argument t because the body reaches a definite height at every instant of the flight. In turn, t is a function of the argument s because to every height reached by the body there correspond two definite values of t (one during the upward flight, the other during its fall).

Definition 2. If to every value of the argument there corresponds one value of the function, the function is termed *single-valued*; if there correspond two or more values, then it is called *multiple-valued (double-valued, triple-valued,* etc.*)*.

In the second example, s is a single-valued function of the argument t, and the quantity t is a double-valued function of the argument s.

A function will be considered single-valued unless it is specifically stated to be multiple-valued.

Example 3. The sum (s) of angles of a polygon is a function of the number (n) of the sides. The argument n can only take on integral values of 3 or more. The dependence of s upon n is expressed by the formula

$$s=\pi(n-2)$$

(the radian is taken as the unit of angular measurement). In turn, n is a function of the argument s; the dependence of n upon s is expressed by the formula

$$n=\frac{s}{\pi}+2.$$

The argument s can only take on values which are multiples of π (π, 2π, 3π, etc.).

Example 4. The side x of a square is a function of the area of the square, $S\left(x=\sqrt{S}\right)$. The argument can assume any positive values.

Note 2. The argument is always a variable. The function is too, as a rule. But a function can also be constant. For instance, the distance of a moving point from a fixed point is a function of the time of motion and, as a rule, varies. But in the motion of a point about the circumference of a circle the distance from the centre does not change.

When a function is a constant quantity, the argument and the function cannot be interchanged (in our example, the duration of motion about the circumference is not a function of the distance from the centre).

197. Ways of Representing Functions

A function is considered to be specified (known) if for every value of the argument (from among possible values) one can find the corresponding value of the function. There are three frequently used modes of representing functions: (a) tabular, (b) graphical, and (c) analytical.

(a) *Tabular*: this mode includes such familiar tables as those of logarithms, square roots, etc.; see also Example 1, Sec. 196. It gives the numerical value of the function directly. That is its advantage over the other methods.

However, it has drawbacks: (1) a table is hard to survey as a whole; (2) it often lacks certain needed values of the argument.

(b) *Graphical*: this method consists in displaying the curve (graph) in which abscissas depict the values of the argument and ordinates give the corresponding values of the function. To facilitate graphical display, the scales on the axes are frequently different.

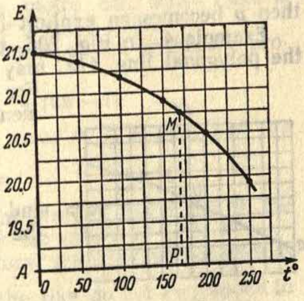

Fig. 205

Example 1. Fig. 205 gives a graphical depiction of the dependence of the modulus of elasticity E of forged iron (in tons per cm^2) upon the temperature t of iron. The scales on the axis of abscissas (t) and the axis of ordinates (E) are labelled with numbers. The curve permits us, for example, to read the modulus of elasticity $E \approx 20.75$ tons/cm^2 at $t=170°$ C.

The advantage of the graphical mode is its surveyability as a whole and the continuity of variation of the argument. The disadvantages are: restricted degree of accuracy and difficulty in reading off values of the function with sufficient accuracy.

(c) *Analytical*: this mode consists in specifying a function by one or several formulas.

Example 2. The functional relationship between the radius r of a circle and the circumference s is given by the formula

$$s=2\pi r \tag{1}$$

Example 3. The functional relationship between the volume V (m^3) and the pressure p (tons/m^2) of 1 kg of air at 0° C is given by the formula

$$pV=8.000 \tag{2}$$

If a relation between x and y is expressed by an equation solved for y, the quantity y is called an ***explicit*** function of the argument x, otherwise, it is an *implicit* function. In Example 2 the quantity s is an explicit function of the argument r, while r is an implicit function of the argument s. In Example 3, p is an implicit function of the argument V and V is an implicit function of the argument p. If Eq. (2) is written in the form

$$p=\frac{8.000}{V} \tag{3}$$

then p becomes an explicit function of the argument V.

Example 4. In Fig. 206, the function given graphically by the polygonal line ABC may be represented by two formulas. For $x<2$ (i. e. for the line segment BA) take the formula

$$y=\frac{1}{2}x$$

and for $x>2$ (i. e. for BC) take the formula

$$y=\frac{1}{3}+\frac{1}{3}x$$

For $x=2$, both formulas yield $y=1$ (point B).

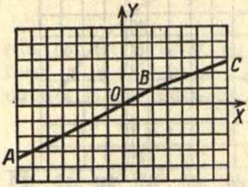

Fig. 206

Example 5. The distance (by road) between points A and B is 90 km. A motor car covered the first half of the distance from A to B at a speed of 0.6 km/min, the second at a speed of 0.9 km/min. Let s (km) be the distance of the car from point A. The time t (min) in transit is a function of the argument s. Two formulas will suffice to specify it:

$$t=\frac{s}{0.6} \qquad \text{for } 0\leqslant s\leqslant 45$$

$$t=75+\frac{s}{0.9} \qquad \text{for } 45\leqslant s\leqslant 90$$

198. The Domain of Definition of a Function

1. The collection of all values which (under the conditions of the problem at hand) the argument x of a function $f(x)$ can assume is called the *domain of definition* (or simply, *domain*) of the function.

Note. A value of x that does not lie within the collection mentioned above is not associated with any value of the function.

Example 1. Given the conditions of Example 5, Sec. 197, the domain of the function $t=f(s)$ is the set of all the numbers from 0 to 90 (including the end-points 0 and 90):

$$0 \leqslant s \leqslant 90$$

Indeed, to every distance from 0 to 90 km there corresponds a definite time t of transit of the motor car, while there is no value of t corresponding to $s < 0$ and $s > 90$.

Example 2. The sum of the terms of the arithmetic progression

$$s=1+3+5+\ldots+(2n-1)$$

is a function of the number of terms n; it is expressed by the formula

$$s=n^2$$

In itself, this formula is meaningful for any n. But in the given problem, n can assume only the values 1, 2, 3, 4, ... The domain is the set of all natural numbers (values such as $n=\frac{1}{2}$, $n=-5$, $n=\sqrt{3}$ and the like do not correspond to any values of the function).

2. A function is frequently specified by a formula without any indication of the domain of definition; then it is assumed that the domain is the set of all values of the argument for which the formula is meaningful.

Example 3. The function s is given by the formula $s=n^2$ (without any indication of the domain). It is assumed that the domain of definition is the set of all real numbers (cf. Example 2).

Example 4. The function y is given by the formula

$$y=\sqrt{x-2}+\sqrt{7-x}$$

which is meaningful only for $2 \leqslant x \leqslant 7$. The domain is the set of all numbers from 2 to 7 (including the boundary points). The graph in Fig. 207 lies wholely above the segment $A'B'$.

Example 5. The function y is given by the formula $y=\frac{1}{x}$. The domain of definition is the set of all numbers except zero. For the value $x=0$, the graph has no point (Fig. 208).

Example 6. The domain of the function $y=\sqrt{x}$ is the collection of all positive numbers and zero (Fig. 209).

3. When the domain of a function is the collection of natural numbers, the function is called *integral*; speaking of the values of an integral function, we say that they form a *sequence* or are *terms of a sequence*.

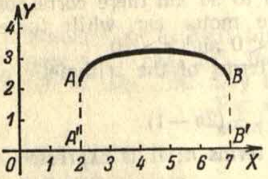

Fig. 207

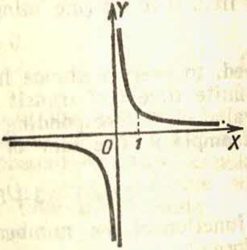

Fig. 208

Example 7. The function $t_n=1\cdot2\cdot3\ \ldots\ n$ is integral. The values $t_1=1$, $t_2=1\cdot2=2$, $t_3=1\cdot2\cdot3=6$, ... form a sequence.

The product $1\cdot2\cdot3\ \ldots\ n$ is denoted by $n!$ (read "*n factorial*") so that this function may be represented by the formula

$$t_n=n!$$

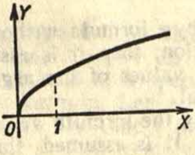

Fig. 209

Example 8. The function $u=\frac{1}{2^n}$, where n takes on the values 1, 2, 3, ..., is integral. The values $u_1=\frac{1}{2}$, $u_2=\frac{1}{4}$, $u_3=\frac{1}{8}$, ... (terms of a geometric progression) form a sequence.

Example 9. The function $s=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}$ (the sum of n terms of a geometric progression) is integral. The values $s_1=\frac{1}{2}$, $s_2=\frac{3}{4}$, $s_3=\frac{7}{8}$, ... form a sequence.

199. Intervals

The domain of definition of functions considered in analysis is usually in the form of one or several "intervals".

An *interval* (a, b) is a collection of numbers x contained between the numbers a and b; in the notation (a, b), the first letter ordinarily denotes the smaller number, the second letter, the larger number, so that

$$a < x < b$$

The numbers a and b are termed the *end-points* of the interval It frequently happens that the end-points a and b, or one of them, are adjoined to the set of points of the interval. An interval to which both end points have been adjoined is called a *closed interval.*

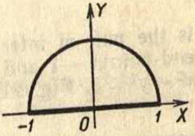

Fig. 210

The interval (a, ∞) is the collection of all numbers greater than a; the interval $(-\infty, a)$ is the collection of all numbers less than a; the interval $(-\infty, \infty)$ is the collection of all real numbers.

Example 1. Under the conditions of Example 5, Sec. 197, the domain of the function t is the closed interval $(0, 90)$, in other words, the argument s can assume all values satisfying the inequality

$$0 \leqslant s \leqslant 90$$

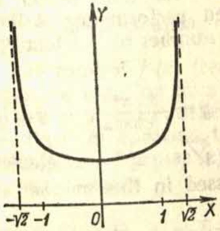

Fig. 211

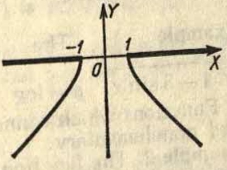

Fig. 212

Example 2. The domain of the function $y=\sqrt{1-x^2}$ is the closed interval $(-1, 1)$ The graph (semicircle) lies above this interval (Fig. 210).

Example 3. The domain of the function

$$y=\frac{1}{\sqrt{2-x^2}}$$

is the interval $(-\sqrt{2}, \sqrt{2})$ (open). The function is not defined on the end-points of the interval (it becomes infinite). The graph (Fig. 211) lies above the interior points of the interval. There are no points above the end-points of the interval and exterior to it.

Example 4. The domain of the function

$$y=-\sqrt{x^2-1}$$

is the pair of intervals $(-\infty, -1)$ and $(1, +\infty)$ with adjoined end-points -1 and 1. The graph (lower half of the hyperbola $x^2-y^2=1$, Fig. 212) lies under these intervals.

200. Classification of Functions

(a) Functions are divided into *single-valued* and *multiple-valued* types (Sec. 196, Definition 2).

(b) Functions represented by formulas are divided into *explicit* and *implicit* types (Sec. 197).

(c) Functions may be *elementary* or *nonelementary*.[1]

A list of the so-called *basic elementary functions* is given in Sec. 201. Each of them represents some kind of "operation" on the argument (squaring, extracting a cube root, taking a logarithm, finding the sine, etc.). New functions (which are also elementary) result from repeated performance of these operations together with any limited number of the four operations of arithmetic.

Example 1. The functions $y=\frac{3+x^2}{1+\log x}$, $y=\log \sin\sqrt[3]{1-3\sin x}$, $y=\log\log(3+2\sqrt[3]{\sin x})$ are elementary. Functions which cannot be expressed in this manner are termed nonelementary.

Example 2. The function $s=1+2+3+\ldots+n$ is an elementary function because it may be expressed by the formula $s=\frac{(1+n)n}{2}$, which contains a limited number of elementary operations.

Example 3. The function $s=1\cdot 2\cdot 3\ldots n$ is nonelementary because it cannot be expressed by a *limited* number of elementary operations (the greater n, the greater number of multiplications have to be performed; it is impossible to transform the expression $1 \cdot 2 \cdot 3 \ldots n$ to an elementary form).

---

[1] The nature of this subdivision is more historical than mathematical.

Note. We have intentionally refrained from subdividing functions into algebraic and transcendental since an exact definition of an algebraic function can only be given on the basis of more sophisticated concepts (continuity or differentiability). What is more, there is no need, within the scope of this book, to differentiate between algebraic and transcendental functions.

201. Basic Elementary Functions

(1) *Power* function $y=x^n$ (where n is a constant real number). For $n=0$, the power function is a constant quantity ($y=1$) (cf. Sec. 196, Note 2).

(2) *Exponential* function $y=a^x$, where a is a positive number [1] (the *base number*).

(3) *Logarithmic* function $y=\log_a x$, where a is a positive number different from unity [2] *(logarithmic base)*.

(4) *Trigonometric* functions $y=\sin x$, $y=\cos x$, $y=\tan x$, $y=\cot x$, $y=\sec x$, $y=\operatorname{cosec} x$.

(5) *Circular (inverse trigonometric)* functions:

$y=\arcsin x, \quad y=\arccos x, \quad y=\arctan x$

$y=\operatorname{arccot} x, \quad y=\operatorname{arcsec} x, \quad y=\operatorname{arccsc} x$ (or $y=\operatorname{arccosec} x$)

202. Functional Notation

The symbol $f(x)$ (read: "f of x") is an abbreviation of the phrase "a function of x".

If two or more different functions of x are being considered, then, in addition to $f(x)$, we can use such notations as $f_1(x)$, $f_2(x)$, $F(x)$, $\varphi(x)$, $\Phi(x)$

The *notation*

$$y=f(x) \tag{1}$$

expresses the fact that the quantity y is equal to some function of x, or that y is a function of the argument x.

The symbol $f(x)$ can be used to designate both an unknown function and a known function.

Examples. (1) The notation $f(x)=\log x$ expresses the fact that the function $f(x)$ is a logarithmic function.

(2) $\varphi(x)=x^n$ states that the function $\varphi(x)$ is a power function.

---

[1] Some writers exclude $a=1$ (here y is a constant).

[2] For base $a=1$, no number (except unity) has a logarithm.

(3) The notation $F(x)=\varphi(x)+f(x)$ means that the function $F(x)$ is the sum of the functions $\varphi(x)$ and $f(x)$. If $f(x)=\log x$ and $\varphi(x)=x^n$, then $F(x)=\log x+x^n$.

(4) The notation $f_1(x)=f_2(x)$ signifies that the functions $f_1(x)$ and $f_2(x)$ are equal (either identically or only for certain values of x).

(5) The notation $u=\varphi(v)$ means that the quantity u is some function of the argument v.

The letter f (or F, φ etc.) used in these notations is called the *function symbol*.

If it is necessary to state that y is dependent upon x in the same way that u is upon v, then the same function symbol is used in notation; thus,

$$u=\varphi(v) \text{ and } y=\varphi(x) \tag{2}$$

or

$$u=F(v) \text{ and } y=F(x) \tag{3}$$

and so forth.

Thus, if the relationship of u and v is expressed by the formula $u=\pi v^2$, then the relationship of y and x is, by virtue of (2), expressed by the formula $y=\pi x^2$. If $u=\frac{\log v}{1+v}$, then $y=\frac{\log x}{1+x}$, etc.

Examples. (6) If $f(x)=\sqrt{1+x^2}$, then $f(t)=\sqrt{1+t^2}$.

(7) If $F(\alpha)=1-\tan^2\alpha$, then $F(\beta)=1-\tan^2\beta$, $F(\gamma)=1-\tan^2\gamma$, etc.

(8) If $f(x)=4$ (i.e. for all values of the argument the function has one and the same value; cf. Sec. 196, Note 2), then $f(y)=4$, $f(z)=4$, etc.

The notations $f(1)$, $f(\sqrt{3})$, $f(a)$, etc. state that we take the values of the function $f(x)$ for $x=1$, for $x=\sqrt{3}$, $x=a$, etc. or the values of the function $f(y)$ for $y=1$, $y=\sqrt{3}$, $y=a$, etc.

Examples. (9) If $f(x)=\sqrt{x^2+1}$, then

$$f(1)=\sqrt{2},\quad f(\sqrt{3})=2,\quad f(a)=\sqrt{a^2+1}$$

(10) If $\varphi(\alpha)=\frac{1}{1+\sin^2\alpha}$, then $\varphi(0)=1$, $\varphi\left(\frac{\pi}{2}\right)=\frac{1}{2}$, $\varphi(\pi)=1$, $\varphi\left(\frac{\pi}{4}\right)=\frac{2}{3}$.

203. The Limit of a Sequence

The number b is called the *limit of the sequence* (Sec. 198, Item 3) $y_1, y_2, \dots, y_n, \dots$ if the term y_n approaches b without bound as the number n increases.

The exact meaning of the expression "approaches without bound" is explained below (immediately after Example 1).

The *notation* [1]

$$\lim y_n = b$$

or, expanded,

$$\lim_{n \to \infty} y_n = b$$

The symbol $n \to \infty$ stresses the fact that the number n increases without bound (tends to infinity).

Example 1. Consider the sequence

$$y_1 = 0.3, \quad y_2 = 0.33, \quad y_3 = 0.333, \ \dots \tag{1}$$

The term y_n approaches $\frac{1}{3}$ without bound (in decimals: 0.3, 0.33, ... which give increasingly exact values of the fraction $\frac{1}{3}$). Hence, $\frac{1}{3}$ is the limit of the sequence (1):

$$\lim y_n = \frac{1}{3}$$

Note. The difference $y_n - \frac{1}{3}$ is successively equal to

$$y_1 - \frac{1}{3} = -\frac{1}{30}, \quad y_2 - \frac{1}{3} = -\frac{1}{300}, \quad y_3 - \frac{1}{3} = -\frac{1}{3000} \tag{2}$$

i.e.

$$y_n - \frac{1}{3} = -\frac{1}{3 \cdot 10^n} \tag{3}$$

The *unbounded* nature in the approach of y_n to $\frac{1}{3}$ is expressed by the fact that the absolute magnitude of the difference (3) beyond some number N remains less than *any* (preassigned) positive number ε. Thus, if we specify $\varepsilon = 0.01$, then $N = 2$, which means that starting with the second number the absolute value $\left| y_n - \frac{1}{3} \right|$ remains less than 0.01. If $\varepsilon = 0.005 \left(= \frac{1}{200} \right)$ is specified, then, again, $N = 2$. If $\varepsilon = 0.001$, then $N = 3$; if $\varepsilon = 0.00001$, then $N = 5$, etc.

It is now easy to understand the exact statement of the definition given at the beginning of this section.

---

[1] The abbreviation lim stands for the Latin *limes*=limit.

Definition. The number b is called the *limit of the sequence* $y_1, y_2, \dots, y_n, \dots$ if the absolute value of the difference $y_n - b$ beyond some number N remains less than any preassigned positive number ε:

$$|y_n - b| < \varepsilon \quad \text{for} \quad n \geqslant N$$

(the integer N depends on the magnitude of ε).

Example 2. In the sequence $y_n = 2 + \frac{(-1)^n}{n}$ (i.e. $y_1 = 1$, $y_2 = 2\frac{1}{2}$, $y_3 = 1\frac{2}{3}$, $y_4 = 2\frac{1}{4}, \dots$) the term y_n tends to 2 as the number n increases. Hence, 2 is the limit of the sequence.

Indeed, we have $|y_n - 2| = \frac{1}{n}$, while $\frac{1}{n}$, beyond some integer, remains less than a preassigned positive number ε (if $\varepsilon = 2$, then from the first integer on; if $\varepsilon = 0.02$, then from the 51st integer on, etc.)

Example 2 shows that the terms of a sequence can oscillate about the limit and (see Example 3) can also be equal to the limit.

Example 3. The sequence

$$y_1 = 0,\ y_2 = 1,\ y_3 = 0,\ y_4 = \frac{1}{2},\ y_5 = 0,\ y_6 = \frac{1}{3},\ \dots$$

specified by the formula $y_n = \frac{1}{n} + \frac{(-1)^n}{n}$ has the limit $b = 0$.

Indeed, the absolute value $|y_n - 0| = \left|\frac{1}{n} + \frac{(-1)^n}{n}\right|$, beyond some integer, remains less than any preassigned positive number ε (if $\varepsilon = \frac{1}{3}$, then from the seventh integer; if $\varepsilon = 0.01$, then from the 201st integer, etc).

Example 4. The sequence $y_n = (-1)^n$ has no limit: the terms $y_1 = -1$, $y_2 = 1$, $y_3 = -1$, $y_4 = 1$, etc. do not approach any constant number.

204. The Limit of a Function

The number b is called the limit of the function $f(x)$ as $x \longrightarrow a$ (read: "as x approaches a", or "as x tends to a"), if as x approaches a either from the right or the left, the value of $f(x)$ approaches (tends to) b without bound. [1]

---

[1] The mathematical meaning of the expression "approaches without bound" is explained in Sec. 205, but the present definition (with account taken of Note 1) is quite sufficient for an understanding of the sequel.

Notation:

$$\lim_{x \to a} f(x) = b$$

Note 1. It is assumed that the function $f(x)$ is defined within some interval containing the point $x=a$ (at all points on the left and on the right of a); at the point $x=a$ itself, the function $f(x)$ is either defined or is not (the latter case is no less important than the former).

Example 1. Consider the function $f(x)=\frac{4x^2-1}{2x-1}$ (it is defined at all points with the exception of $x=\frac{1}{2}\Big)$. Take $x=6$. Then $f(x)=\frac{4\cdot 6^2-1}{2\cdot 6-1}=13$. As x approaches 6 (from the right or from the left) the numerator $4x^2-1$ tends to 143 and the denominator tends to 11. The fraction as a whole tends to $\frac{143}{11}=13$. The number 13 (equal to the value of the function at $x=6$) is at the same time the limit of the function as $x \longrightarrow 6$:

$$\lim_{x \to 6} \frac{4x^2-1}{2x-1}=13$$

Example 2. Consider the same function $f(x)=\frac{4x^2-1}{2x-1}$ but take $x=\frac{1}{2}$. The function $f(x)$ is not defined here (the formula yields the indeterminate form $\frac{0}{0}\Big)$. But the limit of the function exists as $x \longrightarrow \frac{1}{2}$. It is equal to 2.

Indeed, the expression $\frac{4x^2-1}{2x-1}$ is indeterminate *only for* $x=\frac{1}{2}$, but *as x approaches* $\frac{1}{2}$ it is quite determinate and is always equal to $2x+1$. This expression tends to the number 2. Hence,

$$\lim_{x \to \frac{1}{2}} \frac{4x^2-1}{2x-1}=2.$$

Note 2. The graph of the function $y=\frac{4x^2-1}{2x-1}$ is the straight line UV (Fig. 213) devoid of the point $A\left(\frac{1}{2}\ \ 2\right)$. The graph of the function $y=2x+1$ is the same straight line UV taken in its entirety.

Example 3. The function $f(x)=\cos\frac{\pi}{x}$ (it is defined at all points except $x=0$) does not have a limit as $x\to 0$. This is evident from the graph (Fig. 214): when the abscissa approaches zero, the ordinate does not approach anything (the point of the graph performs infinite oscillations with a constant amplitude).

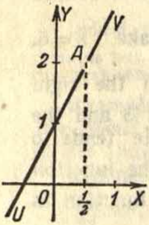

Fig. 213

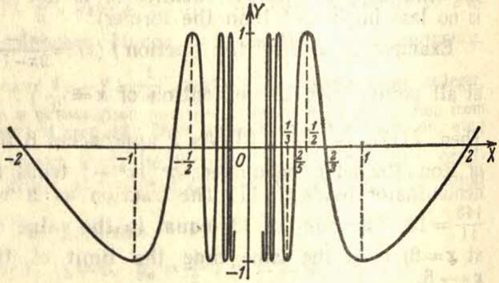

Fig. 214

205. The Limit of a Function Defined

The unbounded nature of the approach of a variable quantity to a constant is expressed (cf. Sec. 203) by the fact that, from some instant onwards, their difference is less than any preassigned positive number. Accordingly, the definition in Sec. 204 can be stated precisely:

Definition. The number b is called the *limit* of the function $f(x)$ as $x\to a$, if the absolute value of the difference $f(x)-b$ remains less than any preassigned positive number ε every time that the absolute value of the difference $x-a$, for $x\neq a$, is less than some positive number δ (dependent on ε).

More briefly (but less rigorously): the number b is the limit of the function $f(x)$ as $x\to a$, if the absolute value $|f(x)-b|$ is arbitrarily small when $|x-a|$ is sufficiently small.

Example. The number 2 is the limit of the function $f(x)=\frac{4x^2-1}{2x-1}$ as $x\to\frac{1}{2}$ (cf. Sec. 204, Example 2).

Indeed, let us require that

$$\left|\frac{4x^2-1}{2x-1}-2\right|$$

$\left(\text{for } x \neq \frac{1}{2}\right)$ be less than ε. We get the inequality

$$|2x-1|<\varepsilon$$

which is equivalent to the inequality

$$\left|x-\frac{1}{2}\right|<\frac{\varepsilon}{2}.$$

Hence, the absolute value of the difference $\frac{4x^2-1}{2x-1}-2$ remains less than any preassigned positive number ε every time that the absolute value of the difference $x-\frac{1}{2}$ is less than $\frac{\varepsilon}{2}$. In the given instance, $\delta=\frac{\varepsilon}{2}$.

206. The Limit of a Constant

Definition. The limit of a constant quantity b is the quantity b itself.

This definition is introduced so that the basic theorems on limits (Sec. 213) should hold true in all cases without exception. It agrees with the definitions of Secs. 203 and 205 (the quantity $|b-b|=0$ is less than any positive number ε).

207. Infinitesimals

An *infinitesimal* is a quantity whose limit is equal to zero.

Example 1. The function x^2-4 is an infinitesimal as $x \longrightarrow 2$ and as $x \longrightarrow -2$. As $x \longrightarrow 1$, the same function is not an infinitesimal.

Example 2. The function $1-\cos\alpha$ is an infinitesimal as $\alpha \longrightarrow 0$, because $\lim\limits_{\alpha \to 0}(1-\cos\alpha)=0$.

In words: "the quantity $1-\cos\alpha$ is infinitely small for infinitely small α".

Example 3. The quantity $\frac{4x^2-1}{2x-1}$ as $x \longrightarrow \frac{1}{2}$ is not an infinitesimal because its limit is equal to 2 (Sec. 204, Example 2).

Example 4. The integral function $y=\frac{1}{n!}$ (Sec. 198, Example 7) is an infinitesimal because the limit of the sequence $\frac{1}{1}, \frac{1}{1\cdot 2}, \frac{1}{1\cdot 2\cdot 3}, \ldots$ is equal to zero.

Note 1. The statements "the number b is the limit of the variable y" and "the difference $y-b$ is an infinitesimal" are equivalent.

Example 5. We have $\lim\limits_{x \to \frac{1}{2}} \frac{4x^2-1}{2x-1}=2$. The same fact may be expressed as "the quantity $\frac{4x^2-1}{2x-1}-2$ is an infinitesimal".

Note 2. Of all constant quantities, only zero is an infinitesimal (cf. Sec. 206).

208. Infinities

An *infinite quantity* is a variable whose absolute value increases without bound.

The exact meaning of the phrase "increases without bound" will be given at the end of this section.

Example 1. The integral function $y=n!$ is an infinitely large quantity because the terms of the sequence 1, $1 \cdot 2$, $1 \cdot 2 \cdot 3$, ... increase without bound.

Example 2. The function $\frac{1}{x}$ is an infinite quantity for infinitesimal x because as x approaches zero the absolute value of $\frac{1}{x}$ increases without bound.

Example 3. The function $\tan x$ is an infinite quantity as $x \longrightarrow \frac{\pi}{2}$.

No constant quantity can be an infinitely large quantity.

Note. The expression "the absolute value of the quantity y increases without bound" means that from some instant onwards $|y|$ remains greater than any preassigned positive number. Accordingly, the concept of an infinitely large quantity can be defined rigorously as follows:

Definition 1. An integral function y is an infinitely large quantity if the absolute value of y_n beyond some number N remains greater than any preassigned positive number M (cf. Sec. 203).

Definition 2. The function $f(x)$ is an infinitely large quantity as $x \to a$ if the absolute value of $f(x)$ remains greater than any preassigned positive number M every time that the absolute value of the difference $x-a$ is less than some positive number δ (dependent on M) (cf. Sec. 205).

209. The Relationship Between Infinities and Infinitesimals

If y is an infinitely large quantity, then $\frac{1}{y}$ is an infinitely small quantity (infinitesimal); if y is an infinitesimal, then $\frac{1}{y}$ is an infinitely large quantity.

Example 1. The quantity $\frac{3}{x-2}$ is infinitely large as $x \longrightarrow 2$. The reciprocal fruction $\frac{x-2}{3}\left(=1 : \frac{3}{x-2}\right)$ is infinitesimal as $x \longrightarrow 2$.

Example 2. The quantity $\tan x$ is infinitesimal as $x \longrightarrow 0$, the quantity $\frac{1}{\tan x} = \cot x$ is infinitely large as $x \longrightarrow 0$.

210. Bounded Quantities

A quantity is called *bounded* if its absolute value does not exceed some (constant) positive number M.

Example 1. The function $\sin x$ is a bouded quantity on the entire real number axis because $|\sin x| \leqslant 1$.

Example 2. The function $\frac{1}{x-2}$ is bounded in the interval (3, 5) but is not bounded in the interval (2, 5) because the argument x can tend to 2 within the interval (2, 5), and then the function is infinitely large (Fig. 215).

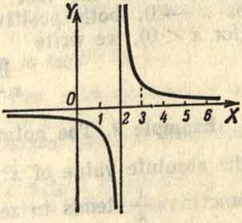

Fig. 215

Every constant quantity is bounded. Every infinitely large quantity is unbounded.

Note. An unbounded quantity may not be infinitely large. Thus, the integral function $n+(-1)^n$ is not infinitely large because for odd n it is always zero; but neither is it bounded because for even n, from some point onwards, it remains greater than any positive number M.

211. An Extension of the Limit Concept

If a variable quantity s is infinitely great, then we say that s "approaches infinity" or "has an infinite limit".

Notation:

$$s \longrightarrow \infty \text{ or } \lim s = \infty \tag{1}$$

If from some instant onwards 1) an infinitely large quantity remains positive, then we say that it "tends to plus infinity" and we write

$$s \longrightarrow +\infty \text{ or } \lim s = +\infty \tag{2}$$

If from some instant onwards an infinitely large quantity remains negative, we say that it "tends to minus infinity" and we write

$$s \longrightarrow -\infty \text{ or } \lim s = -\infty \tag{3}$$

In place of (1) we often find more expressive the notation

$$s \longrightarrow \pm\infty \text{ or } \lim s = \pm\infty \tag{4}$$

Example 1. As $x \longrightarrow 0$ the function $\cot x$ has an infinite limit:

$$\lim_{x \to 0} \cot x = \infty$$

To stress the fact that the function $\cot x$ can assume, as $x \longrightarrow 0$, both positive values (for $x > 0$) and negative (for $x < 0$), we write

$$\lim_{x \to 0} \cot x = \pm\infty$$

Example 2. The notation $\lim\limits_{x \to \infty} \frac{1}{x} = 0$ means that when the absolute value of x increases without bound, then the function $\frac{1}{x}$ tends to zero.

Example 3. We can write

$$\lim_{x \to +\infty} 2^x = +\infty$$

or

$$\lim_{x \to +\infty} 2^x = \infty$$

The latter notation leaves the question of the sign of the function 2^x open. But one cannot write $x \to \infty$ in place of $x \to +\infty$ in the left-hand members. The former notation would include the case when

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1) This expression is made precise in the same way as in Sec. 208 (definitions 1 and 2).

$x \to -\infty$, but then the function 2^x would tend to zero and not to infinity, i. e.

$$\lim_{x \to -\infty} 2^x = 0$$

Note. An infinite quantity does not have a limit in the earlier established meaning (Secs. 203-205) for the simple reason that one cannot, for instance, say that "the difference between $f(x)$ and ∞ remains less than any preassigned positive number". Thus the introduction of an infinite limit *extends* the limit concept. In contrast to an infinite limit, the limit defined earlier is called a *finite* limit.

212. Basic Properties of Infinitesimals

It is assumed here that the quantities under consideration are functions of *one and the same* argument.

Theorem I. The sum of two, three and, in general, any fixed number of infinitesimals is an infinitesimal.

Note 1. If the number of terms is not fixed, but varies together with the variation of the argument, then Theorem I may become invalid. Thus, if we have n terms equal separately to $\frac{1}{n}$, then as $n \longrightarrow \infty$ each term is infinitely small, but the sum $\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = \frac{1}{n} \cdot n$ is equal to 1.

Note 2. The difference between two infinitesimals is an infinitesimal (a particular case of Theorem I).

Theorem II. The product of a bounded quantity (Sec. 210) by an infinitesimal is an infinitesimal.

In particular, the product of a constant quantity by an infinitesimal and also the product of two infinitesimals is an infinitesimal.

Theorem III. The quotient of an infinitesimal divided by a variable quantity tending to a limit *not equal to zero* is an infinitesimal.

Note 3. If the limit of the divisor is equal to zero, i.e. if the dividend and divisor are both infinitely small, then the quotient may not be an infinitesimal. Thus, the quantities x^2 and x^3 are infinitely small as $x \to 0$. The quotient $x^3 : x^2 = x$ is also infinitely small, but the quotient $x^2 : x^3 = \frac{1}{x}$ is infinitely great. The quantities $6x^2 + x^3$ and $2x^2$ are infinitely small as $x \to 0$, but the limit of the quotient $(6x^2 + x^3) : 2x^2$ is equal to 3.

213. Basic Limit Theorems

It is assumed here that all the given quantities (summands, factors, dividend, and divisor) depend on *one and the same* argument x and have *finite* limits (as $x \to a$ or as $x \to \infty$).

Theorem I. The limit of a sum of two, three and, generally, any fixed number of terms is equal to the sum of the limits of the separate terms (cf. Sec. 212, Note 1).

More concisely: *the limit of a sum is equal to the sum of the limits:*

$$\lim (u_1 + u_2 + \ldots + u_k) = \lim u_1 + \lim u_2 + \ldots + \lim u_k \quad (1)$$

Here, the symbol $x \to a$ (or $x \to \infty$) is assumed for each limit sign.

Theorem Ia (particular case of Theorem I):

$$\lim (u_1 - u_2) = \lim u_1 - \lim u_2 \quad (2)$$

Theorem II. The limit of a product of two, three and, generally, any fixed number of factors is equal to the product of their limits:

$$\lim (u_1 u_2 \ldots u_k) = \lim u_1 \cdot \lim u_2 \ldots \lim u_k \quad (3)$$

Theorem IIa. A constant factor may be taken outside the sign of the limit:

$$\lim cu = c \lim u \quad (4)$$

Theorem III. The limit of a quotient is equal to the quotient of the limits if the limit of the divisor is not equal to zero:

$$\lim \frac{u}{v} = \frac{\lim u}{\lim v} \ (\lim v \neq 0) \quad (5)$$

Example 1.

$$\lim_{x \to 5} \frac{x+4}{x-2} = \lim_{x \to 5} (x+4) : \lim_{x \to 5} (x-2) = 9:3 = 3$$

If the limit of the divisor is zero and the limit of the divident is nonzero, then the quotient has an infinite limit.

Example 2.

$$\lim_{x \to 2} \frac{x+4}{x-2} = \infty$$

Here

$$\lim_{x \to 2} (x-2) = 0 \quad \text{and} \quad \lim_{x \to 2} (x+4) = 6 \neq 0$$

Note 1. If both the dividend and the divisor tend to zero, then the quotient can have either an infinite limit or a finite limit (Sec. 212, Note 3). It can also have no limit.

Thus, $\lim\limits_{x \to 0} x^2 \cos \frac{\pi}{x} = 0$ and $\lim\limits_{x \to 0} x^2 = 0$, but the quotient $x^2 \cos \frac{\pi}{x} : x^2 = \cos \frac{\pi}{x}$ does not have a limit as $x \to 0$ (Sec. 204, Example 3).

Note 2. When $\lim v = 0$, but $\lim u \neq 0$, then Theorem III holds true if it is interpreted in a broader sense: namely, that the notation $\lim\limits_{x \to a} f(x) = \frac{c}{0}$ (c is a number not equal to zero) is to be understood in the sense that $\lim\limits_{x \to a} f(x) = \infty$.

Example 3. Find $\lim\limits_{x \to 2} \frac{x+4}{x-2}$.

The limit of the divisor is zero, the limit of the dividend is 6. Taking the notation $\frac{6}{0}$ in the indicated meaning, we obtain

$$\lim_{x \to 2} \frac{x+4}{x-2} = \frac{6}{0} = \infty$$

(cf. Example 2).

Note 3. When $\lim v = 0$ and $\lim u = 0$, Theorem III is *inapplicable* since the expression $\frac{0}{0}$ is indeterminate. However, even in this case Theorem III cannot yield an *incorrect* result. For example, let it be required to find

$$\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x - 1}$$

Applying Theorem III formally, we obtain $\frac{0}{0}$. This indeterminate expression serves as a signal that the direct route is closed and a detour route has to be sought (see Sec. 204, Example 2).

It is of course impossible to cancel out the zeros and write 1 in place of $\frac{0}{0}$.

214. The Number e

The integral function $u_n = \left(1 + \frac{1}{n}\right)^n$ increases as $n \to \infty$, but remains bounded.[1] But every increasing, yet bounded, quantity has a (finite) limit. The limit to which $\left(1+\frac{1}{n}\right)^n$ tends as $n \to \infty$ is e:

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[1] It might seem that the unbounded increase in the exponent would imply an unbounded increase in the function $\left(1 + \frac{1}{n}\right)^n$. But the growth in the exponent is compensated for by the fact that the

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{1}$$

The number e (which is irrational) is equal (to six significant figures) to

$$e = 2.71828$$

In many cases it is advantageous to take the number e as a logarithmic base (cf. Sec. 242).

The function $\left(1+\frac{1}{n}\right)^n$ has the number e as its limit not only for integral values of n but even when n approaches infinity ranging over the entire number line in continuous fashion. What is more, the argument n can assume both positive and negative values provided only that it increases without bound in absolute value. To bring this circumstance out more vividly, let us replace the letter n by the letter x and write

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e \tag{2}$$

(see Sec. 211) or

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \tag{3}$$

---

base $1+\frac{1}{n}$ tends to 1. It is useful to verify this computationally using five-place tables of logarithms:

$$\left(1+\frac{1}{5}\right)^5 = 2.48, \quad \left(1+\frac{1}{10}\right)^{10} = 2.59, \quad \left(1+\frac{1}{50}\right)^{50} = 2.69$$

$$\left(1+\frac{1}{100}\right)^{100} = 2.71$$

It is possible to prove the bounded character of $\left(1+\frac{1}{n}\right)^n$ by means of the binomial formula. The first term is 1, the second is also 1, the third is equal to $\frac{n(n-1)}{2}\cdot\frac{1}{n^2}$ for any n less than $\frac{1}{2}$, the fourth is always less than $\frac{1}{2^2}$, the fifth is less than $\frac{1}{2^3}$, etc. And so any value of u_n is less than

$$1+1+\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots\right)$$

that is to say, it is less than 3.

215. The Limit of $\frac{\sin x}{x}$ as $x \to 0$

If x is the radian measure of the angle, then

$$\lim_{x \to 0} \frac{x}{\sin x} = 1 \text{ and } \lim_{x \to 0} \frac{\sin x}{x} = 1 \quad (1)$$

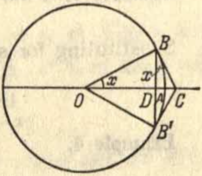

Fig. 216

Explanation. Let us take the radius OA (Fig. 216) as unit length. Then $x = \overset{\frown}{AB}$, $\sin x = BD$. We have $x : \sin x = \overset{\frown}{AB} : BD = \overset{\frown}{B'AB} : B'B$. The arc $\overset{\frown}{B'AB}$ is greater than the chord $B'B$. Therefore $x : \sin x > 1$. On the other hand, the arc $\overset{\frown}{B'AB}$ is less than $BC + B'C = 2BC$, i. e. $\overset{\frown}{AB} < BC$. Hence, $x : \sin x < BC : BD = \sec x$ (from the triangle DBC)

Hence, the ratio $\frac{x}{\sin x}$ lies between unity and $\sec x$. But the magnitude of $\sec x$ itself tends to unity as $x \to 0$, and hence $\frac{x}{\sin x}$ surely does.

216. Equivalent Infinitesimals

Definition. Two infinitesimals are called *equivalent* if the limit of their ratio is equal to unity.

Example 1. The quantities x and $\sin x$, which are infinitesimal as $x \to 0$, are equivalent because (Sec. 215) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$. The quantities $2x$ and $\sin 2x$ are equivalent. The quantities x^2 and $\sin^2 x$ are also equivalent.

Example 2. The infinitesimals $\alpha^2 + 3\alpha^3$ and $\alpha^2 - 4\alpha^3$ $(\alpha \to 0)$ are equivalent because

$$\lim_{\alpha \to 0} \frac{\alpha^2 + 3\alpha^3}{\alpha^2 - 4\alpha^3} = \lim_{\alpha \to 0} \frac{1 + 3\alpha}{1 - 4\alpha} = 1$$

The equivalence of infinitesimals is denoted by the same symbol $\approx$ as approximate equality. Thus,

$$\sin x \approx x, \ \sin 2x \approx 2x, \ \sin^2 x \approx x^2, \ \alpha^2 + 3\alpha^3 \approx \alpha^2 - 4\alpha^3$$

Note. Indeed, equivalent quantities are approximately equal (the equality is the more exact, the closer to zero the equivalent quantities approach). Thus, for $\alpha = 0.01$ the quantity $\alpha^2 + 3\alpha^3$ is equal to 0.000103; and $\alpha^2 - 4\alpha^3$ is 0.000096. The difference amounts to 0.000007, which is about 7% of one of the equivalent quantities. The closer they are to zero, the smaller is the percentage.

Theorem. The limit of a quotient (ratio) of two infinitesimals remains the same if one of them (or both) is replaced by an equivalent quantity.

Example 3. Find $\lim\limits_{x \to 0} \frac{\sin 2x}{x}$.

Substituting for $\sin 2x$ the equivalent quantity $2x$, we get

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{2x}{x} = 2$$

Example 4.

$$\lim_{x \to 0} \frac{\sin 2x}{\sin 5x} = \lim_{x \to 0} \frac{2x}{5x} = \frac{2}{5}$$

Example 5. Find

$$\lim_{x \to 0} \frac{1-\cos x}{x}$$

Solution. We have

$$1-\cos x = 2\sin^2 \frac{x}{2}$$

and since

$$\sin^2 \frac{x}{2} \approx \left(\frac{x}{2}\right)^2$$

it follows that

$$\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{2\left(\frac{x}{2}\right)^2}{x} = 0$$

217. Comparison of Infinitesimals

Definition 1. If the ratio $\frac{\beta}{\alpha}$ of two infinitesimals is itself infinitely small $\left[\text{i.e. if } \lim \frac{\beta}{\alpha} = 0, \text{ and, hence (Sec. 209), } \lim \frac{\alpha}{\beta} = \infty\right]$, then β is termed a quantity of *higher order* relative to α; and α is a quantity of *lower order* with respect to β.

Definition 2. If the ratio $\frac{\beta}{\alpha}$ of two infinitesimals tends to a finite limit not equal to zero, then α and β are called infinitesimals of the *same order.* [1]

Note. Equivalent infinitesimals always have one and the same order [2]

Example 1. As $x \to 0$, the quantity x^5 is of higher order than x^3 because $\lim\limits_{x \to 0} \frac{x^5}{x^3} = 0$. Conversely, x^3 is of lower order than x^5 because $\lim\limits_{x \to 0} \frac{x^3}{x^5} = \infty$.

Example 2. As $x \to 0$, the quantities $\sin x$ and $2x$ are of the same order because (Sec. 215)

$$\lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2}$$

Example 3. As $x \to 0$, the quantity $1 - \cos x$ is of higher order than $\sin x$ since (Sec. 216, Example 5)

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin x} = 0$$

As $\alpha \to 0$, each of the quantities $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \ldots$ is of lower order than any successor. Therefore, the following definition is the keystone of the further classification of infinitesimals.

Definition 3. An infinitesimal β is of the *m*th order with respect to an infinitesimal α if β is of the same order as α^m, i.e. (see Definition 2) if the ratio $\frac{\beta}{\alpha^m}$ has a finite limit not equal to zero.

Example 4. As $x \to 0$, the infinitesimal $\frac{1}{4}x^3$ is of third order with respect to x because $\lim\limits_{x \to 0} \left(\frac{1}{4}x^3 : x^3\right) = \frac{1}{4}$, the infinitesimal $\frac{1}{7}x^2$ is of second order, the infinitesimal $\sqrt{x}$ is of order $\frac{1}{2}$.

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[1] In place of the ratio $\frac{\beta}{\alpha}$ we can take the reciprocal ratio $\frac{\alpha}{\beta}$, since it too will have a finite limit not equal to zero $\left(\text{if } \lim \frac{\beta}{\alpha} = m,\right.$ then $\left.\lim \frac{\alpha}{\beta} = \frac{1}{m}\right)$.

[2] The converse does not hold true. Thus, the quantities $2x$ and $3x$ have, as $x \to 0$, the same order $\left(\lim\limits_{x \to 0} \frac{2x}{3x} = \frac{2}{3}\right)$ but they are not equivalent.

Example 5. The infinitesimal $1-\cos\alpha$ $(\alpha\to 0)$ is of second order with respect to α since (see note on Definition 2)

$$1-\cos\alpha=2\sin^2\frac{\alpha}{2}\approx 2\left(\frac{\alpha}{2}\right)^2$$

Example 6. The infinitesimal $\frac{1}{4}\alpha^3+1000\alpha^4$ $(\alpha\to 0)$ is of third order, that is, the same as the term $\frac{1}{4}\alpha^3$, the order of which is lower than that of the other term. That always occurs in the case of a sum of two or more terms.

Example 7. The infinitesimal $x^3\sin^2 x$ $(x\to 0)$ is of fifth order with respect to x (the number 5 is the sum of the orders of the factors; this always occurs in the case of a product of two or more factors).

Theorem 1. The difference $\alpha-\beta$ of two equivalent infinitesimals α and β is of higher order with respect to either α or β.

Example 8. We have $x\approx\sin x$ as $x\to 0$. Therefore, $x-\sin x$ is of higher order with respect to x (and also with respect to $\sin x$).

Theorem 2 (converse). If the difference of the infinitesimals α and β is of higher order with respect to one of them (then it is of higher order with respect to the other as well), then $\alpha\approx\beta$.

Example 9. The infinitesimals $\alpha^2+3\alpha^3$ and α^2 $(\alpha\to 0)$ differ by $3\alpha^3$: this is a quantity of higher order than α^2. And so

$$\alpha^2+3\alpha^3\approx\alpha^2$$

217a. The Increment of a Variable Quantity

Definition. If a variable z assumes the value $z=z_1$ and then $z=z_2$, then the difference z_2-z_1 is called the *increment* of z. The increment may be positive, negative, or zero. The increment is denoted by the Greek letter Δ (delta) (the symbol Δz reads "delta z"). It denotes the change in z, "increment of the quantity z"; we have

$$\Delta z=z_2-z_1$$

The increment of a constant is zero.

Example. The initial value of the argument $x=3$, the increment of the argument $\Delta x=-2$. Find the corresponding increment Δy of the function $y=x^2$.

Solution. Since $x_1=3$ and $x_2-x_1=-2$, it follows that $x_2=1$. The function $y=x^2$ first takes on the value $y_1=3^2=9$, and then $y_2=1^2=1$.

The increment of the function is $\Delta y=y_2-y_1=1-9=-8$.

218. The Continuity of a Function at a Point

Definition. A function $f(x)$ is called *continuous at a point* $x=a$ if the following two conditions are fulfilled:

1. For $x=a$ the function $f(x)$ has a definite value b.
2. As $x \to a$, the function has a limit which is also equal to b.

If even one of these conditions is violated, the function is called *discontinuous* at the point $x=a$.

Example 1. The function $f(x)=\frac{1}{x-3}$ is continuous at the point $x=5$ (M in Fig. 217) because (1) at $x=5$ it has a definite value $f(5)=\frac{1}{2}$; (2) as $x \to 5$ it has a limit which is also equal to $\frac{1}{2}$. The function is discontinuous at the point $x=3$ because the first condition is not fulfilled (the function does not have a definite value). Neither is the second condition fulfilled.

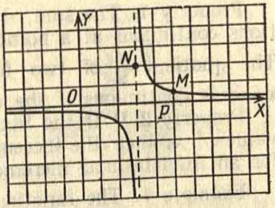

Fig. 217

Example 2. Let us specify the function $\varphi(x)$ as follows:

$$\varphi(x)=\frac{1}{x-3} \quad \text{for } x \neq 3,$$

$$\varphi(x)=2 \quad \text{for } x=3$$

This function (its graph is obtained from the graph of Example 1 by adjoining the point N; see Fig. 217) is also discontinuous at the point $x=3$. This time the first condition is fulfilled but the second is not: the function $\varphi(x)$ has an infinite limit as $x \to 3$.

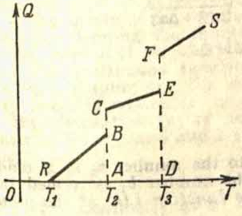

Fig. 218

Example 3. The quantity Q of heat imparted to a body is a function of the temperature T of the body. Fig. 218 depicts the graph of this function. The line RB corresponds to the solid state (T_1 is the initial temperature, T_2 the melting point), the line CE corresponds to the liquid state (T_3 is the temperature of vapourization), the line FS corresponds to the gaseous state. The function Q is discontinuous at $T=T_2$ and $T=T_3$; it does not have a definite value at these points. Thus, the melting point T_2 is associated with all possible quantities of heat from $Q = AB$ to $Q = AC$.

219. The Properties of Functions Continuous at a Point

Property 1. The sum, difference and product of two functions continuous at a point $x = a$ are continuous at this point. The quotient $\frac{u}{v}$ of two functions continuous at the point $x = a$ is continuous if the divisor v does not vanish for $x = a$.

Property 2. 1) If the function $f(x)$ is continuous for some value of x, then the increment in the function is infinitesimal for an infinitesimal increment in the argument.

Example 1. The function $f(x) = \frac{1}{x-3}$ is continuous at the point $x = 5$, and $f(5) = \frac{1}{2}$ (Sec. 218, Example 1). For $x = 5 + \Delta x$ the function has the value

$$f(5 + \Delta x) = \frac{1}{2 + \Delta x}$$

The increment in the function is

$$f(5 + \Delta x) - f(5) = -\frac{\Delta x}{2(2 + \Delta x)}$$

It is infinitesimal for an infinitesimal Δx.

219a. One-Sided (Unilateral) Limits. The Jump of a Function

If the value of a function $f(x)$ tends to the number b_1 as x tends to a from the side of small values, then the number b_1 is termed the *left-hand limit* (or *limit on the left*) *of the function* $f(x)$ at the point $x = a$ and is written

$$\lim_{x \to a-0} f(x) = b_1 \tag{1}$$

If $f(x)$ tends to b_2 as x tends to a from the side of larger values, then b_2 is called the *right-hand limit* (or *limit on the right*) *of the function* $f(x)$ as $x \to a$ and is written

$$\lim_{x \to a+0} f(x) = b_2 \tag{2}$$

The quantity $|b_2 - b_1|$ is called a *discontinuity* (a *jump* or *saltus*).

---

1) Property 2 may be taken for a definition of continuity of a function at a point (equivalent to definition of Sec. 218).

The left and right limits are generically termed a "one-sided (unilateral) limit".

Example 1. The function Q depicted in Fig. 218 has at the point T_2 a left-hand limit AB and a right-hand limit AC. The jump is shown as $BC=AC-AB$.

Example 2. The function $f(x)=\frac{1}{1+2^{1/x}}$ (Fig. 219) has, at the point $x=0$, a right-hand limit $b_2=0$ and a left-hand limit $b_1=1$. The jump is equal to unity.

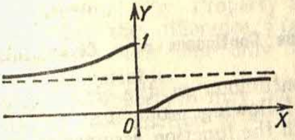

Fig. 219

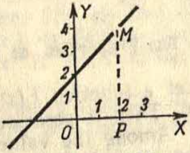

Fig. 220

Two one-sided limits of a function $f(x)$ at a point $x=a$ may be equal. If the function is defined at the point $x=a$, it is continuous at this point.

Example 3. The function $f(x)=\frac{x^2-4}{x-2}$ has one-sided limits at the point $x=2$ both equal to 4. But at the point $x=2$ itself the function is not defined and is therefore discontinuous. The graph (Fig. 220) is a straight line $y=x+2$, without the point $M(2, 4)$. If we further agree that $f(2)=4$, then $f(x)$ will become continuous. The graph will include the point M as well.

If by means of the supplementary condition defining the function $f(x)$ at the point a it is possible to convert a discontinuous function into a continuous function, the discontinuity is called *removable*. In Example 3 the discontinuity is removable while in Examples 1 and 2 it is non-removable.

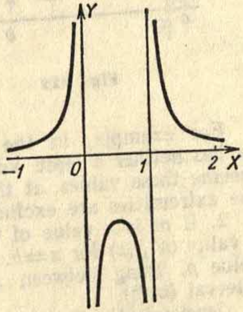

Fig. 221

220. The Continuity of a Function on a Closed Interval

Definition. A function is called *continuous on a closed interval* if it is continuous at every point of the interval including the two end-points.

We similarly define the continuity of a function in open intervals.

Example. Consider the function $\frac{1}{4x(x-1)}$ (Fig. 221). It is continuous on the closed interval $\left(1\frac{1}{2}, 2\right)$, but is discontinuous on the closed interval (0, 1) because both end-points $x=0$ and $x=1$ are points of discontinuity. It is also discontinuous on the closed interval (1, 2) since one end-point $x=1$ is a point of discontinuity. It is also discontinuous on the closed interval $\left(\frac{1}{2}, 2\right)$ since there is a point of discontinuity ($x=1$) inside the interval.

221. The Properties of Functions Continuous on a Closed Interval

Let a function $f(x)$ be continuous on the closed interval (a, b). Then it possesses the following properties:

1. Among the values which the function assumes at points of the given interval there is a greatest and a least.

Note 1. Among the values which the function $f(x)$ assumes at points of an *open* interval (a, b) there may not be a greatest or a least value.

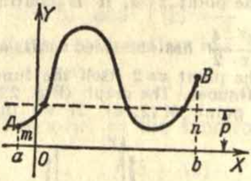

Fig. 222

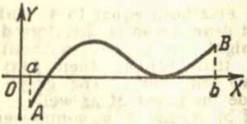

Fig. 223

For example, in the *open* interval (1, 3) the function $2x$ has neither a least value nor a greatest value (it could assume these values at the end-points $x=1$ and $x=3$, but the extremities are excluded from the open interval).

2. If m is a value of the function $f(x)$ for $x=a$ and n is a value of $f(x)$ for $x=b$, then the function $f(x)$ assumes any value p, lying between m and n, at least once inside the interval (a, b).

Geometrically, any straight line drawn parallel to the axis of abscissas above the point A but below the point B (Fig. 222) will meet the curve AB at least once (three times in Fig. 222).

Note 2. A discontinuous function may not have Property 2 (see Figs. 218 and 219).

2a. In particular, if the function has a positive value at one end of the interval and a negative value at the other end, then it will vanish at least once within the interval.

Geometrically, if one of the points A, B (Fig. 223) lies above the x-axis and the other below the x-axis, then the curve AB will meet the x-axis at least once (twice in Fig. 223).

3. If the variables x and x' vary so that the difference $x-x'$ is infinitesimal, then the difference $f(x)-f(x')$ is also infinitesimal.

Note 3. If x' is a constant c, the difference $f(x)-f(c)$ is an infinitesimal, by Property 2 of Sec. 219. By Property 3 of Sec. 221, the difference $f(x)-f(x')$ is infinitesimal for infinitesimal $x-x'$ not only when x' is constant but also when x' is variable.

Note 4. Property 3 may not hold true in the case of continuity of the function in an *open* interval. Thus, the function $\frac{1}{x}$ is continuous in the interval (0, 1) devoid of the end-point $x=0$. Let x and x' vary so that $x'=2x$ as $x \to 0$. Then the difference $x-x'$ is infinitesimal, but the difference $f(x)-f(x')=\frac{1}{x}-\frac{1}{2x}=\frac{1}{2x}$ is infinitely great.

DIFFERENTIAL CALCULUS

222. Introductory Remarks

The source of differential calculus lies in two problems: (1) finding the tangent to an arbitrary line (Sec. 225), (2) finding the velocity, given an arbitrary law of motion (Sec. 223).

Both problems led to one and the same computational problem which lies at the heart of differential calculus. The problem is that of finding, on the basis of a given function $f(t)$, a certain function $f'(t)$ (which later became known as the *derivative*) representing the rate of change of the function $f(t)$ with respect to the variation of the argument (a precise definition of a derivative is given in Sec. 224).

It was in this general form that the problem was posed by Newton and, in similar form, by Leibniz in the 70s and 80s of the seventeenth century. But even during the preceding half century, Fermat, Pascal and other scholars had actually given rules for finding the derivatives of many functions.

Newton and Leibniz brought this development to its culmination. They introduced the general concepts of derivative 1) and differential 2) and also the symbols which greatly simplified computations. They refined the apparatus of differential calculus and applied it to the solution of numerous problems in geometry and mechanics. It was only in the 19th century that the whole system was placed on a rigorously logical basis (see Sec. 191).

223. Velocity 3)

In order to determine the velocity of a train, we note the point at which it is located at time $t=t_1$ and then at time $t=t_2$. Let these be the distances $s=s_1$ and $s=s_2$. The increment (Sec. 217a) in distance $\Delta s=s_2-s_1$ is divided by

---

1) Newton used the term "fluxion" The term "derivative" was introduced at the end of the 18th century by Arbogast.

2) The term "differential" (from the Latin *differentia*) was given by Leibniz.

3) This section introduces Sec. 224

the increment in the time $\Delta t = t_2 - t_1$. The quotient

$$\frac{\Delta s}{\Delta t} \tag{1}$$

yields the average *velocity* of the train for the interval (t_1, t_2). In the case of nonuniform motion, the average velocity does not describe the rate of motion at time $t = t_1$ with sufficient exactitude. But the smaller Δt, the more exact is this speed. For this reason, the *speed at time* $t = t_1$ is the limit to which the ratio $\frac{\Delta s}{\Delta t}$ tends as $\Delta t \to 0$:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \tag{2}$$

Example. *Free fall of a body*. We have

$$s = \frac{1}{2} gt^2 \tag{3}$$

Since $t_2 = t_1 + \Delta t$, it follows that

$$\Delta s = s_2 - s_1 = \frac{1}{2} g (t_1 + \Delta t)^2 - \frac{1}{2} gt_1^2$$

Hence

$$v = \lim_{\Delta t \to 0} \frac{\frac{1}{2} g (t_1 + \Delta t)^2 - \frac{1}{2} gt_1^2}{\Delta t} \tag{4}$$

Having computed the limit, we find

$$v = gt_1 \tag{5}$$

The notation t_1 is introduced to bring out the *constancy of t when computing the limit*. Since t_1 is an arbitrary value of time, the subscript 1 can best be dropped; then from the formula

$$v = gt \tag{5a}$$

it is evident that the velocity v (like the distance s) is a function of the time. The form of the function v depends completely on the form of the function s, so that the function s generates ("derives", as it were) the function v. Hence the name, the "derivative function".

224. The Derivative Defined [1]

Let $y=f(x)$ be a continuous function (of the argument x) defined in the interval (a, b) and let x be some point of this interval. We give to the argument x an increment Δx (positive or negative). The function $y=f(x)$ will receive the increment Δy, equal to

$$\Delta y=f(x+\Delta x)-f(x) \tag{1}$$

If Δx is infinitesimal, then Δy is also infinitesimal (Sec. 219).

The limit to which the ratio $\frac{\Delta y}{\Delta x}$ tends as $\Delta x \longrightarrow 0$, i.e.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \tag{2}$$

is itself a function of the argument x (cf. Sec. 223). This function is called the *derivative* of the function $f(x)$ and is denoted by $f'(x)$ or y'.

Briefly, a *derivative function* is the limit [2] to which tends the ratio of an infinitesimal increment in the function to a corresponding infinitesimal increment in the argument.

Note. In the process of finding the limit (2), the quantity x is regarded as a constant.

Example 1. Find the value of the derivative of the function $y=x^2$ for $x=7$.

Solution. For $x=7$ we have $y=7^2=49$. Give to the argument x an increment Δx. The argument becomes equal to $7+\Delta x$, and the function becomes $(7+\Delta x)^2$.

The increment Δy of the function is

$$\Delta y=(7+\Delta x)^2-7^2=14\Delta x+\Delta x^2$$

The ratio of this increment to the increment Δx is

$$\frac{\Delta y}{\Delta x}=\frac{14\Delta x+\Delta x^2}{\Delta x}=14+\Delta x$$

Now find the limit to which $\frac{\Delta y}{\Delta x}$ tends as $\Delta x \longrightarrow 0$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}=\lim_{\Delta x \to 0}(14+\Delta x)=14$$

The desired value of the derivative is 14.

---

[1] It is advisable to read Sec. 223 first.

[2] For cases when the limit does not exist, see Sec. 231.

Example 2. Find the derivative of the function $y=x^2$ (for an arbitrary value of x). Give to the argument an increment Δx. The argument becomes $x+\Delta x$. The increment Δy of the function is $(x+\Delta x)^2-x^2=2x\,\Delta x+\Delta x^2$. The ratio $\frac{\Delta y}{\Delta x}$ is equal to $\frac{(x+\Delta x)^2-x^2}{\Delta x}=2x+\Delta x$. The derivative function is the limit of this ratio as $\Delta x \longrightarrow 0$:

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x$$

The sought-for derivative $y'=2x$. For $x=7$ we get $y'=14$ (cf. Example 1).

Example 3. Find the derivative of the function $y=\sin x$ (the argument is expressed in radian measure).

Solution. Give to the argument an increment Δx. The increment of the function is

$$\Delta y=\sin(x+\Delta x)-\sin x=2\cos\left(x+\frac{\Delta x}{2}\right)\cdot\sin\frac{\Delta x}{2}$$

The ratio $\frac{\Delta y}{\Delta x}$ is

$$\frac{2\cos\left(x+\frac{\Delta x}{2}\right)\cdot\sin\frac{\Delta x}{2}}{\Delta x}=\cos\left(x+\frac{\Delta x}{2}\right)\frac{2\sin\frac{\Delta x}{2}}{\Delta x}$$

The limit of this ratio as $\Delta x\longrightarrow 0$ (Secs. 213, 215) is equal to

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)\lim_{\Delta x\to 0}\frac{2\sin\frac{\Delta x}{2}}{\Delta x}=\cos x$$

Hence, $y'=\cos x$.

225. Tangent Line

The *tangent line* to the curve L at the point M (Fig. 224) is the straight line $T'MT$ with which the secant line MM' tends to coincidence[1] when the point M', always on L, tends to M (either from the right or from the left).

Note. From Fig. 225 it is evident that the tangent can have, besides the point of tangency, points common to the curve and the tangent.

---

[1] The expression "tends to coincidence" means that the acute angle between the fixed straight line $T'MT$ and the rotating line MM' tends to zero.

If the curve L is the graph of a function $y=f(x)$, then the slope of the tangent is equal to the value of the derivative function at the corresponding point.[1]

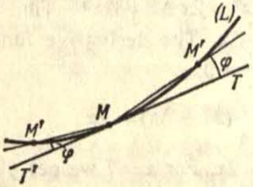

Fig. 224

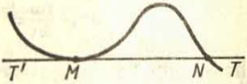

Fig. 225

This is clear from Fig. 226. The slope k of the secant line is $k=\frac{QM'}{MQ}=\frac{\Delta y}{\Delta x}$. If M' tends to M, then k has as a limit the slope m of the tangent. Hence, $m=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$, i.e. (Sec. 224) $m=f'(x)$.

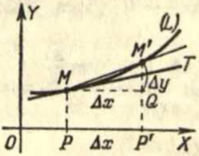

Fig. 226

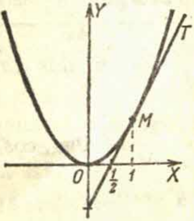

Fig. 227

Example 1. Find the slope and the equation of the tangent to the parabola $y=x^2$ at the point $M(1, 1)$ (Fig. 227)

Solution. We have $y'=2x$ (Sec. 224, Example 2). For $x=1$ we get $y'=2$. The desired slope of the tangent $m=2$. The equation of the tangent will be $y-1=m(x-1)$, i. e. $y=2x-1$.

---

[1] If the graph has no tangent, the function $f(x)$ has no derivative, and vice versa.

Example 2. Find the equation of the tangent to the curve $y=\sin x$ (sine curve, Fig. 228) at the point $O(0, 0)$.

Solution. We have $y'=\cos x$ (Sec. 224, Example 3). For $x=0$ we get $y'=1$. The equation of the tangent is $y=x$.

Note that the sine curve lies on both sides of the tangent line $T'OT$.

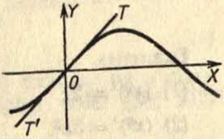

Fig. 228

Example 3. The slope of the straight line $y=ax+b$ (it is equal to a) is the derivative of the function $y=ax+b$ (the tangent to a straight line is the line itself).

226. The Derivatives of Some Elementary Functions

1. *The derivative of a constant quantity* is equal to zero

$$(a)'=0 \tag{1}$$

Physical meaning (Sec. 223): the velocity of a fixed point is zero.

Geometrical meaning: the slope of the straight line $y=a$ (UV in Fig. 229) is zero (cf. Sec. 225, Example 3).

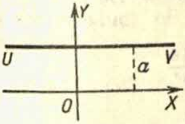

Fig. 229

Note. For some values of x a function can have a zero derivative without being a constant. Thus, the derivative $(\sin x)'=\cos x$ (Sec. 224, Example 3) is zero for $x=\frac{\pi}{2}$, $x=-\frac{3\pi}{2}$, etc.

But if the derivative $f'(x)$ is *identically* zero, then the function $f(x)$ must definitely be constant (Sec. 265, Theorem 1).

2. The *derivative of an independent variable* is unity:

$$(x)'=1 \tag{2}$$

Geometrical meaning: the slope of the straight line $y=x$ is equal to unity.

Physical meaning: if the distance covered by a body is numerically equal to the time spent in motion, then the velocity is numerically equal to unity.

3. The *derivative of the linear function* $y=ax+b$ is the constant quantity a:

$$(ax+b)'=a \tag{3}$$

4. The *derivative of a power function* is equal to the product of the exponent by the power function with exponent decreased by one

$$(x^n)' = nx^{n-1} \tag{4}$$

Examples.

(1) $(x^2)' = 2x$.

(2) $(x^3)' = 3x^2$.

(3) $(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

(4) $\left(\frac{1}{x^2}\right)' = (x^{-2})' = -2x^{-3} = -\frac{2}{x^3}$.

227. Properties of a Derivative

1. A constant factor may be taken outside the sign of the derivative:

$$[af(x)]' = af'(x)$$

Examples.

(1) $(3x^2)' = 3(x^2)' = 3\cdot 2x = 6x$.

(2) $\left(\frac{5}{x^2}\right)' = 5\left(\frac{1}{x^2}\right)' = 5\left(-\frac{2}{x^3}\right) = -\frac{10}{x^3}$.

(3) $(\sqrt{2x})' = \sqrt{2}(\sqrt{x})' = \frac{\sqrt{2}}{2\sqrt{x}} = \frac{1}{\sqrt{2x}}$.

2. The derivative of an algebraic sum of some fixed number of functions is equal to the algebraic sum of their derivatives:

$$[f_1(x) + f_2(x) - f_3(x)]' = f_1'(x) + f_2'(x) - f_3'(x)$$

Examples.

(4) $(0.3x^2 - 2x + 0.8)' = (0.3x^2)' - (2x)' + (0.8)' = 0.6x - 2$

(the derivative of the last term is zero; Sec. 226, Item 1).

(5) $\left(\frac{3}{x^2} - 6\sqrt{x}\right)' = \left(\frac{3}{x^2}\right)' - 6(\sqrt{x})' = -\frac{6}{x^3} - \frac{3}{\sqrt{x}}$.

228. The Differential

Definition. Let the increment (Sec. 217a) in the function $y=f(x)$ be split up into a sum of two terms:

$$\Delta y = A\Delta x + \alpha \tag{1}$$

where A is not dependent on Δx (i. e. is constant for a given value of the argument x) and α is of higher order (Sec. 217) than Δx (as $\Delta x \to 0$).

Then the first ("principal") term, which is proportional to Δx, is called the *differential* of the function $f(x)$ and is denoted by dy or $df(x)$.

Example 1. Take the function $y=x^3$. Then[1]

$$\Delta y = 3x^2\,\Delta x + (3x\,\Delta x^2 + \Delta x^3) \tag{2}$$

Here, the coefficient $A=3x^2$ is not dependent on Δx, so that the first term is proportional to Δx; the other term, $\alpha = 3x\,\Delta x^2 + \Delta x^3$ however is of higher (second) order with respect to Δx. Hence, the term $3x^2\,\Delta x$ is the differential of the function x^3

$$dy = 3x^2\,\Delta x \quad \text{or} \quad d(x^3) = 3x^2\,\Delta x \tag{3}$$

Theorem 1. The coefficient A is equal to the derivative $f'(x)$; in other words, the differential of a function is equal to the product of the derivative by the increment in the argument:

$$dy = y'\,\Delta x \tag{4}$$

or

$$df(x) = f'(x)\,\Delta x \tag{4a}$$

Example 2. In Example 1 we found that $d(x^3)=3x^2\,\Delta x$. The coefficient $3x^2$ is the derivative of the function x^3.

Example 3. If $y=\frac{1}{x}$, then $y'=-\frac{1}{x^2}$ (Sec. 226, Item 4). Therefore $dy=-\frac{\Delta x}{x^2}$.

Let us verify this. We have $\Delta y=\frac{1}{(x+\Delta x)}-\frac{1}{x}=\frac{-\Delta x}{x(x+\Delta x)}$. If we split up this expression into two terms, the first being $-\frac{\Delta x}{x^2}$, then the second will be $\frac{\Delta x^2}{x^2(x+\Delta x)}$. The latter term is of higher (second) order with respect to Δx.[1]

---

[1] The notation Δx^2 is the same as $(\Delta x)^2$ (parentheses are dropped). If it is necessary to indicate the increment of the function x^2, then we write $\Delta(x^2)$.

Theorem 2. If the derivative is not equal to zero, then the differential of the function and its increment are equivalent (as $\Delta x \longrightarrow 0$); if the derivative is zero (the differential is then also zero), they are not equivalent.

Example 4. If $y=x^2$, then $\Delta y=2x\,\Delta x+\Delta x^2$ and $dy=2x\,\Delta x$. For $x=3$ the quantities $\Delta y=6\Delta x+\Delta x^2$ and $dy=6\Delta x$ are equivalent, for $x=0$ the quantities $\Delta y=\Delta x^2$ and $dy=0$ are not equivalent.

The equivalence of the differential and the increment is frequently employed in approximate calculations (as a rule, it is easier to compute a differential than a derivative).

Example 5. We have a metal cube with edge $x=10.00$ cm. When heated, the edge increased by $\Delta x=0.01$ cm. How much did the volume V of the cube increase?

Solution. We have $V=x^3$ so that $dV=3x^2\,\Delta x=3\cdot 10^2\cdot 0.01=3$ (cm^3). The increase in the volume ΔV is equivalent to the differential dV so that $\Delta V \approx 3$ cm^3. The total computation would have yielded $\Delta V=10.01^3-10^3=3.003001$. But in this result all the digits, except the first, are unreliable and so we have to round off to 3 cm^3 in any case.

Other examples of the employment of a differential in approximate computations are given in Sec. 243 (Example 4) and Sec. 248.

229. The Mechanical Interpretation of a Differential

Let $s=f(t)$ be the distance of a rectilinearly moving point from its initial position (t is the time in transit). The increment Δs is the distance covered by the point during the time interval Δt, while the differential $ds=f'(t)\,\Delta t$ (Sec. 228, Theorem 1) is the distance the point would have covered during time Δt if it had maintained the speed $f'(t)$ reached at time t. For an infinitesimal Δt the imagined distance ds differs from the true distance Δs by an infinitesimal of order higher than Δt. If the velocity at time t is not equal to zero, then ds yields an approximation of the small displacement of the point (cf. Sec. 228, Theorem 2).

---

[1] It is assumed that $x \neq 0$ $\left(\text{for } x=0 \text{ the function } \frac{1}{x} \text{ itself is not defined}\right)$.

230. The Geometrical Interpretation of a Differential

Let curve L (Fig. 230) be the graph of a function $y=f(x)$. Then

$$\Delta x = MQ, \quad \Delta y = QM'$$

The tangent line MN divides the segment Δy into two parts QN and NM'. The former is proportional to Δx and is equal to $QN = MQ \cdot \tan \angle QMN = \Delta x f'(x)$ (see Sec. 225), i. e. QN is the differential dy.

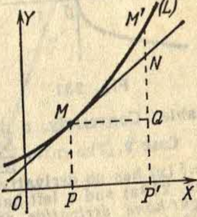

Fig. 230

The latter part NM' yields the difference $\Delta y - dy$; it is of higher order with respect to Δx. In the given case, when $f'(x) \neq 0$ (the tangent line is not parallel to the x-axis), the segments QM' and QN are equivalent (Sec. 228, Theorem 2). In other words, NM' is also of higher order with respect to $\Delta y = QM'$. This is evident from the figure (as M' approaches M, the segment NM' comprises an ever smaller portion of the line segment QM').

Thus, the differential of a function is graphically depicted as the increment in the ordinate of the tangent line.

231. Differentiable Functions

A continuous function which (at a given point) has a differential is called *differentiable* at that point.

A discontinuous function cannot have either a derivative or a differential at a point of discontinuity (the graph does not have a tangent line; see Fig. 214 on page 264 and Fig. 219 on page 279).

A function which is continuous at some point may not have a differential at that point. Below we consider three characteristic cases.

Case 1. The function $y=f(x)$ has an *infinite derivative* at the given point, i. e.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = +\infty$$

or

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\infty$$

(thus, Δy is of lower order than Δx). The graph has a vertical tangent line.

Notation (by convention):

$$f'(x)=\infty$$

Example 1. The function $f(x)=\sqrt[3]{x}$ (Fig. 231) is not differentiable at the point $x=0$. The quantity

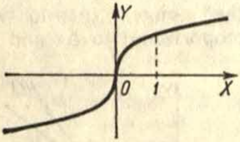

Fig. 231

$$\frac{\Delta y}{\Delta x}=\frac{\sqrt[3]{0+\Delta x}-\sqrt[3]{0}}{\Delta x}$$

has an infinite limit $+\infty$ as $\Delta x\to 0$.

At the point $x=0$ the tangent line coincides with the y-axis.

Note 1. A function which at a given point has a *finite* derivative is differentiable. Conversely, a differentiable function has a finite derivative.

Case 2. The ratio $\frac{\Delta y}{\Delta x}$ has no limit as $\Delta x\to 0$ (i. e. the function $y=f(x)$ has no derivative), but it has a right-hand limit (as $\Delta x\to +0$, Sec. 219a) and a left-hand limit (as $\Delta x\to -0$). The former is called a *right-hand derivative* and is denoted by $f'(x+0)$ and the second is called a *left-hand derivative* and is denoted by $f'(x-0)$.

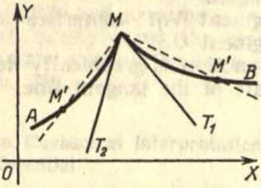

Fig. 232

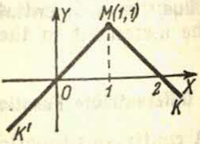

Fig. 233

At the point of interest (M in Fig. 232) the graph has no tangent line, but it has a *right-hand tangent line* MT_1 and a *left-hand tangent line* MT_2; that is, the secant line MM' tends to coincidence with MT_1 when M' tends to M from the right, and with MT_2 when M' tends to M from the left.

Example 2. The function $f(x)=1-|1-x|$ (Fig. 233) is not differentiable at the point $x=1$. The line $K'MK$ has no tangent line at the point $M(1, 1)$. The right-hand derivative $f'(1+0)=-1$, the left-hand derivative $f'(1-0)=1$.

Case 3. The function $y=f(x)$ has no left-hand or right-hand derivative (or has neither). The graph does not have a corresponding one-sided tangent.

Example 3. The function given by the formula $f(x)=x\sin\frac{1}{x}$ (Fig. 234) and redefined as $f(0)=0$ $\left(\text{the expression } \sin\frac{1}{x} \text{ is meaningless for } x=0\right)$

is continuous at the point $x=0$. However, when M' tends to O from the right (or the left), the secant line OM' oscillates between the straight lines UV ($y=x$) and $U'V'$ ($y=-x$) and does not tend to either straight

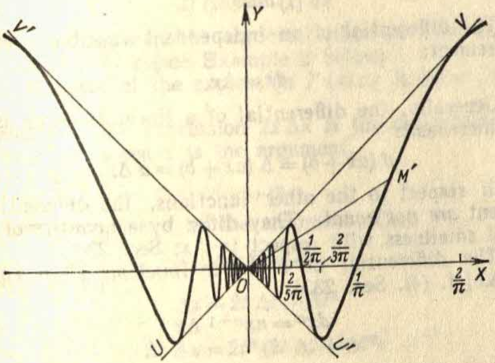

Fig. 234

line. The graph, at point O, has neither right-hand nor left-hand tangent line, and the function $f(x)$ has neither right-hand nor left-hand derivative.

Note 2. One can even think up continuous functions that have no derivative at any point at all.[1] Hence, the existence of a derivative does not follow logically from the continuity of a function. This was first pointed out by the great Russian mathematician N. I. Lobachevsky.[2]

---

[1] We can neither construct nor even imagine a curve graphically depicting such a function, for our conception of a curve involves an abstraction from the properties of real objects and it is intimately bound up with the concept of direction. Even in Example 3, the "line" $y=x\sin\frac{1}{x}$ is devoid of direction at the point $x=0$. Here, however our imagination is aided by the fact that the graph has a definite direction in any neighbourhood of point O.

[2] Nikolai Ivanovich Lobachevsky (1792-1856) created non-Euclidean geometry and made valuable contributions to algebra and analysis. He was also a prominent public figure, and an outstanding teacher who did much in the sphere of education. His whole life and work are closely bound up with Kazan University from which he graduated and at which he was professor and rector.

232. The Differentials of Some Elementary Functions

1. The differential of a constant is zero:

$$da = 0 \tag{1}$$

2. The differential of an independent variable is equal to its increment:

$$dx = \Delta x \tag{2}$$

3. Generally, the differential of a linear function is equal to its increment:

$$d\,(ax + b) = \Delta\,(ax + b) = a\,\Delta x \tag{3}$$

With respect to the other functions, the differential and increment *are not equal*. (They differ by a quantity of higher order of smallness with respect to Δx; Sec. 228.)

4. The differential of a power function x^n is equal to $nx^{n-1}\,\Delta x$ [cf. (4), Sec. 233]:

$$dx^n = nx^{n-1}\,\Delta x \tag{4}$$

233. Properties of a Differential

1. A constant factor may be taken outside the sign of the differential:

$$d\,[a\,f\,(x)] = a\,df\,(x) \tag{1}$$

2. The differential of an algebraic sum of a fixed number of functions is equal to the algebraic sum of their differentials:

$$d\,[f_1\,(x) + f_2\,(x) - f_3\,(x)] = df_1\,(x) + df_2\,(x) - df_3\,(x) \tag{2}$$

3. The differential of a function is equal to the product of the derivative by the differential of the argument:

$$df\,(x) = f'\,(x)\,dx \tag{3}$$

This follows from Sec. 228 (Theorem 1) and Sec. 232, Item 2

In particular (cf. Sec. 232, Item 4),

$$d\,(x^n) = nx^{n-1}\,dx \tag{4}$$

234. The Invariance of the Expression $f'(x)\,dx$

The expression $f'(x)\,\Delta x$ represents (Sec. 228, Theorem 1) the differential $df\,(x)$ when x is regarded as the argument. But if the quantity x itself is regarded as a function of some argument t, then the expression $f'(x)\,\Delta x$, as a rule, *does not* *represent the differential* (see Example 1 below); the only exception is the case of a linear relation: $x=at+b$.

On the contrary, formula (3), Sec. 233,

$$df(x)=f'(x)\,dx \tag{1}$$

is true both when x is the argument (then $dx=\Delta x$) and when x is a function of t (see Example 2 below).

This property of the expression $f'(x)\,dx$ is called its *invariance*.

Example 1. The expression $2x\,\Delta x$ is the differential of the function $y=x^2$ when x is the argument.

Now put

$$x=t^2 \tag{2}$$

and we will consider t as the argument. Then

$$y=x^2=t^4 \tag{3}$$

From (2) we find

$$\Delta x=2t\,\Delta t+\Delta t^2 \tag{4}$$

Hence

$$2x\,\Delta x=2t^2(2t\,\Delta t+\Delta t^2) \tag{5}$$

This expression is not proportional to Δt and therefore now $2x\,\Delta x$ is not a differential. The differential of the function y is found from (3):

$$dy=4t^3\,\Delta t \tag{6}$$

Comparing (5) and (6) we see that $2x\,\Delta x$ and dy differ by the quantity $2t^2\,\Delta t^2$, which is of second order with respect to Δt.

Example 2. The expression $2x\,dx$ is the differential of the function $y=x^2$ *for any argument* t. For example, let $x=t^2$. Then

$$dx=2t\,\Delta t$$

Hence

$$2x\,dx=2t^2\cdot 2t\,\Delta t=4t^3\,\Delta t$$

Comparing with (6), we see that

$$dy=2x\,dx$$

235. Expressing a Derivative in Terms of Differentials

The derivative of a function y with respect to the argument x is equal to the ratio of the differential of the variable y to the differential of the variable x:

$$y'_x=\frac{dy}{dx}$$

The subscript x on the symbol y' emphasizes the fact that *when we are seeking the derivative*, the argument is x. The differentials dy and dx *may be taken with respect to any argument* (see Sec. 234).

An extremely convenient notation for a derivative is often the expression $\frac{dy}{dx}$ and similar expressions: $\frac{df(x)}{dx}$ (the derivative of the function $f(x)$ with respect to x), $\frac{d\varphi(t)}{dt}$ (the derivative of the function $\varphi(t)$ with respect to t), $\frac{d(3x^2+2x+1)}{dx} = 6x+2$ and so forth.

The following conventional notations are also employed: $\frac{d}{dx}f(x)$, $\frac{d}{dx}(3x^2+2x+1)$ and so forth, which are particularly convenient when taking the derivative of a complicated expression.

236. The Function of a Function (Composite Function)

A quantity y is called a *function of a function (composite function)* if it is regarded as the function of some (auxiliary) variable u, which in turn depends on an argument x:

$$y=f(u), \quad u=\varphi(x) \tag{1}$$

In this way, y is a function of x, and this may be written as

$$y=f[\varphi(x)] \tag{2}$$

If $f(u)$ and $\varphi(x)$ are continuous functions, then the function $f[\varphi(x)]$ is also continuous.

Example. If $y=u^3$ and $u=1+x^2$, then y is a composite function of x, and we write

$$y=(1+x^2)^3$$

237. The Differential of a Composite Function

Finding the differential of a composite function does not require any special rules (due to the invariance of the expression $f'(x)\,dx$, Sec. 234).

Example 1. Find the differential of the function $y=(1+x^2)^3$.

Solution. Regarding y as a composite function ($y=u^3$, $u=1+x^2$), we have

$$dy=3u^2\,du, \quad du=2x\,dx$$

Whence

$$dy = 3(1+x^2)^2 \cdot 2x\,dx = (6x + 12x^3 + 6x^5)\,dx$$

The same result is obtained directly:

$$dy = d(1 + 3x^2 + 3x^4 + x^6) = (6x + 12x^3 + 6x^5)\,dx$$

Note. In actual practice, no special designation is introduced for the auxiliary variable u. In Example 1, the procedure is:

$$d(1+x^2)^3 = 3(1+x^2)^2 \cdot d(1+x^2) = 3(1+x^2)^2\,2x\,dx$$

Example 2. Find $d\sqrt{a^2-x^2}$.

Solution.

$$d\sqrt{a^2-x^2} = d(a^2-x^2)^{\frac{1}{2}} = \frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}\,d(a^2-x^2) =$$

$$= -\frac{x\,dx}{\sqrt{a^2-x^2}}$$

238. The Derivative of a Composite Function

The derivative of a function of a function is equal to the derivative of the function with respect to the auxiliary variable multiplied by the derivative of the auxiliary variable with respect to the argument:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad (1)$$

Example 1. Find the derivative of the function

$$y = \sqrt{a^2 - x^2}$$

(with respect to the argument x).

Putting

$$y = u^{\frac{1}{2}}, \quad u = a^2 - x^2$$

we have

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{a^2-x^2}}, \quad \frac{du}{dx} = -2x$$

By formula (1) we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{a^2-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{a^2-x^2}}$$

Note. When using the notation $(\sqrt{a^2-x^2})'$, beginners frequently make the following mistake. Knowing that $(\sqrt{x})'=\frac{1}{2\sqrt{x}}$, they write the result as $\frac{1}{2\sqrt{a^2-x^2}}$ and forget to multiply by $(a^2-x^2)'=-2x$. The error is due to imperfect notation (it is not seen with respect to what variable the derivative is taken). Therefore, it is advisable at the beginning to write as follows:

$$\frac{d}{dx}\sqrt{a^2-x^2}=\frac{1}{2\sqrt{a^2-x^2}}\frac{d}{dx}(a^2-x^2)=\frac{1}{2\sqrt{a^2-x^2}}(-2x)\quad .$$

When sufficient skill has been developed, the intermediate transformation is done mentally.

The best safeguard against mistakes is a preliminary computation of the differential $d\sqrt{a^2-x^2}$. Obtaining (Sec. 237, Example 2) $\frac{-x\,dx}{\sqrt{a^2-x^2}}$, we take the coefficient of dx (i. e. we divide by dx) and find for the derivative the expression $\frac{-x}{\sqrt{a^2-x^2}}$.

Example 2. Find the derivative of the function $y=\sin^2 2x$.

Solution. Here we have a chain of three relations:

$$y=u^2,\quad u=\sin v,\ v=2x$$

By analogy with (1) we have $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$. Taking into account that $\frac{du}{dv}=\frac{d\sin v}{dv}=\cos v$ (Sec. 224, Example 3), we find

$$\frac{dy}{dx}=2u\cdot\cos v\cdot 2=4\sin 2x\cdot\cos 2x=2\sin 4x$$

To avoid mistakes, it is best to proceed as follows:

$d\sin^2 2x=2\sin 2x\cdot d\sin 2x=2\sin 2x\cos 2x\cdot d(2x)=4\sin 2x\cdot\cos 2x\cdot dx$

Dividing by dx we obtain

$$\frac{d\sin^2 2x}{dx}=4\sin 2x\cos 2x$$

239. Differentiation of a Product

Rule. The differential of a product of two functions is equal to the sum of the products of each of the functions by the differential of the other:

$$d(uv)=u\,dv+v\,du \tag{1}$$

For three factors we have

$$d(uvw)=vw\cdot du+uw\cdot dv+uv\cdot dw \tag{2}$$

and similarly for a greater number of factors.

The derivative of a product is computed by the same rule (the word "differential" is both times replaced by the word "derivative"):

$$(uv)' = uv' + vu' \tag{1a}$$

$$(uvw)' = vwu' + uwv' + uvw' \tag{2a}$$

Example 1. Find the differential and the derivative of the function $(2x^2+3x)(x^3-2)$.

Solution.

$$\begin{aligned} d[(2x^2+3x)(x^3-2)] &= (2x^2+3x)\,d(x^3-2)+(x^3-2)\,d(2x^2+3x) = \\ &= (2x^2+3x)\,3x^2\,dx+(x^3-2)(4x+3)\,dx = \\ &= (10x^4+12x^3-8x-6)\,dx \end{aligned}$$

The coefficient $10x^4+12x^3-8x-6$ is the derivative. By formula (1a) we would have found

$$[(2x^2+3x)(x^3-2)]' = (2x^2+3x)(x^3-2)' + (x^3-2)(2x^2+3x)'$$

and so forth.

Example 2.

$$d\left(x\sin\frac{1}{x}\right) = xd\sin\frac{1}{x} + \sin\frac{1}{x}\cdot dx =$$

$$= x\cos\frac{1}{x}\,d\left(\frac{1}{x}\right) + \sin\frac{1}{x}\,dx = \frac{-x\cos\frac{1}{x}}{x^2}\,dx + \sin\frac{1}{x}\,dx =$$

$$= \left(-\frac{1}{x}\cos\frac{1}{x} + \sin\frac{1}{x}\right)dx$$

Whence

$$\frac{d}{dx}\left(x\sin\frac{1}{x}\right) = -\frac{1}{x}\cos\frac{1}{x} + \sin\frac{1}{x} \tag{3}$$

Note. It is assumed that $x \neq 0$. For $x=0$ the function $x\sin\frac{1}{x}$ is not defined. But even if it is redefined (Sec. 231, Example 3), it is not differentiable for $x=0$ [as $x \to 0$, the derivative (3) does not tend to any limit; see Fig. 234].

240. Differentiation of a Quotient (Fraction)

Rule. The differential of a fraction is equal to the product of the denominator by the differential of the numerator minus the product of the numerator by the differential of the denominator, the whole expression divided by the denominator squared:

$$d\frac{u}{v} = \frac{v\,du - u\,dv}{v^2} \tag{1}$$

The same rule holds for the derivative of a fraction (the word "differential" is replaced in each case by the word "derivative")

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \tag{1a}$$

Example 1. Find y' if $y = \frac{2x+1}{x^2+1}$

We have

$$y' = \frac{(x^2+1)(2x+1)' - (2x+1)(x^2+1)'}{(x^2+1)^2} = \frac{(x^2+1)\,2 - (2x+1)\,2x}{(x^2+1)^2}$$

i. e.

$$y' = \frac{2(-x^2 - x + 1)}{(x^2+1)^2}$$

Example 2. Find $d\sqrt{\frac{1+x}{1-x}}$.

First consider the given expression as a composite function $\left(y = \sqrt{u};\ u = \frac{1+x}{1-x}\right)$:

$$d\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\, d\,\frac{1+x}{1-x} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\,\frac{(1-x)\,dx + (1+x)\,dx}{(1-x)^2}$$

Simplifying, we get

$$d\sqrt{\frac{1+x}{1-x}} = \frac{dx}{(1-x)\sqrt{1-x^2}}$$

241. Inverse Function

If from the relation $y = f(x)$ there follows the relation $x = \varphi(y)$, then the function $\varphi(y)$ is called an *inverse* function of $f(x)$.

Example 1. The inverse of the function $y = x^2$ is the (double-valued) function $x = \pm\sqrt{y}$.

Example 2. The inverse of the function $y = \sin x$ is the (infinitely multiple-valued) function $x = \arcsin y$ (defined for all values of y less than unity in absolute value).

Note. As a rule, an inverse function is multiple-valued.[1] The multiple-valuedness can be avoided if we narrow the range of variation of the argument of the initial function. For instance, in Example 1 we can eliminate the negative values of the argument x and then the inverse function $x = +\sqrt{y}$ will be single-valued.

---

[1] The only exceptions are those cases when the value of the direct function, as the argument increases, either constantly increases or constantly decreases (such functions are termed *monotonic*).

If the earlier notations of the variables are retained, the graph of the function $y=f(x)$ also serves as the graph of the inverse function $x=\varphi(y)$.

Ordinarily, however, the notations of the variables change roles and the argument of the inverse function is denoted by x, like the argument of the direct function.

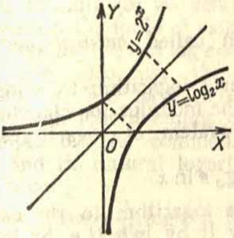

Fig. 235

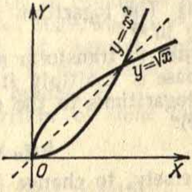

Fig. 236

Example 3. The inverse (single-valued) function of $y=x^2$ is $y=\sqrt{x}$; the inverse function of $y=2^x$ is the function $y=\log_2 x$.

In this notation, the graphs of the initial and inverse functions are symmetric with respect to the straight line $y=x$ (Fig. 235).

The derivative of an inverse function. The derivative of an inverse function is equal to unity divided by the derivative of the original function:[1]

$$\frac{dx}{dy}=1:\frac{dy}{dx} \qquad (1)$$

Example 4. Let us consider the function $y=x^2$ for positive values of x. The inverse function (Fig. 236) is $x=\sqrt{y}$. We have

$$\frac{dy}{dx}=2x,\quad \frac{dx}{dy}=\frac{1}{2x}=\frac{1}{2}\,\frac{1}{\sqrt{y}}$$

---

[1] If the derivative $\frac{dy}{dx}$ vanishes, then formula (1) may be understood in the sense that the inverse function has an infinite derivative at the point in question, i.e. $\lim\limits_{\Delta y\to 0}\frac{\Delta x}{\Delta y}=\infty$ (see Sec. 231, Case 1; cf. Sec. 213, Note 2).

242. Natural Logarithms

The formula for differentiating a logarithmic function (Sec. 243) is of the most elementary form when the base is the number

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\approx 2.71828$$

(Sec. 214). The logarithm is then called *natural* and is denoted by ln.[1]

In order to transform a natural logarithm to a logarithm to any base a, multiply it by the modulus for changing from natural logarithms to the other system of logarithms (equal to $\log_a e$):

$$\log_a x=\log_a e\ln x \tag{1}$$

Conversely, to change from a logarithm to the base a to the natural logarithms, multiply it by $\ln a$ (i.e. by $\log_e a$):[2]

$$\ln x=\ln a\log_a x \tag{2}$$

Mnemonic rule: Writing the formula (1) in complete form, we get $\log_a x=\log_a e\log_e x$. Discard the log signs and form fractions of the remaining letters $\frac{x}{a}$, $\frac{e}{a}$, $\frac{x}{e}$; then the first is the product of the last two. The same holds for formula (2).

The modulus for changing from *natural to common* logarithms is denoted by M:

$$M=\log e=0.43429 \tag{3}$$

(it is easy to remember the first four digits: $M=0.4343$). Formulas (1) and (2) take the form[3]

$$\log x=M\ln x, \tag{4}$$

$$\ln x=\frac{1}{M}\log x \tag{5}$$

---

[1] The initial letters of the Latin words *logarithmus naturalis*. The number e is irrational; more, it is transcendental, that is to say, it cannot be the root of any algebraic equation with rational coefficients. Also transcendental are the natural logarithms of all integers, and also the common logarithms of all integers (except 1, 10, 100, 1000, etc.). The transcendence of the number e was proved in 1871 by the French mathematician Hermite, the transcendence of the logarithms was proved by the Soviet mathematician A. Gelfond in 1934.

[2] The quantities $\log_a e$ and $\log_e a$ are reciprocal ($\log_a e\cdot\log_e a=1$).

[3] To avoid confusion as to when to multiply by M and when by $\frac{1}{M}$, remember that the common logarithm of any number is *less* than the natural logarithm (for example, $\ln 10\approx 2.3$, while $\log 10=1$).

where

$$\frac{1}{M}=\ln 10 \approx 2.3026 \tag{6}$$

For multiplication by M and $\frac{1}{M}$ there are special tables (p. 843).

Example 1. Find $\ln 100$.

Using formula (5) we get $\ln x \approx 2.3026 \cdot 2 \approx 4.605$.

Example 2. Compute e^3 using tables of common logarithms.

We have $\log (e^3)=3 \log e=3M=1.3029$, whence $e^3 \approx 20.09$.

One can also use the table of natural logarithms (pp. 839-842). We have $\ln (e^3)=3$; it is necessary to interpolate to find four decimals of the number e^3.

Example 3. The common logarithm of some number is 0.5041; find its natural logarithm.

We have

$$\ln x=\frac{1}{M} \log x \approx 2.303 \cdot 0.5041 \approx 1.161$$

This product may be found with the aid of the table on p. 843; namely,

$$\begin{array}{l} \frac{1}{M}\cdot 0.50 \quad\ \approx 1.1513 \\ \frac{1}{M}\cdot 0.0041 \approx 0.0094 \\ \hline \frac{1}{M}\cdot 0.5041 \approx 1.161 \end{array}$$

243. Differentiation of a Logarithmic Function

The differential and derivative of a natural logarithm (Sec. 242) are expressed by the formulas

$$d \ln x=\frac{dx}{x}, \tag{1}$$

$$\frac{d}{dx} \ln x=\frac{1}{x} \tag{2}$$

If the base of the logarithm is an arbitrary number, then [1]

$$d \log_a x=\log_a e \frac{dx}{x}, \tag{3}$$

$$\frac{d}{dx} \log_a x=\log_a e \cdot \frac{1}{x} \tag{4}$$

---

[1] Formula (3) may be obtained from (1) by taking into account (1), Sec. 242.

In particular, for the common logarithms

$$d\log x = \frac{M\,dx}{x}, \tag{3a}$$

$$\frac{d}{dx}\log x = M\cdot\frac{1}{x} \tag{4a}$$

Here, $M \approx 0.4343$ is the modulus for changing from natural logarithms to common logarithms (Sec. 242).

Example 1.

$$\frac{d}{dx}\ln(ax+b) = \frac{1}{ax+b}\cdot\frac{d}{dx}(ax+b) = \frac{a}{ax+b}$$

Example 2.

$$d\ln\frac{1+x}{1-x} = d\ln(1+x) - d\ln(1-x) = \frac{dx}{1+x} + \frac{dx}{1-x} = \frac{2dx}{1-x^2}$$

Example 3. Find the value of the derivative of $\log x$ for $x = 100$.

Formula (4a) yields $(\log x)' = \frac{M}{x} \approx \frac{0.4343}{100} \approx 0.0043$

Example 4. Find log 101 without using tables.

The increment $\Delta\log x$ is approximately equal to the differential $d\log x = \frac{M\,\Delta x}{x}$. For $x = 100$ and $\Delta x = 1$ we obtain $\Delta\log x \approx \frac{0.4343\cdot 1}{100} \approx 0.0043$. Hence,

$$\log 101 = \log 100 + \Delta\log 100 \approx 2 + 0.0043 = 2.0043$$

which coincides with the tabular value.

244. Logarithmic Differentiation

When differentiating expressions which are in a form convenient for taking logarithms, the latter operation may be performed first.

Example 1. Differentiate the function $y = xe^{-x^2}$

(1) Taking logs to the base e, we get

$$\ln y = \ln x - x^2 \tag{1}$$

(2) Now differentiate both sides of (1):

$$\frac{dy}{y} = \frac{dx}{x} - 2x\,dx$$

(3) Substituting for y the expression xe^{-x^2}, we get

$$dy = xe^{-x^2}\left(\frac{1}{x} - 2x\right)dx = e^{-x^2}(1-2x^2)\,dx$$

Example 2. Differentiate the function $y = x^x$.
Take the following steps.

(1) $\ln y = x \ln x$,

(2) $\frac{y'}{y} = x(\ln x)' + \ln x = 1 + \ln x$,

(3) $y' = y(1+\ln x) = x^x(1+\ln x)$

Example 3. Differentiate the function

$$y = \sqrt{\frac{1+x}{1-x}}$$

(cf. Sec. 240, Example 2).

(1) $\ln y = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$,

(2) $\frac{y'}{y} = \frac{1}{2}\,\frac{1}{1+x} + \frac{1}{2}\,\frac{1}{1-x} = \frac{1}{1-x^2}$,

(3) $y' = \sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{1-x^2} = \frac{1}{(1-x)\sqrt{1-x^2}}$.

Example 4. Differentiate the function

$$y = \frac{(x+1)^2}{(x+2)^3\,(x+3)^4}$$

(1) $\ln y = 2\ln(x+1) - 3\ln(x+2) - 4\ln(x+3)$,

(2) $\frac{y'}{y} = \frac{2}{x+1} - \frac{3}{x+2} - \frac{4}{x+3}$,

(3) $y' = \frac{(x+1)^2}{(x+2)^3\,(x+3)^4}\left(\frac{2}{x+1} - \frac{3}{x+2} - \frac{4}{x+3}\right) =$

$$= -\frac{(x+1)(5x^2+14x+5)}{(x+2)^4\,(x+3)^5}.$$

The foregoing method is called *logarithmic differentiation*, and the derivative of the logarithm of the function $y = f(x)$,

$$(\ln y)' = \frac{y'}{y} = \frac{f'(x)}{f(x)}$$

is called the *logarithmic derivative* of the function $f(x)$.

245. Differentiating an Exponential Function

The differential and derivative of the exponential function e^x $\left[\text{where } e=\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x \approx 2.71828\right]$ are expressed by the formulas [1]

$$de^x = e^x dx, \quad \frac{d}{dx}e^x = e^x \tag{1}$$

(the derivative of the function e^x is equal to the function itself). For an arbitrary base a we have

$$da^x = a^x \ln a\, dx, \quad \frac{d}{dx}a^x = a^x \ln a \tag{2}$$

In particular,

$$d\,10^x = 10^x \frac{1}{M}dx, \quad \frac{d}{dx}10^x = 10^x\frac{1}{M} \tag{2a}$$

Here, $\frac{1}{M} = \ln 10 \approx 2.3026$.

Example 1.

$$\frac{d}{dx}(e^{3x}) = e^{3x}\frac{d}{dx}(3x) = 3e^{3x}$$

Example 2.

$$d\left(xe^{-x^2}\right) = x\,de^{-x^2} + e^{-x^2}dx = xe^{-x^2}d\left(-x^2\right) + e^{-x^2}dx = \\ = e^{-x^2}\left(1-2x^2\right)dx$$

Example 3.

$$d\frac{e^t-e^{-t}}{e^t+e^{-t}} = \frac{(e^t+e^{-t})\,d\,(e^t-e^{-t})-(e^t-e^{-t})\,d\,(e^t+e^{-t})}{(e^t+e^{-t})^2} = \\ = \frac{(e^t+e^{-t})^2-(e^t-e^{-t})^2}{(e^t+e^{-t})^2}dt = \frac{4dt}{(e^t+e^{-t})^2}$$

Example 4.

$$d\,7^{t^2} = 7^{t^2}\ln 7\, d\left(t^2\right) = 2t\,7^{t^2}\ln 7\, dt$$

---

[1] Formulas (1) and (2) may be obtained by logarithmic differentiation (Sec. 244) or by regarding the exponential function as the inverse of the logarithmic function (Sec. 241).

246. Differentiating Trigonometric Functions[1]

Differentials	Derivatives
I. $d \sin x = \cos x \, dx,$	$\frac{d}{dx} \sin x = \cos x,$
II. $d \cos x = -\sin x \, dx,$	$\frac{d}{dx} \cos x = -\sin x,$
III. $d \tan x = \frac{dx}{\cos^2 x},$	$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x},$
IV. $d \cot x = -\frac{dx}{\sin^2 x},$	$\frac{d}{dx} \cot x = -\frac{1}{\sin^2 x}$

These formulas should be memorized. The following two need not be:

V. $d \sec x = \tan x \cdot \sec x \, dx,$ $\frac{d}{dx} \sec x = \tan x \cdot \sec x,$

VI. $d \operatorname{cosec} x = -\cot x \cdot \operatorname{cosec} x \, dx,$ $\frac{d}{dx} \operatorname{cosec} x = -\cot x \cdot \operatorname{cosec} x$

Example 1.

$$d \sin 2x = \cos 2x \, d(2x) = 2 \cos 2x \, dx$$

Example 2.

$$\frac{d}{dx} \ln \sqrt{\sin 2x} = \frac{1}{2} \frac{d}{dx} \ln \sin 2x = \frac{1}{2 \sin 2x} \cdot \frac{d}{dx} \sin 2x = \cot 2x$$

Example 3.

$$\frac{d}{d\varphi} \ln \tan \varphi = \frac{1}{\tan \varphi} \cdot \frac{d}{d\varphi} \tan \varphi = \cot \varphi \frac{1}{\cos^2 \varphi} = \frac{2}{\sin 2\varphi}$$

Example 4.

$$\frac{d}{dx} x^{\sin x} = x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right)$$

This is obtained by logarithmic differentiation (Sec. 244). Putting $y = x^{\sin x}$, we find $\ln y = \sin x \ln x$, whence $\frac{1}{y} \frac{d}{dx} y = \cos x \ln x + \frac{\sin x}{x}$.

---

[1] For the derivation of formula (I) see Sec. 224, Example 3; formula II is derived in similar fashion. Formulas III and IV are derived by means of the relations

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

247. Differentiating Inverse Trigonometric Functions [1]

	Differentials	Derivatives
I.	$d\arcsin x = \frac{dx}{\sqrt{1-x^2}}$,	$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$,
II.	$d\arccos x = -\frac{dx}{\sqrt{1-x^2}}$,	$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$,
III.	$d\arctan x = \frac{dx}{1+x^2}$,	$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$,
IV.	$d\,\mathrm{arccot}\, x = -\frac{dx}{1+x^2}$,	$\frac{d}{dx}\mathrm{arccot}\, x = -\frac{1}{1+x^2}$

These formulas should be memorized. The following two need not be:

V.	$d\,\mathrm{arcsec}\, x = \frac{dx}{x\sqrt{x^2-1}}$,	$\frac{d}{dx}\mathrm{arcsec}\, x = \frac{1}{x\sqrt{x^2-1}}$,
VI.	$d\,\mathrm{arccsc}\, x = -\frac{dx}{x\sqrt{x^2-1}}$,	$\frac{d}{dx}\mathrm{arccsc}\, x = -\frac{1}{x\sqrt{x^2-1}}$

Example 1.

$$d\arcsin\frac{x}{a} = \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \frac{dx}{\sqrt{a^2-x^2}} \tag{1}$$

Example 2.

$$d\arctan\frac{x}{a} = \frac{d\left(\frac{x}{a}\right)}{1+\left(\frac{x}{a}\right)^2} = \frac{a\,dx}{a^2+x^2} \tag{2}$$

Example 3.

$$\frac{d}{dx}\arctan\frac{3x+5}{2} = \frac{d}{dx}\left(\frac{3x+5}{2}\right) : \left[1+\left(\frac{3x+5}{2}\right)^2\right] =$$

$$= \frac{3}{2}\cdot\frac{4}{9x^2+30x+29} = \frac{6}{9x^2+30x+29}$$

Example 4.

$$d\arccos\frac{3}{4x-1} = d\left(\frac{3}{4x-1}\right) : -\sqrt{1-\left(\frac{3}{4x-1}\right)^2} =$$

$$= -\frac{3\cdot4\cdot dx}{(4x-1)^2} : -\frac{\sqrt{(4x-1)^2-9}}{|4x-1|} = \frac{6dx}{|4x-1|\sqrt{4x^2-2x-2}}$$

---

[1] Formulas I-VI are derived from the corresponding formulas of Sec. 246 (see Sec. 241).

Note. The function arccos $\frac{3}{4x-1}$ is only defined for $\left|\frac{3}{4x-1}\right| \leqslant 1$, i.e. either for $x \geqslant 1$ or for $x \leqslant -\frac{1}{2}$. It is not defined in the interval $\left(-\frac{1}{2}, 1\right)$. If one formally substitutes some unsuitable value (say, $x=0$) in the expression of the differential, it will turn out to be imaginary.

247a. Some Instructive Examples

The examples given below serve to illuminate some of the more subtle questions that arise in the differentiation of inverse trigonometric functions.

Example 1.

$$d \arctan \frac{1}{x} = \frac{1}{1+\frac{1}{x^2}} \cdot d\left(\frac{1}{x}\right) = -\frac{dx}{1+x^2}$$

The expression obtained coincides with the differential of the function arccot x. However, the following equality holds only for positive x:

$$\arctan \frac{1}{x} = \operatorname{arccot} x \tag{1}$$

For negative x we have 1)

$$\arctan \frac{1}{x} - \operatorname{arccot} x = -\pi \tag{2}$$

---

1) Formula (2) may be readily verified for the point $x=-1$ for which we have

$$\operatorname{arccot} x = \frac{3\pi}{4}, \quad \arctan \frac{1}{x} = -\frac{\pi}{4}$$

On the other hand, for negative values of x the difference $\arctan \frac{1}{x} - \operatorname{arccot} x$ is constant because its derivative $\left(\frac{1}{1+x^2} - \frac{1}{1+x^2}\right)$ for all $x < 0$ is equal to zero (see Sec. 226, Item 1). Consequently, formula (2) holds true for *all negative* values of x; putting $x=+1$ and reasoning in the same way, we are convinced that formula (1) holds true for *all positive* values of x. For $x=0$, the function $\arctan \frac{1}{x}$ is not defined and, hence, does not have a derivative. That is why one cannot assert that the function $\arctan \frac{1}{x} - \operatorname{arccot} x$ is constant *over the entire number line* (cf. Sec. 265, Theorem 1).

For $x=0$ the function $\arctan \frac{1}{x}$ (Fig. 237) is discontinuous $\left(\text{its limit from the left is } -\frac{\pi}{2}, \text{ from the right } +\frac{\pi}{2}; \text{ cf. Sec. 218}\right)$ and, hence, it is not differentiable, whereas the function $\operatorname{arccot} x$ (Fig. 238) is continuous and its derivative for $x=0$ is -1. The right branch of the graph $y=\arctan \frac{1}{x}$ coincides with the right half of the graph

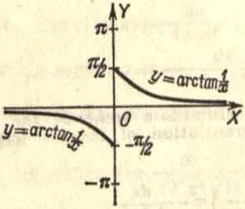

Fig. 237

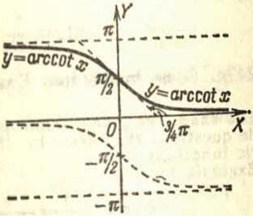

Fig. 238

$y=\operatorname{arccot} x$, while the left branch coincides with the left half of the dashed line in Fig. 238 (this yields the nonprincipal value of the multiple-valued function $y=\operatorname{arccot} x$).

Example 2.

$$\frac{d}{dx}\arctan\frac{1+x}{1-x}=\frac{d}{dx}\left(\frac{1+x}{1-x}\right):\left[1+\left(\frac{1+x}{1-x}\right)^2\right]=$$

$$=\frac{2}{(1-x)^2}\cdot\frac{(1-x)^2}{(1-x)^2+(1+x)^2}=\frac{1}{1+x^2}$$

This expression coincides with the derivative of the function $\arctan x$.

For $x<1$, this function is connected with the given one by the relation [1]

$$\arctan\frac{1+x}{1-x}=\arctan x+\frac{\pi}{4} \tag{3}$$

(Fig. 239), and for $x>1$, by the relation

$$\arctan\frac{1+x}{1-x}=\arctan x-\frac{3\pi}{4} \tag{4}$$

For $x=1$, the function $\arctan\frac{1+x}{1-x}$ has a discontinuity $AB=\pi$, and does not have a derivative.

Example 3.

$$\frac{d}{dx}\arcsin(\sin x)=\frac{1}{\sqrt{1-\sin^2 x}}\cdot\frac{d}{dx}\sin x=\frac{\cos x}{|\cos x|}$$

---

[1] The proof is the same as in the preceding footnote of this section.

This derivative is equal to $+1$ when $\cos x > 0$, and to -1 when $\cos x < 0$. For $x=(2n+1)\frac{\pi}{2}$, when $\cos x=0$, the derivative does not exist.

Note. In the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ we have $\arcsin(\sin x)=x$, in the interval $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$ we have $\arcsin(\sin x)=\pi-x$, in the interval $\frac{3\pi}{2} \leqslant x \leqslant \frac{5\pi}{2}$ we have $\arcsin(\sin x)=x-2\pi$ and so forth (Fig. 240). Therefore, inside the first interval the derivative is equal to 1, inside the second one, to -1, etc. At the points $x=(2k+1)\frac{\pi}{2}$ the derivative has a discontinuity; at each of these points we have one-sided derivatives (cf. Sec. 231, Example 2).

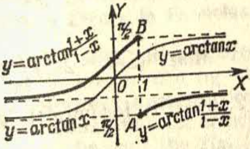

Fig. 239

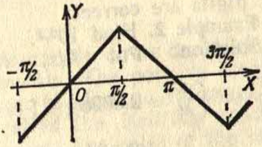

Fig. 240

248. The Differential in Approximate Calculations

It often happens that a function $f(x)$ and its derivative $f'(x)$ may be readily calculated for $x=a$, but not for values of x close to a (here, direct computation of the function is difficult). Then use is made of the approximate formula

$$f(a+h) \approx f(a)+f'(a)\,h \tag{1}$$

It states that the increment $f(a+h)-f(a)$ of the function $f(x)$ for small values of h is approximately equal[1] to the differential $f'(a)\,h$ (cf. Sec. 228, Theorem 2).

Below (Sec. 265) a method is indicated for evaluating the error[2] of formula (1), but the evaluation often involves cumbersome computation. For rough calculations, one often confines oneself to formula (1).

---

[1] If $f'(a)=0$, then formula (1) states that the increment in the function is small compared to h; then, for sufficiently small values of h we can take it that $f(a+h)=f(a)$.

[2] See also Sec. 271, Note.

Example 1. Extract the square root of 3654.

Solution. It is necessary to find the value of the function $f(x)=\sqrt{x}$ for $x=3654$. It is easy to compute the values of $f(x)$ and $f'(x)=\frac{1}{2\sqrt{x}}$ for $x=3600$. Formula (1), for $a=3600$ and $h=54$, yields $\sqrt{3654}\approx 60+\frac{1}{2\cdot 60}\cdot 54\approx 60.45$. Here, all the digits are correct.

Example 2. Find $10^{2.1}$.

Solution. Put $f(x)=10^x$ so that (Sec. 245) $f'(x)=\frac{1}{M}10^x\left(\frac{1}{M}\approx 2.3026\right)$. For $a=2$, $h=0.1$, formula (1) gives

$$10^{2.1}\approx 100+\frac{1}{M}\cdot 100\cdot 0.1\approx 123.0$$

This result is rather rough (to within the fourth significant digit, $10^{2.1}=125.9$).

If we compute $10^{2.01}$ (now $h=0.01$) in the same fashion, we get 102.3. All the digits are correct.

Example 3. Without using tables, find the value of tan 46°.

Solution. Put $f(x)=\tan x$, $a=45^\circ$, $h=1^\circ=0.0175$ radian; then we have $f'(a)=\frac{1}{\cos^2 45^\circ}=2$. Hence, $\tan 46^\circ\approx 1+2\times\times 0.0175=1.0350$.

Only the last digit is incorrect; from the tables we find $\tan 46^\circ=1.0355$.

It is worth noting the following approximate formulas[1] (α is an infinitesimal):

$$\frac{1}{1+\alpha}\approx 1-\alpha,\qquad \frac{1}{1-\alpha}\approx 1+\alpha; \tag{2}$$

$$\frac{1}{(1+\alpha)^2}\approx 1-2\alpha,\qquad \frac{1}{(1-\alpha)^2}\approx 1+2\alpha; \tag{3}$$

$$\sqrt{1+\alpha}\approx 1+\frac{1}{2}\alpha,\qquad \sqrt{1-\alpha}\approx 1-\frac{1}{2}\alpha; \tag{4}$$

$$\frac{1}{\sqrt{1+\alpha}}\approx 1-\frac{1}{2}\alpha,\qquad \frac{1}{\sqrt{1-\alpha}}\approx 1+\frac{1}{2}\alpha; \tag{5}$$

$$\sqrt[3]{1+\alpha}\approx 1+\frac{1}{3}\alpha,\qquad \sqrt[3]{1-\alpha}\approx 1-\frac{1}{3}\alpha; \tag{6}$$

---

[1] Formulas (2)-(6) are special cases of the formula $(1+\alpha)^n\approx 1+n\alpha$, which is obtained from (1) by putting $f(x)=x^n$, $a=1$, $h=\alpha$.

$$\ln(1+\alpha)\approx\alpha,\qquad \ln(1-\alpha)\approx-\alpha; \tag{7}$$

$$e^{\alpha}\approx 1+\alpha,\qquad 10^{\alpha}\approx 1+\frac{1}{M}\alpha; \tag{8}$$

$$\sin\alpha\approx\alpha,\qquad \cos\alpha\approx 1-\frac{1}{2}\alpha^2,\ \tan\alpha\approx\alpha \tag{9}$$

249. Using the Differential to Estimate Errors in Formulas

Data obtained in measurements contain errors due to inaccuracies in the measuring instruments. The positive number which definitely exceeds the error in absolute value (or, at worst, is equal to this error) is called the *limiting absolute error* or, simply, the *limiting error*. The ratio of the limiting error to the absolute value of the quantity being measured is called the *limiting relative error*.

Example 1. The length of a pencil is measured with a ruler having millimetre divisions. The measurement yields 17.9 cm. The error is not known but it is definitely less than 0.1 cm. Therefore we can take 0.1 cm for the limiting error. The limiting relative error is equal to $\frac{0.1}{17.9}$. Rounding this up we get 0.6%.

Finding the limiting error. Suppose a function y is computed from an exact formula $y=f(x)$, but the value of x is obtained by measurement and therefore contains an error. Then the limiting absolute error $|\Delta y|$ of the function is found from the formula

$$|\Delta y|\approx|dy|=|f'(x)|\,|\Delta x| \tag{1}$$

where $|\Delta x|$ is the limiting error of the argument. The quantity $|\Delta y|$ is rounded up (because of the inaccuracy of the formula itself).

Example 2. The side of a square is measured and found to be 46 m. The limiting error is equal to 0.1 m. Find the limiting error for the area of the square.

Solution. We have $y=x^2$ (where x is the side of the square and y is the area). Whence $|\Delta y|\approx 2|x|\,|\Delta x|$. In our example, $x=46$ and $|\Delta x|=0.1$. Hence $|\Delta y|\approx 2\cdot 46\cdot 0.1=9.2$. The limiting absolute error is rounded off to 10 m². The limiting relative error is equal to $\frac{10}{46^2}\approx 0.5\%$.

The limiting relative error $\left|\frac{\Delta y}{y}\right|$ may also be found by means of logarithmic differentiation (Sec. 244) by using the formula

$$\left|\frac{\Delta y}{y}\right| \approx |d \ln y| \tag{2}$$

In particular, for $y=x^n$ $\left(\text{then } d \ln y=\frac{n\,dx}{x}\right)$ we have

$$\left|\frac{\Delta y}{y}\right| \approx n\left|\frac{\Delta x}{x}\right| \tag{3}$$

that is, the limiting relative error of the power x^n is equal to the n-fold limiting relative error of the argument.

Example 3. Under the hypotheses of Example 2, the limiting relative error of the area is equal to $2\cdot\frac{0.1}{46} \approx 0.5\%$.

Example 4. Measuring the edge of a cube yields $x=12.4$ cm. The limiting error is 0.05 cm. What is the limiting relative error for the volume of the cube?

Solution. The limiting relative error for x is equal to $\frac{0.05}{12.4} \approx 0.004$; for x^3 it is equal to $3\cdot 0.004=0.012$.

Rule 1. The limiting relative error of a product of two or several factors is equal to the sum of the limiting relative errors of the factors.

Rule 2. The limiting relative error of a fraction is equal to the sum of the limiting relative errors of the numerator and denominator.

These rules follow from Secs. 239, 240.[1]

Example 5. In seeking the specific weight of a body, we have found its weight $p=20$ g and the weight of the water it displaces, $v=40$ g. The limiting absolute error for p is 0.5 g, for v it is 1 g. Determine the limiting relative error for the specific weight.

Solution. The specific weight y is equal to $\frac{p}{v}$. We have

$$\left|\frac{\Delta y}{y}\right| = \left|\frac{\Delta p}{p}\right| + \left|\frac{\Delta v}{v}\right| = \frac{0.5}{20}+\frac{1}{40}=0.05$$

Example 6. The altitude h and radius of the base r of a cylinder have been measured to within 1%. Find the limiting relative error (1) for the lateral surface S and (2) for the volume V of the cylinder.

---

[1] The formula $d \ln \frac{u}{v}=\frac{du}{u}-\frac{dv}{v}$ yields the limiting relative error $\left|\frac{du}{u}\right|+\left|\frac{dv}{v}\right|$ and not $\left|\frac{du}{u}-\frac{dv}{v}\right|$ because the quantities $\frac{du}{u}$ and $\frac{dv}{v}$ can have different signs.

Solution. We have $S=2\pi rh$. The factor 2π is an exact number; its error is zero. The relative error for S is $\left|\frac{\Delta S}{S}\right| = \left|\frac{\Delta r}{r}\right| + \left|\frac{\Delta h}{h}\right| = 2\%$ and for $V=\pi r^2 h$ it is equal to $2\left|\frac{\Delta r}{r}\right| + \left|\frac{\Delta h}{h}\right| = 3\%$.

250. Differentiation of Implicit Functions

Let an equation relating x and y and satisfied by the values $x=x_0$ and $y=y_0$ define y as an implicit function of x. To find the derivative $\frac{dy}{dx}$ at the point $x=x_0$, $y=y_0$ there is no need to seek the explicit expression of the function. It is sufficient to equate the differentials of both sides of the equation and from the equality obtained to find the ratio $dy:dx$.

Note. An equation connecting x and y can define y as a multiple-valued function $F(x)$ of x. But specifying a pair of values $x=x_0$ and $y=y_0$ isolates one of the many values of the function.

Geometrically, a straight line parallel to OY (Fig. 241) can intersect the curve L at several points M_0, M_1, M_2, ..., but specification of the point M_0 isolates the arc AM_0B (that passes through it) which is a single-valued function.

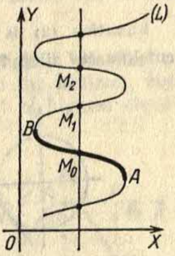

Fig. 241

Example 1. Find the derivative of the implicit function given by the equation $x^2+y^2=25$ at the point $x=4$, $y=-3$.

First method. Solving the equation we get $y=-\sqrt{25-x^2}$ (we choose the minus sign because for $x=4$ we must have $y=-3$). Now we get

$$\frac{dy}{dx}=\frac{x}{\sqrt{25-x^2}}=\frac{4}{3}$$

Second method. Equating the differentials of the right and left sides, we obtain

$$2x\,dx+2y\,dy=0$$

whence

$$\frac{dy}{dx}=-\frac{x}{y}=\frac{4}{3} \qquad (1)$$

We have found the slope of the tangent line M_0T to the circle $x^2+y^2=25$ (Fig. 242) at the point $M_0(4, -3)$. The slope of the radius OM_0 is $-\frac{3}{4}$. The product of the slopes is equal to -1, i.e. $OM_0 \perp M_0T$.

Example 2. Find the derivative $\frac{dy}{dx}$ of the implicit function given by the equation [1]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{2}$$

Differentiating, we find

$$\frac{2x\,dx}{a^2}+\frac{2y\,dy}{b^2}=0$$

whence

$$\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y} \tag{3}$$

Equation (2) is an ellipse. By virtue of (3) the slope of the tangent line MT (Fig. 243) is $-\frac{b^2}{a^2}\frac{x}{y}$ The slope of the diameter MM' is $\frac{y}{x}$. The product of the slopes is equal to $-\frac{b^2}{a^2}$ Hence, (Sec. 55), the directions MT and MM' are conjugate, that is to say the diameter MM' bisects the chords parallel to MT.

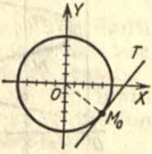

Fig. 242

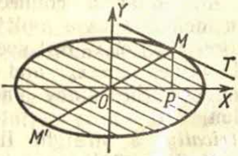

Fig. 243

The diameters of hyperbolas and parabolas possess the same property

251. Parametric Representation of a Curve

Any variable quantity t defining the position of a point on a curve is called a *parameter*.[2]. In mechanics, time is most often taken as the parameter.

---

[1] Eq. (2) defines y as a double-valued function of x, but insofar as the values of both variables will be known, one of the two values of the function is taken (cf. Example 1).

[2] The term "parameter" is used in yet another sense to denote a quantity which for a given curve is invariable but changes when moving from one curve of a given type to another. For example, the quantity p in the equation of the parabola $y^2=2px$ is constant for the given parabola but changes when we pass to another parabola.

The coordinates of a point lying on a curve L are functions of the parameter:

$$x = f(t), \tag{1}$$

$$y = \varphi(t) \tag{2}$$

Eqs. (1) and (2) are called the *parametric equations* of the curve L (cf. Sec. 152).

If it is desired to find an equation relating the coordinates x, y of curve L, one has to eliminate t from Eqs. (1) and (2) (see Examples 1 and 2).

It may happen, however, that the equation obtained after eliminating t represents a curve which the curve L covers only in part (see Example 3)

Example 1. Let O (Fig. 244) be the highest position of a material particle thrown at an angle to the horizon, and let t be the time reckoned from the instant of highest elevation. The position of the point M on the trajectory AOB is determined by the quantity t so that t is the parameter. The parametric equations of the trajectory referred to the XOY system are

$$x = OP = v_0 t, \tag{3}$$

$$y = PM = -\frac{1}{2} g t^2 \tag{4}$$

They state that the point M is in uniform motion with velocity v_0 in the horizontal direction and in uniform accelerated motion (g is the acceleration of gravity) in the vertical direction.

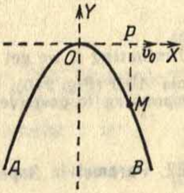

Fig. 244

Eliminating t we get the equation

$$y = -\frac{g}{2v_0^2} x^2 \tag{5}$$

which shows that the motion is along a parabola.

Example 2. The position of a point M on a circle $ABA'B'$ of radius R (Fig. 245) is determined by the magnitude of the angle $\varphi = \angle AOM$ so that φ is the parameter. Setting up axes as shown in Fig. 245, we have the parametric equations of the circle:

$$x=R\cos\varphi, \tag{6}$$

$$y=R\sin\varphi \tag{7}$$

In order to eliminate φ, square (6) and (7) and add:

$$x^2+y^2=R^2 \tag{8}$$

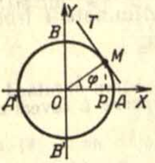

Fig. 245

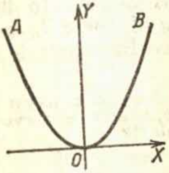

Fig. 246

Example 3. Consider a curve given by the parametric equations

$$x=\sqrt{t},\quad y=\frac{1}{2}t \tag{9}$$

Eliminating t, we get the equation $y=\frac{1}{2}x^2$ which describes a parabola AOB (Fig. 246). The curve (9) is half of this parabola (OB) corresponding to positive values of x.

252. Parametric Representation of a Function

Let there be given two functions of the argument t:

$$x=f(t),\quad y=\varphi(t) \tag{1}$$

Then one of them, say y, is a function of the other.[1] The representation of this function with the aid of equalities (1) is called *parametric*, the auxiliary quantity t being called the *parameter*.

In order to obtain an explicit expression of y as a function of x, one has to solve the equation $x=f(t)$ for t (this is not always possible) and substitute the expression found into the equation $y=\varphi(t)$.

---

[1] As a rule, it is multivalued even when $f(t)$ and $\varphi(t)$ are single-valued.

On the contrary, it is often more convenient to pass from nonparametric representation to parametric. Utilizing the arbitrariness of choice of one of the functions $f(t)$, $\varphi(t)$, we attempt to ensure single-valuedness and, if possible, simplicity of both functions.

The derivative $\frac{dy}{dx}$ is expressed in terms of the parameter t by the formula

$$\frac{dy}{dx}=\frac{d\varphi(t)}{df(t)}=\frac{\varphi'(t)}{f'(t)} \tag{2}$$

In a parametric representation, both variables x and y are on equal terms (cf. Sec. 251).

Example 1. Given two functions:

$$x=R\cos t, \quad y=R\sin t \tag{3}$$

They specify y parametrically as a double-valued function of x (and conversely). From the first equation we find $\cos t=\frac{x}{R}$ so that $\sin t=\pm\sqrt{1-\frac{x^2}{R^2}}$. Substituting into the second equation, we get

$$y=\pm\sqrt{R^2-x^2} \tag{4}$$

This is the equation of a circle (cf. Sec. 251, Example 2). The parameter t is the angle XOM (see Fig. 245). The derivative $\frac{dy}{dx}$ expressed in terms of the parameter t is

$$\frac{dy}{dx}=\frac{d(R\sin t)}{d(R\cos t)}=-\cot t \tag{5}$$

This is the slope of the tangent line MT.

Example 2. The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{6}$$

describes an ellipse and specifies a double-valued function $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$. To represent it parametrically, one can arbitrarily express one of the variables, say x, as a function of t. Putting $\frac{x}{a}=\cos t$, we find $\frac{y}{b}=\pm\sin t$. The sign may be chosen at will. Let us take the plus sign. We obtain the parametric representation

$$x=a\cos t, \quad y=b\sin t \tag{7}$$

The geometrical meaning of the parameter t is evident from Fig. 247 where ANA' is a circle of radius a and N is a point taken on the same vertical line as point M of the ellipse on the same side[1] of the axis AA'. We have $t = \angle AON$. The derivative $\frac{dy}{dx}$ is expressed in terms of t by the formula

$$\frac{dy}{dx} = \frac{d\,(b \sin t)}{d\,(a \cos t)} = -\frac{b}{a} \cot t$$

This is the slope of the tangent line MT.

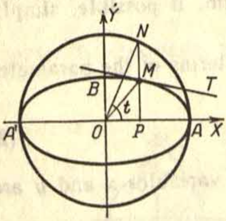

Fig. 247

Note. The ordinary specification of a function $y=f(x)$ may be regarded as a special case of the parametric representation; namely, it may be written in the form

$$x=t, \quad y=f(t)$$

253. The Cycloid

The *cycloid* is a curve described by a point M on the circumference of a circle rolling (without sliding) along a straight line (*directrix* or *base line*) The rolling circle is called the *generatrix*.

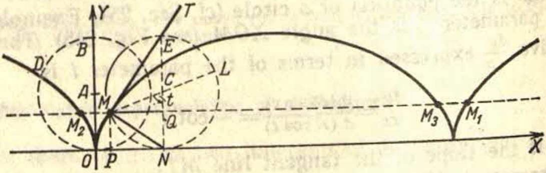

Fig 248

In Fig. 248, the directrix is OX; the generating circle is given in two positions: in the "initial" (ODB) when M touches the directrix and in an "intermediate" position (NME).

Note. The expression "rolls without sliding" means that the point of tangency N is at a distance from the initial position O equal to the arc NM:

$$ON = \overset{\frown}{NM} \tag{1}$$

---

[1] If we take $\frac{y}{b} = -\sin t$, then N must be taken on the other side.

Parametric equations of the cycloid. If the coordinate axes are as indicated in Fig. 248 and if we take for the parameter the angle $t=\angle MCN$, we get the following parametric equations[1] of the cycloid:

$$x=a(t-\sin t), \tag{2}$$

$$y=a(1-\cos t) \tag{3}$$

where a is the radius of the generating circle.

If (3) is solved for t and substituted into (2) we get x as an infinitely multiple-valued function of y:

$$x=2ak\pi \pm \left(a \arccos\frac{a-y}{a}-\sqrt{y(2a-y)}\right) \tag{4}$$

where k is any integer.[2]

The ordinate y is a single-valued, but not an elementary, function of x (see Fig. 248).

The slope k of the tangent line is

$$k=\frac{dy}{dx}=\frac{a \sin t}{a(1-\cos t)} \tag{5}$$

and the slope k' of the straight line NM is

$$k'=\frac{y-y_N}{x-x_N}=\frac{a(1-\cos t)}{-a \sin t} \tag{6}$$

Hence, $kk'=-1$, i.e. $MT \perp MN$. Consequently, to construct a tangent line to the cycloid it is sufficient to join M to the highest point of the generating circle (angle NME is a right angle: the angle in a semicircle is a right angle).

254. The Equation of a Tangent Line to a Plane Curve

Let MT (Fig. 249) be a tangent line to the curve L at the point $M(x, y)$. Denote the running coordinates of the point N lying on the tangent line by X, Y.

---

[1] The value of the angle t can be positive or negative and can have any absolute value: for $0 \leqslant t \leqslant \frac{\pi}{2}$ Eqs. (2)-(3) are easily read from Fig. 248:

$$x=OP=ON-PN=\overset{\smile}{NM}-MQ=at-a\sin t,$$
$$y=PM=NC-QC=a-a\cos t$$

[2] In Fig. 248, the points M, M_1, etc. are associated with the plus sign in front of the parentheses, the points M_2, M_3, etc., with the minus sign.

For any representation of the curve L (explicit, implicit or parametric) the equation of the tangent line may be written in the following symmetric form:

$$\frac{X-x}{dx}=\frac{Y-y}{dy} \tag{1}$$

If the curve L is given by the equation $y=f(x)$, then from (1) we obtain[1]

$$Y-y=f'(x)(X-x) \tag{2}$$

If the curve L is given parametrically, we get

$$\frac{X-x}{x'}=\frac{Y-y}{y'} \tag{3}$$

where x', y' are derivatives with respect to the parameter.

In an implicit representation of the curve L we equate the differentials of both sides of the equation (cf. Sec. 250) and in the equality obtained replace dx, dy by the proportionate quantities $X-x$, $Y-y$.

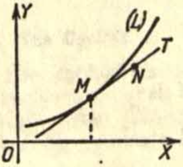

Fig. 249

Example 1. Find the equation of the tangent line to the parabola $y=x^2-3x+2$ at the point (0, 2).

We have $y'=2x-3=-3$. By (2) the desired equation is $Y-2=-3X$.

Example 2. Find the equation of the tangent line to the ellipse

$$x=5\sqrt{2}\cos t, \quad y=3\sqrt{2}\sin t \tag{4}$$

at the point $M(-5, 3)$ (cf. Sec. 252, Example 2).

Solution. To the given point there corresponds the value $t=\frac{3\pi}{4}$. From (4) we have

$$x'=-5\sqrt{2}\sin t=-5, \quad y'=3\sqrt{2}\cos t=-3$$

According to (3), the equation of the tangent line is

$$\frac{X+5}{-5}=\frac{Y-3}{-3}$$

---

[1] It is assumed that the derivative $f'(x)$ at the point M is finite. However, if $f'(x)=\infty$ (Sec. 231, Case 1), then in place of (2) we have the equation

$$X-x=0$$

(the tangent line is parallel to the axis of ordinates).

i. e.
$$3X-5Y+30=0$$

Example 3. Find the equation of the tangent line to the equilateral hyperbola $xy=m^2$ at the point $\left(\frac{m}{2}, 2m\right)$.

Solution. Equating the differentials of both sides of the equation, we obtain
$$x\,dy+y\,dx=0$$
Replacing dx, dy by the quantities $X-x$, $Y-y$, we get
$$x(Y-y)+y(X-x)=0 \qquad (5)$$
Since $xy=m^2$, it follows that (5) may be rewritten as
$$xY+yX=2m^2 \qquad (6)$$
Substituting $x=\frac{m}{2}$, $y=2m$ into (5) or into (6), we obtain
$$Y+4X=4m$$

254a. Tangent Lines to Quadric Curves

	Equation of curve	Equation of tangent line
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$	$\frac{xX}{a^2}+\frac{yY}{b^2}=1,$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$	$\frac{xX}{a^2}-\frac{yY}{b^2}=1,$
Parabola	$y^2=2px,$	$yY=p(X+x)$

255. The Equation of a Normal

The *normal* at a point M of a curve L (Fig. 250) is the perpendicular MN to the tangent line MT.

According to Eq. (1), Sec. 254, the equation of the normal is of the form
$$(X-x)\,dx+(Y-y)\,dy=0 \qquad (1)$$
In accordance with Eqs. (2) and (3), Sec. 254, we obtain the equation of the normal in the following formulas:
$$Y-y=-\frac{1}{f'(x)}(X-x), \qquad (2)$$
$$(X-x)\,x'+(Y-y)\,y'=0 \qquad (3)$$

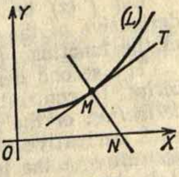

Fig. 250

In an implicit representation of the curve L we equate the differentials of both sides of the equation and eliminate dx and dy by means of (1).

Example 1. Find the equation of the normal to the parabola $y=\frac{1}{2}x^2$ at the point $(-2, 2)$

We have $y'=x=-2$; according to (2) the desired equation is

$$Y-2=\frac{1}{2}(X+2)$$

Example 2. The equation of the normal to the cycloid

$$x=a(t-\sin t),\quad y=a(1-\cos t) \tag{4}$$

(Sec. 253) is, according to (3), of the form

$$(X-x)(1-\cos t)+(Y-y)\sin t=0 \tag{5}$$

or, utilizing (4),

$$X(1-\cos t)+Y\sin t-at(1-\cos t)=0 \tag{6}$$

This equation is satisfied for $X=at$, $Y=0$; hence, the normal passes (see Fig. 248) through the point $N(at, 0)$ of the generating circle.

Example 3. Find the equation of the normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiating, we obtain

$$\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}=0 \tag{7}$$

Eliminating the differentials from (7) and (1), we get

$$\frac{(X-x)\,y}{b^2}=\frac{(Y-y)\,x}{a^2}$$

256. Higher-Order Derivatives

Let $f'(x)$ be a derivative of the function $f(x)$; then the derivative of the function $f'(x)$ is called the *second derivative* of the function $f(x)$ and is denoted by $f''(x)$.

The second derivative is also called a *second-order derivative*. In contrast, the function $f'(x)$ is called a *first-order derivative*, or the *first derivative*.

A derivative of the second derivative is called the *third derivative* of the function $f(x)$ (or the *third-order derivative*). It is denoted by $f'''(x)$.

In similar manner we define the derivatives of the fourth order $f^{\mathrm{IV}}(x)$, fifth order $f^{\mathrm{V}}(x)$ and so forth (numbers are used instead of dashes to save space and Roman numerals are used to avoid confusion with exponents).

A derivative of the nth order is symbolized by $f^{(n)}(x)$.

If a function is denoted by a single letter, say y, then its successive derivatives are denoted by

$$y',\ y'',\ y''',\ y^{\mathrm{IV}},\ y^{\mathrm{V}},\ \ldots,\ y^{(n)}$$

Example 1. Find the successive derivatives of the function $f(x)=x^4$.

Solution. $f'(x)=4x^3$, $f''(x)=(4x^3)'=12x^2$, $f'''(x)=24x$, $f^{\mathrm{IV}}(x)=24$, $f^{\mathrm{V}}(x)=0$.

Subsequent derivatives are also equal to zero.

Example 2. If $y=\sin x$, then

$$y'=\cos x=\sin\left(x+\frac{\pi}{2}\right),\ y''=-\sin x=\sin(x+\pi),$$

$$y'''=-\cos x=\sin\left(x+\frac{3\pi}{2}\right),\ \ldots,\ y^{(n)}=\sin\left(x+n\frac{\pi}{2}\right)$$

The values of the derivatives for a given value of the argument $x=a$ are denoted by $f'(a)$, $f''(a)$, $f'''(a)$, etc. In Example 1 we have $f'(2)=32$, $f''(2)=48$ and so forth.

Example 3. If $f(x)=\ln(1+x)$, then

$$f'(x)=\frac{1}{1+x},\ f''(x)=-\frac{1}{(1+x)^2},\ f'''(x)=\frac{1\cdot 2}{(1+x)^3},$$

$$f^{\mathrm{IV}}(x)=-\frac{1\cdot 2\cdot 3}{(1+x)^4},\ \ldots,\ f^{(n)}(x)=\frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

Consequently,

$$f(0)=0,\ f'(0)=1,\ f''(0)=-1,\ f'''(0)=2!,$$
$$f^{\mathrm{IV}}(0)=-3!,\ \ldots,\ f^{(n)}(0)=(-1)^{n+1}(n-1)!$$

257. Mechanical Meaning of the Second Derivative

Let a point be in rectilinear motion. Covering a distance s in time t, it acquires a velocity v. Let this velocity change, the increment during the time interval $(t,\ t+\Delta t)$ being Δv. Then the ratio $\frac{\Delta v}{\Delta t}$ yields the change in velocity per (average) unit of time and is called the *average acceleration*. This relation describes the rate of change of the velocity at time t the more precisely, the smaller Δt is. Therefore, the *acce-*

leration (at time t) is the limit of the ratio $\frac{\Delta v}{\Delta t}$ as $\Delta t \to 0$, that is, the derivative $\frac{dv}{dt}$. But the velocity v itself is a derivative: $\frac{ds}{dt}$. Therefore, acceleration is the second derivative of the distance with respect to the time.

Example. The motion of an undamped oscillation of a membrane is given by the equation

$$s = a \sin \frac{2\pi t}{T} \tag{1}$$

(T is the period of oscillation, a is the amplitude, and s is the deviation of a point of the membrane from the position of rest).

The rate of motion is

$$v = s' = \frac{2\pi a}{T} \cos \frac{2\pi t}{T} \tag{2}$$

The acceleration is

$$v' = s'' = \frac{-4\pi^2 a}{T^2} \sin \frac{2\pi t}{T} \tag{3}$$

Comparing (2) and (3) we see that

$$s'' = -\frac{4\pi^2}{T^2} s \tag{4}$$

thus, the elastic force of oscillation (it is proportional to the acceleration by Newton's second law) is proportional to the deviation and has opposite direction.

258. Higher-Order Differentials

Let us consider a number of equidistant values of an argument:

$$x, \ x+\Delta x, \ x+2\Delta x, \ x+3\Delta x, \ \ldots$$

and the corresponding values of the function:

$$y = f(x), \quad y_1 = f(x+\Delta x), \quad y_2 = f(x+2\Delta x),$$
$$y_3 = f(x+3\Delta x), \ \ldots$$

We introduce the notations

$$\Delta y = f(x+\Delta x) - f(x),$$
$$\Delta y_1 = f(x+2\Delta x) - f(x+\Delta x),$$
$$\Delta y_2 = f(x+3\Delta x) - f(x+2\Delta x)$$

etc. The quantities Δy, Δy_1, Δy_2, ... are called the *first differences* of the function $f(x)$. The *second differences* are the quantities $\Delta y_1 - \Delta y$, $\Delta y_2 - \Delta y_1$, etc. They are denoted by $\Delta^2 y$ (read: delta two y), $\Delta^2 y_1$, etc.

$$\Delta^2 y = \Delta y_1 - \Delta y,$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

The *third differences* are defined similarly: $\Delta^3 y = \Delta^2 y_1 - \Delta^2 y$, etc.

Example 1. Let $f(x) = x^3$ and $x = 2$. The first differences will be

$$\Delta y = (2+\Delta x)^3 - 2^3 = 12\Delta x + 6\Delta x^2 + \Delta x^3,$$
$$\Delta y_1 = (2+2\Delta x)^3 - (2+\Delta x)^3 = 12\Delta x + 18\Delta x^2 + 7\Delta x^3,$$
$$\Delta y_2 = (2+3\Delta x)^3 - (2+2\Delta x)^3 = 12\Delta x + 30\Delta x^2 + 19\Delta x^3,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The second differences:

$$\Delta^2 y = \Delta y_1 - \Delta y = 12\Delta x^2 + 6\Delta x^3,$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = 12\Delta x^2 + 12\Delta x^3,$$
$$\cdots\cdots\cdots\cdots\cdots$$

The third differences:

$$\Delta^3 y = \Delta^2 y_1 - \Delta^2 y = 6\Delta x^3,$$
$$\cdots\cdots\cdots\cdots$$

For an infinitesimal Δx, the first difference is, as a rule, of first order with respect to Δx, the second difference is of second order, the third, of third order, etc.

In Sec. 228 we called the principal term of the first difference ($12\Delta x$ in Example 1) the differential of the function. We will now call it the *first differential*. The *second differential* is then the principal term of the second difference; it is proportional to Δx^2 ($12\Delta x^2$ in Example 1); the *third differential* is the principal term of the third difference, which term is proportional to Δx^3 ($6\Delta x^3$ in Example 1), etc. Let us formulate this exactly.

Definition. Let the second difference $\Delta^2 y$ of the function $y = f(x)$ be split up into a sum of two terms:

$$\Delta^2 y = B\Delta x^2 + \beta$$

where B is independent of Δx and the term β is of higher order of smallness with respect to Δx^2. Then the term $B\Delta x^2$ is called the *second differential* of the function y and is denoted by d^2y or $d^2f(x)$. The differentials of higher orders are defined in similar fashion.

Theorem 1. The coefficient B of Δx^2 in the expression of the second differential is equal to the second derivative $f''(x)$. The coefficient C of Δx^3 in the expression of the third differential $C\,\Delta x^3$ is equal to the third derivative $f'''(x)$, etc.

Example 2. If $f(x)=x^3$, then $f''(x)=6x$. Accordingly, $d^2(x^3)=6x\,\Delta x^2$. For $x=2$ we have $d^2(x^3)=12\Delta x^2$ (cf. Example 1). Further, $f'''(x)=6$ (for any value of x); accordingly, $d^3(x^3)=6\Delta x^3$.

Theorem 1 may be formulated differently as follows.

Theorem 1a. A differential of the nth order is equal to the product of the nth derivative by the nth power of the increment of the independent variable:

$$d^n f(x)=f^{(n)}(x)\,\Delta x^n \tag{1}$$

Since for the independent variable we have

$$\Delta x=dx$$

it follows that

$$d^n f(x)=f^{(n)}(x)\,dx^n \text{ [1]} \tag{2}$$

Example 3. $d(x^4)=4x^3\,dx$, $d^2(x^4)=12x^2\,dx^2$, $d^3(x^4)=24x\,dx^3$, $d^4(x^4)=24dx^4$, $d^5(x^4)=0$, $d^6(x^4)=d^7(x^4)=\ldots=0$ (cf. Sec. 256, Example 1).

Example 4. $d^n(\sin x)=\sin\left(x+n\frac{\pi}{2}\right)dx^n$ (cf. Sec. 256, Example 2).

Theorem 2. If we consider the differential dx of the argument x as a quantity independent of x, then the second differential of the function $f(x)$ is equal to the differential of its first differential:

$$d(df(x))=d^2f(x) \tag{3}$$

Under the same condition, the third differential is the differential of the second, etc.

---

[1] If x is not an independent variable, then formula (1) does not, as a rule, hold for any value of n, even for $n=1$ (cf. Sec. 234). But in this case, even formula (2), which is always true for $n=1$, does not, as a rule, hold for differentials of higher order ($n=2, 3, \ldots$). In other words, the expressions $f''(x)\,dx^2$, $f'''(x)\,dx^3$, ... *are not invariant.*

Thus, if $f(x)=x^3$, then the expression $6x\,dx^2$ represents $d^2(x^3)$ when x is an independent variable. But if we put $x=t^2$ and take t instead of x for the independent variable, then $f(x)=t^6$ and we get $6x\,dx^2=24t^4\,dt^2$, whereas $d^2f(x)=30t^4\,dt^2$.

Example 5. Let $f(x)=x^4$. We have $df(x)=4x^3\,dx$. If we consider dx as independent of x, then it must be regarded as a constant when differentiating. Hence, $d(4x^3\,dx)=d(4x^3)\,dx=12x^2\,dx^2$. But this is the second differential of the function x^4 (Example 3). Then $d[d^2(x^3)]=$ $=d(12x^2\,dx^2)=d(12x^2)\,dx^2=24x\,dx^3$; this is the third differential of x^4, etc.

The second differential of a linear function of an independent variable is equal to zero:

$$d^2(ax+b)=0$$

In particular, the second differential of the independent variable is zero: $d^2x=0$.

The third differential of a quadratic function is zero:

$$d^3(ax^2+bx+c)=0$$

Generally, the $(n+1)$th differential of a polynomial of degree n is zero.

259. Expressing Higher Derivatives in Terms of Differentials

The expression of a second derivative in terms of differentials[1] is of the form

$$y''=\frac{dx\,d^2y-dy\,d^2x}{dx^3} \qquad (1)$$

It holds for any choice of the argument.

If we take x for the argument (then $d^2x=0$), it follows that

$$y''=\frac{d^2y}{dx^2} \qquad (2)$$

This expression also follows from (2), Sec. 258 (for $n=2$). The following expressions are a consequence of the same formula:

$$y'''=\frac{d^3y}{dx^3},\ y^{\text{IV}}=\frac{d^4y}{dx^4},\ \ldots,\ y^{(n)}=\frac{d^ny}{dx^n} \qquad (3)$$

---

[1] We have $y''=\frac{dy'}{dx}$, substitute $y'=\frac{dy}{dx}$; in differentiating apply Theorem 2, Sec. 258.

provided that x is the independent variable. Their general expressions are complicated.¹)

Note. A derivative of the nth order is frequently *denoted by* $\frac{d^ny}{dx^n}$ irrespective of which quantity is taken as the argument. But one *cannot substitute*, into this expression, expressions of the variables y and x in terms of a parameter.

260. Higher Derivatives of Functions Represented Parametrically

Let y be a function of x given by the equations

$$x=\varphi(t), \quad y=f(t) \tag{1}$$

The derivatives of first and second orders are found from the formulas ²)

$$y'=\frac{f'(t)}{\varphi'(t)}\,, \tag{2}$$

$$y''=\frac{\varphi'(t)f''(t)-f'(t)\varphi''(t)}{[\varphi'(t)]^3} \tag{3}$$

The expressions of subsequent derivatives are involved; ³) when the functions $f(t)$ and $\varphi(t)$ are given, the computation is more simply carried out step by step, as in the following example.

Example. Let

$$x=a\cos t, \quad y=b\sin t$$

Then (cf. Sec. 252, Example 2)

$$y'=d(b\sin t):d(a\cos t)=-\frac{b}{a}\cot t$$

---

¹) We have $y'''=\frac{dy''}{dx}$; substitute expression (1). The result is most conveniently given in the form

$$y'''=\left[dx\begin{vmatrix} dx & dy \\ d^3x & d^3y \end{vmatrix}-3d^2x\begin{vmatrix} dx & dy \\ d^2x & d^2y \end{vmatrix}\right]:dx^5 \tag{4}$$

Subsequent expressions are still more complicated.

²) Formula (3) is derived like formula (1), Sec. 259, and may be obtained from the latter by replacing the differentials by the corresponding derivatives with respect to the parameter.

³) See footnote 1.

Further

$$y'' = d\left(-\frac{b}{a}\cot t\right):d\,(a\cos t) = -\frac{b}{a^2\sin^3 t},$$

$$y''' = d\left(-\frac{b}{a^2\sin^3 t}\right):d\,(a\cos t) = -\frac{3b\cos t}{a^3\sin^5 t}$$

and so forth.

261. Higher Derivatives of Implicit Functions

In order to find the successive derivatives of a function y (of an argument x) given implicitly by some equation, one has to differentiate this equation successively, i. e. equate the differentials (or derivatives) of the right and left sides. We obtain a series of equalities; from the first we find the expression of y' in terms of x and y, the second (taking into account the expression of y' that was found) yields the expression of y'' in terms of x and y, the third (taking into account the expressions of y', y'') yields y''', etc. Simplifications are possible in special cases.

Example. Find the derivatives, up to third order, of the function $y = f(x)$ given by the equation

$$x^2 + y^2 = 25 \tag{1}$$

and determine the values of these derivatives at the point (3, 4).

Solution. Equating the differentials, we obtain

$$x\,dx + y\,dy = 0 \tag{2}$$

whence

$$x + yy' = 0 \tag{2a}$$

Equating the differentials of both sides of (2a), we get

$$dx + y'\,dy + yy''\,dx = 0 \tag{3}$$

whence

$$1 + y'^2 + yy'' = 0 \tag{3a}$$

We differentiate once again

$$2y'\,dy' + y''\,dy + yy'''\,dx = 0 \tag{4}$$

whence

$$3y'y'' + yy''' = 0 \tag{4a}$$

From (2a) we get

$$y' = -\frac{x}{y} \tag{5}$$

From (3a) we get $y''=-\frac{1+y'^2}{y}$; taking into account (5), we have

$$y''=-\frac{x^2+y^2}{y^3} \qquad (6)$$

From (4a), taking into account (5) and (6), we get

$$y'''=\frac{-3x\,(x^2+y^2)}{y^5} \qquad (7)$$

Substituting $x=3$, $y=4$ into (5), (6) and (7), we get

$$y'=-\frac{3}{4},\quad y''=-\frac{25}{64},\ y'''=-\frac{225}{1024}$$

Note 1. Here the computation may be simplified. By virtue of the equation $x^2+y^2=25$, formula (6) takes the form $y''=-\frac{25}{y^3}$. From this $y'''=-\frac{d}{dx}\left(\frac{25}{y^3}\right)=\frac{75}{y^4}\,y'=-\frac{75x}{y^5}$.

Note 2. There is no need to derive (3a) from (3), (4a) from (4), etc. The derivatives may be taken at once. However, a preliminary calculation of the differentials is a guarantee against certain mistakes common to beginners (for the derivative of y'^2, they write $2y'$ in place of $2y'y''$ and so on; cf. Sec. 238, Note).

262. Leibniz Rule

In order to form the expression of the nth derivative of the product uv (with respect to any argument), expand $(u+v)^n$ by the binomial theorem (Newton's) and in the expansion obtained replace all powers by derivatives of the appropriate order, zero powers ($u^0=v^0=1$) that are assumed in the extreme terms of the expansion being replaced by the functions themselves.

By this rule we get

$$(uv)'=u'v+uv', \qquad (1)$$

$$(uv)''=u''v+2u'v'+uv'', \qquad (2)$$

$$(uv)'''=u'''v+3u''v'+3u'v''+uv''', \qquad (3)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(uv)^{(n)}=u^{(n)}v+nu^{(n-1)}v'+\frac{n\,(n-1)}{1\cdot 2}\,u^{(n-2)}v''+\ldots$$

$$\ldots+\frac{n\,(n-1)\ \ldots\ (n-k+1)}{k!}\,u^{(n-k)}v^{(k)}+\ldots+uv^{(n)} \qquad (4)$$

This rule, which was perceived by Leibniz, is proved by the method of complete mathematical induction.

Example 1. Find the tenth derivative of the function $e^x x^2$.

Solution. Using formula (4) (for $u=e^x$, $v=x^2$, $n=10$) we get

$$(e^x x^2)^{\mathrm{X}} = (e^x)^{\mathrm{X}} x^2 + 10\,(e^x)^{\mathrm{IX}}\,(x^2)' + 45\,(e^x)^{\mathrm{VIII}}\,(x^2)'' + \ldots$$

The subsequent terms need not be written out since the derivatives of x^2 of third and higher orders are equal to zero. Taking into account that all the derivatives of e^x are equal to e^x, we obtain

$$(e^x x^2)^{\mathrm{X}} = e^x\,(x^2+20x+90)$$

Example 2. Find the values of all derivatives of $f(x)=\arctan x$ for $x=0$.

Solution. We have

$$f'(x)=\frac{1}{1+x^2} \tag{5}$$

so that

$$f(0)=0, \quad f'(0)=1 \tag{6}$$

The direct computation of higher derivatives is an involved operation. But if we represent (5) in the form

$$f'(x)\,(1+x^2)=1$$

and apply the Leibniz rule ($u=f'(x)$, $v=1+x^2$), we get

$$f^{(n+1)}(x)\,(1+x^2)+nf^{(n)}(x)\,2x+n\,(n-1)\,f^{(n-1)}(x)=0$$

For $x=0$ we have

$$f^{(n+1)}(0)+n\,(n-1)\,f^{(n-1)}(0)=0 \tag{7}$$

Since $f^{(0)}(0)=f(0)=0$, the values of all the derivatives of even order are equal to zero:

$$f''(0)=f^{\mathrm{IV}}(0)=f^{\mathrm{VI}}(0)=\ldots=0 \tag{8}$$

Since $f'(0)=1$, from (7) we get, successively,

$$\begin{aligned}
f'''(0)&=-1\cdot 2f'(0)=-(2!),\\
f^{\mathrm{V}}(0)&=-3\cdot 4f'''(0)=+(4!),\\
f^{\mathrm{VII}}(0)&=-5\cdot 6f^{\mathrm{V}}(0)=-(6!),\\
&\ldots\ldots\ldots\ldots\ldots\\
f^{(2k+1)}(0)&=-(2k-1)\,2kf^{(2k-1)}(0)=(-1)^k\,(2k!)
\end{aligned}$$

From (3a) we get $y''=-\frac{1+y'^2}{y}$; taking into account (5), we have

$$y''=-\frac{x^2+y^2}{y^3} \tag{6}$$

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$$(uv)'''=u'''v+3u''v'+3u'v''+uv''', \tag{3}$$

. . . .

$$(uv)^{(n)}=u^{(n)}v+nu^{(n-1)}v'+\frac{n\,(n-1)}{1\cdot 2}\,u^{(n-2)}v''+\ldots$$

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and apply the Leibniz rule ($u=f'(x)$, $v=1+x^2$), we get

$$f^{(n+1)}(x)\,(1+x^2)+nf^{(n)}(x)\,2x+n\,(n-1)\,f^{(n-1)}(x)=0$$

For $x=0$ we have

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$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^{(2k+1)}(0)=-(2k-1)\,2kf^{(2k-1)}(0)=(-1)^k\,(2k!)$$

263. Rolle's Theorem [1]

Theorem. Let a function $f(x)$, differentiable in a closed interval (a, b), vanish at the end-points of the interval. Then the derivative $f'(x)$ will vanish at least once inside the interval.

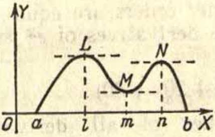

Fig. 251

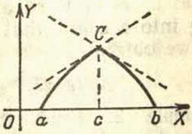

Fig. 252

In Fig 251, between the points $x=a$ and $x=b$, where the curve of the function $f(x)$ cuts the x-axis, there are three points L, M and N where the tangent is parallel to the x-axis [i. e. $f'(x)=0$].

In Fig. 252, between $x=a$ and $x=b$ there is not a single point with "horizontal" tangent. The reason is that at the point C the graph has no tangent, i. e. the function $f(x)$ is not differentiable at the point $x=c$ (there are two one-sided derivatives here, Sec. 231).

Note 1. If a differentiable function $f(x)$ has the same values at $x=a$ and $x=b$, even though not equal to zero, then the derivative $f'(x)$ still vanishes in the interior of the interval (a, b).

Note 2. Rolle's theorem also holds true even in the case when $f(x)$ is differentiable only at interior points of the interval (a, b); at the end-points, the function $f(x)$ may not be differentiable but only continuous.

Rolle's theorem is ordinarily stated for these most general conditions; this complicates the statement of the theorem and makes it difficult to grasp the *basic* content. Later on (Secs. 264, 266, 283) we will state the conditions of a number of theorems under less than the most general assumptions (which are given as notes).

---

[1] M. Rolle (1652–1719), a contemporary of Newton and Leibniz, considered differential calculus to be logically inconsistent and, naturally, could not have stated "Rolle's theorem". Rolle stated an algebraic theorem from which follows the consequence: if a and b are roots of the equation $x^n+p_1x^{n-1}+p_2x^{n-2}+\dots+p_{n-1}x+p_n=0$, then between a and b there is a root of the equation $nx^{n-1}+(n-1)p_1x^{n-2}+\dots+p_{n-1}=0$. This proposition is a special case of "Rolle's theorem" (the left side of the second equation is the derivative of the left side of the first equation). Hence the name (historically inaccurate) "Rolle's theorem".

264. Lagrange's Mean-Value Theorem[1]

Statement of the theorem. If a function $f(x)$ is differentiable in a closed interval (a, b), then the ratio $\frac{f(b)-f(a)}{b-a}$ is equal to the value of the derivative $f'(x)$ at some interior point $x=\xi$[2] of the interval (a, b):

$$\frac{f(b)-f(a)}{b-a}=f'(\xi) \qquad (1)$$

Geometrical interpretation. The ratio $\frac{f(b)-f(a)}{b-a}=\frac{KB}{AK}$ (Fig. 253) is the slope of the chord AB, and $f'(\xi)$ is the slope of the tangent NT. Lagrange's theorem asserts that there is at least one point N between A and B on the arc $\overset{\smile}{AB}$ where the tangent is parallel to the chord AB, provided that there is a tangent at *every point* of the arc $\overset{\smile}{AB}$.

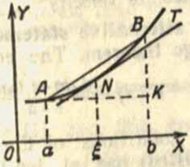

Fig. 253

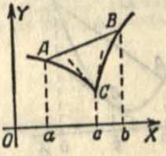

Fig. 254

From Fig. 254 it is clear that if this condition is not fulfilled the theorem may not hold true. There is no tangent at point C (there are only one-sided tangents: right and left). The function $f(x)$ depicted by the graph ACB is nondifferentiable at $x=c$, and the Lagrange theorem does not hold true: the derivative $f'(x)$ is not equal to the ratio $\frac{f(b)-f(a)}{b-a}$ for any intermediate value ξ.

---

[1] Lagrange, Joseph Louis (1736-1813), great French scientist, founder of analytical mechanics, one of the creators of the calculus of variations.

[2] The Greek letter ξ (xi) is the standard notation for the "mean value" of an argument (i. e. the value contained within a given interval).

Mechanical interpretation. Let $f(t)$ be the distance of a point at time t from an initial position. Then $f(b)-f(a)$ is the distance covered from time $t=a$ to time $t=b$, the ratio $\frac{f(b)-f(a)}{b-a}$ is the average velocity during this interval of time. The Lagrange theorem asserts that at some intermediate time the velocity of the point is equal to the average (mean) velocity of motion provided that at each instant the point has a definite velocity.

The theorem may not hold if this condition is not fulfilled. For example, if a point moves the first hour at 20 metres an hour, and the second at 30 m/hr, then the mean velocity of motion is 25 m/hr; the point did not have that velocity once during the two hours. The theorem was violated because at the end of the first hour the point did not have a definite velocity.[1]

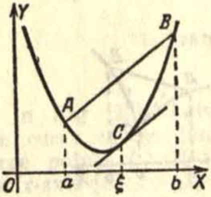

Fig. 255

An alternative statement of the Lagrange theorem. The equation

$$f'(x)=\frac{f(b)-f(a)}{b-a}$$

(if the conditions of the theorem are fulfilled) has at least one root $x=\xi$ within the interior of the interval (a, b).

The position of this root (or roots) depends on the type of function $f(x)$. If it is a quadratic function (and the graph is a parabola; Fig. 255), we obtain a first-degree equation; its root lies precisely at the midpoint of (a, b), or

$$\xi=\frac{b+a}{2}$$

For other functions, this property is only approximately fulfilled; namely, if a has a constant value and b tends to a, then one of the roots, as a rule,[2] tends to the midpoint of the interval (a, b), i. e. $\lim \frac{\xi-a}{b-a}=\frac{1}{2}$ as $b \to a$.

Example 1. Let $f(x)=x^2$. Then $f'(\xi)=2\xi$. Formula (1) takes the form

$$\frac{b^2-a^2}{b-a}=2\xi$$

---

[1] Actually, the transition from 20 m/hr to 30 m/hr ordinarily takes place gradually, not instantaneously, and then there is an instant when the velocity is equal to 25 m/hr.

[2] The only exceptions are cases when the second derivative $f''(a)$ is zero or does not exist.

whence

$$\xi=\frac{a+b}{2}$$

i. e. ξ lies exactly at the midpoint of the interval (a, b).

Example 2. Let $f(x)=x^3$, then $f'(x)=3x^2$. Take $a=10$, $b=12$. We then have

$$\frac{f(b)-f(a)}{b-a}=364$$

According to Lagrange's theorem, the equation $3x^2=364$ should have a root between 10 and 12. Indeed, its positive root $x=\sqrt{121\frac{1}{3}}\approx 11.015$ lies in the interval (10, 12) and, what is more, very close to the midpoint.

Note. The Lagrange theorem also holds true when the function $f(x)$ is differentiable only at interior points of the interval (a, b) (being nondifferentiable and only continuous at the end-points).

265. Formula of Finite Increments

Formula (1), Sec. 264, may be rewritten as

$$f(b)-f(a)=f'(\xi)(b-a) \tag{1}$$

or, in other notation,

$$f(a+h)-f(a)=f'(\xi)h \tag{2}$$

This is the *formula of finite increments*, which is also written as

$$f(a+h)=f(a)+f'(\xi)h \tag{3}$$

Application to approximate calculations. In Sec. 248 we employed the approximate formula

$$f(a+h)\approx f(a)+f'(a)h \tag{4}$$

to compute $f(a+h)$. The *exact* formula (3) enables us (though the value of ξ is unknown) to estimate the error in formula (4). Now if we put $\xi=\frac{a+b}{2}$ in formula (3), then as a rule (cf. Sec. 264) it yields a much better approximation than (4), though it ceases to be exact.

Example. Find log 101 without using tables.

Assuming $f(x)=\log x$, we have $f'(x)=\frac{M}{x}$ $(M=0.43429)$. For $a=100$ and $h=1$, formula (4) yields

$$\log 101\approx\log 100+M\cdot\frac{1}{100}\cdot 1=2.0043429 \tag{5}$$

To estimate the error, use the exact formula (3). This yields

$$\log 101 = \log 100 + M \cdot \frac{1}{\xi} \cdot 1 \tag{6}$$

Here, ξ lies between 100 and 101, so that $\frac{1}{\xi} > \frac{1}{101}$. The error of formula (5) is $M\left|\frac{1}{100} - \frac{1}{\xi}\right|$ and this is definitely less than $M\left(\frac{1}{100} - \frac{1}{101}\right)$; that is to say, it is less than 0.00004. Such is the limiting error of formula (5) (the true error is half that).

But if in formula (6) we put $\xi = \frac{1}{2}(100 + 101) = 100.5$, then we get

$$\log 101 \approx \log 100 + M \cdot 0.00995025 \cdot 1 = 2.0043213 \tag{7}$$

Here only the last digit is incorrect; its true value is greater by unity.

Corollaries to formula (1). From the definition of a derivative it follows directly that the derivative of a constant is zero. A consequence of formula (1) is the following inverse theorem.

Theorem 1. If in an interval (m, n) the derivative $f'(x)$ is everywhere equal to zero, then in this interval the function $f(x)$ is a constant [i. e. for any values (a, b) in this interval[1] the values of the function $f(x)$ are the same].

Explanation. By hypothesis, the function $f(x)$ is differentiable in the interval (m, n) and all the more so in the interval (a, b). Hence, we can apply to it (Sec. 264) formula (1). In (1) we have to put $f'(\xi) = 0$ (by hypothesis). This yields $f(b) = f(a)$.

From Theorem 1 there follows directly

Theorem 2. If the derivatives of two functions $f(x)$ and $\varphi(x)$ are everywhere equal in an interval (m, n), then *in this interval* the values of both functions differ by a constant quantity.

---

[1] By hypothesis, the function $f(x)$ is defined *throughout* the interval (m, n), otherwise it would not have a derivative everywhere. If, contrary to hypothesis, $f(x)$ is defined at all points of (m, n) except, say, two points $x = k$ and $x = l$ $(k < l)$, then it may turn out that the function is constant only in each of the (open) intervals (m, k), (k, l) and (l, n) *separately*, but changes its value when passing from one to another (see Examples 1 and 2, Sec. 247a).

266. Generalized Mean-Value Theorem (Cauchy)

Cauchy's theorem.1) Let the derivatives $f'(t)$ and $\varphi'(t)$ of two functions $f(t)$ and $\varphi(t)$, differentiable in a closed interval (a, b), not be simultaneously zero anywhere in the interior of the interval. Also let one of the functions $f(t)$, $\varphi(t)$ have distinct values at the end-points of the interval [say $\varphi(a) \neq \varphi(b)$]. Then the increments $f(b)-f(a)$ and $\varphi(b)-\varphi(a)$ of the given functions are to each other as their derivatives at some interior point $t=\tau$ (Greek letter tau) of the interval (a, b):

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f'(\tau)}{\varphi'(\tau)} \tag{1}$$

Lagrange's formula [formula (1), Sec. 264] is a special case of formula (1) when $\varphi(t)=t$.

Geometrical interpretation. Same as for Lagrange's theorem, only the curve ACB (Fig. 256) is given by the parametric equations

$$x=\varphi(t), \quad y=f(t)$$

We have

$$OA'=\varphi(a), \quad OB'=\varphi(b);$$
$$AA'=f(a), \quad BB'=f(b)$$

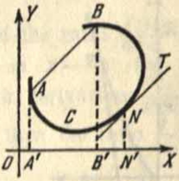

Fig. 256

The ratio $\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}$ is the slope of the chord AB, the ratio $\frac{f'(t)}{\varphi'(t)}=\frac{dy}{dx}$ is the slope of the tangent NT.

In Fig. 256, the tangent NT is parallel to the chord AB, the point N lies on the arc AB (but its projection N' on the x-axis does not lie on the segment $A'B'$; the same goes for the projection on the y-axis).

Note 1. If, contrary to hypothesis, we had $f(a)=f(b)$ and $\varphi(a)=\varphi(b)$, then the left side of (1) would be indeterminate.

Note 2. The Cauchy theorem requires that $f'(t)$ and $\varphi'(t)$ should not be zero simultaneously *in the interior* of the interval (a, b), but at one of the end-points (or at both) they can simultaneously be zero (or even not exist, so long as $f(x)$ and $\varphi(x)$ are continuous at both end-points).

Example 1. Consider the functions

$$f(t)=t^3 \quad \text{and} \quad \varphi(t)=t^2$$

in the interval (0, 2). At the end-point $t=0$, the derivatives

$$f'(t)=3t^2 \quad \text{and} \quad \varphi'(t)=2t$$

vanish, but both are nonzero in the interior of the interval. Each of the functions $f(t)$, $\varphi(t)$ has distinct values at the end-points $t=0$ and $t=2$. The conditions of the Cauchy theorem are fulfilled. Hence the

---

1) Cauchy, Augustin-Louis (1789-1857), celebrated French mathematician and physicist. Cauchy posed the problem of constructing mathematical analysis on a rigorous logical basis. In the main, he solved this problem.

ratio

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f(2)-f(0)}{\varphi(2)-\varphi(0)}=\frac{2^3}{2^2}=2$$

must be equal to the ratio

$$\frac{f'(t)}{\varphi'(t)}=\frac{3t^2}{2t}=\frac{3}{2}t$$

at some point $t=\xi$ lying between $a=0$ and $b=2$. Indeed, the equation

$$\frac{3}{2}t=2$$

has a root $t=\frac{4}{3}$, which lies in the interior of the interval $(0, 2)$.

Example 2. Consider the same functions $f(t)=t^3$ and $\varphi(t)=t^2$ in the interval $\left(-1\frac{1}{2}, 2\right)$. For $a=-1\frac{1}{2}$, $b=2$ we have

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{b^3-a^3}{b^2-a^2}=\frac{b^2+ab+a^2}{b+a}=\frac{13}{2}$$

The equation

$$\frac{3}{2}t=\frac{13}{2}$$

has a unique root $t=4\frac{1}{3}$, but it is exterior to the interval $\left(-1\frac{1}{2}, 2\right)$. The Cauchy theorem did not hold because the point $t=0$, where both derivatives $f'(t)$, $\varphi'(t)$ are equal to zero, now lies inside the interval (a, b). Geometrically, the picture is as follows: the parametric equations $x=t^2$, $y=t^3$ describe a semicubical parabola AOB (Fig. 257); to the values $a=-1\frac{1}{2}$, $b=2$ there correspond points $A\left(2\frac{1}{4}, -3\frac{3}{8}\right)$ and $B(4, 8)$. On the arc AOB of the curve $x=t^2$, $y=t^3$ (semicubical parabola) there are no points where the tangent could be parallel to the chord AB (such a point exists outside the arc AB above point B).

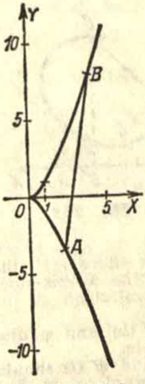

Fig. 257

Mechanical interpretation. Let t be the time and

$$s_P=f(t)$$

and

$$s_Q=\varphi(t)$$

be the distances of two rectilinearly moving bodies P and Q from their initial positions. Then $f'(t)$ and $\varphi'(t)$ are the velocities v_P and v_Q of the bodies P and Q. By the hypothesis of the Cauchy theorem, v_P and v_Q are not zero simultaneously. The theorem states that the distances covered by the bodies in the time interval (a, b) are to one another as the velocities at some intermediate instant[1] (the same for both bodies).

267. Evaluating the Indeterminate Form $\frac{0}{0}$

If some function is not defined at a point $x=a$, but has a limit as $x \longrightarrow a$, then finding this limit is called *evaluating the indeterminate form*. In particular, evaluating the indeterminate form $\frac{0}{0}$ is the name for finding the limit of the ratio $\frac{f(x)}{\varphi(x)}$ when the functions $f(x)$, $\varphi(x)$ are infinitesimal as $x \longrightarrow a$.

L'Hospital's rule.[2] To find the limit of the ratio $\frac{f(x)}{\varphi(x)}$ of two functions which are infinitesimal as $x \longrightarrow a$ (or as $x \longrightarrow \infty$), we can consider the ratio of their derivatives $\frac{f'(x)}{\varphi'(x)}$. If it tends to a limit (finite or infinite), then the ratio $\frac{f(x)}{\varphi(x)}$ also tends to that limit.[3]

---

[1] Let us explain this pictorially. Suppose during the time interval (a, b) body P covers twice the distance that Q does ($s_P=2s_Q$). If both motions are uniform, then at *any* intermediate time we have $v_P=2v_Q$. Now let one of the motions (or both of them) be nonuniform. It cannot be that, always, $v_P > 2v_Q$ (for then the distance covered by P would exceed that covered by Q by more than a factor of two). Likewise, it is impossible that, always $v_P < 2v_Q$. Therefore, if at first v_P exceeds $2v_Q$, then later v_P is less than $2v_Q$ (and vice versa). Hence, at some intermediate time we must have that $v_P=2v_Q$. At that time we have $v_P:v_Q=s_P:s_Q$ because by hypothesis the case when $v_P=v_Q=0$ is excluded (for then the ratio $v_P:v_Q$ would be indeterminate).

[2] L'Hospital (1661-1704), author of the first printed manual on differential calculus (1696) where the rule is formulated (but less rigorously than as given here). In compiling this manual, l'Hospital made use of the manuscript of his teacher John Bernoulli. This rule is mentioned in the manuscript and so the name "l'Hospital's rule" is historically inaccurate.

[3] In the statement of the rule, the requirement is ordinarily included that the derivative $\varphi'(x)$ be nonzero in some neighbourhood of the point $x=a$. This requirement is superfluous since the rule itself states that the ratio $\frac{f'(x)}{\varphi'(x)}$ has a limit as $x \longrightarrow a$, and by virtue of the definition of limit (Sec. 205) this is only possible when $\varphi'(x) \neq 0$ near $x=a$.

Example 1. Evaluate $\lim\limits_{x \to 1} \frac{x^2-1}{x^3-1}$.

The functions $f(x) = x^2 - 1$ and $\varphi(x) = x^3 - 1$ are infinitesimal as $x \longrightarrow 1$. Consider the ratio $\frac{f'(x)}{\varphi'(x)} = \frac{2x}{3x^2}$. It approaches the limit $\frac{2}{3}$ as $x \longrightarrow 1$. According to the l'Hospital rule, $\frac{x^2-1}{x^3-1}$ tends to the same limit. Indeed,

$$\lim_{x \to 1} \frac{x^2-1}{x^3-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \lim_{x \to 1} \frac{x+1}{x^2+x+1} = \frac{2}{3}$$

If not only the functions $f(x)$, $\varphi(x)$, but also their derivatives $f'(x)$, $\varphi'(x)$ are infinitesimal as $x \longrightarrow a$, then one can again apply the l'Hospital rule in order to find the limit of $\frac{f'(x)}{\varphi'(x)}$.

Example 2. Evaluate $\lim\limits_{x \to 1} \frac{x^3-3x+2}{x^3-x^2-x+1}$.

The numerator and denominator are infinitesimal. By the l'Hospital rule

$$\lim_{x \to 1} \frac{x^3-3x+2}{x^3-x^2-x+1} = \lim_{x \to 1} \frac{3x^2-3}{3x^2-2x-1}$$

Here the numerator and denominator are again infinitesimals. Apply the l'Hospital rule once again:

$$\lim_{x \to 1} \frac{3x^2-3}{3x^2-2x-1} = \lim_{x \to 1} \frac{6x}{6x-2} = \frac{3}{2}$$

Example 3. Find $\lim\limits_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

Using the l'Hospital rule successively, we twice get a ratio of infinitesimals:

$$\frac{f'(x)}{\varphi'(x)} = \frac{e^x + e^{-x} - 2}{1 - \cos x}, \quad \frac{f''(x)}{\varphi''(x)} = \frac{e^x - e^{-x}}{\sin x}$$

The third time we obtain the ratio

$$\frac{f'''(x)}{\varphi'''(x)} = \frac{e^x + e^{-x}}{\cos x}$$

It has the limit 2 as $x \to 0$. Hence,

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = 2$$

Note 1. Theoretically, the possibility is not precluded that all derivatives of both functions $f(x)$, $\varphi(x)$ will be infinitesimals. Such cases do not occur in real-world problems.

It is useful to combine the application of l'Hospital's rule with transformations that facilitate finding the limit.

Example 4. Find $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$.

Following the l'Hospital rule, we seek the limit of the ratio

$$\frac{f'(x)}{\varphi'(x)} = \frac{\frac{1}{\cos^2 x} - \cos x}{3 \sin^2 x \cos x} \text{ as } x \to 0$$

Here, $f'(x)$ and $\varphi'(x)$ are infinitesimals, but it is not advisable to seek $\lim\limits_{x \to 0} \frac{f''(x)}{\varphi''(x)}$. It is better to transform $\frac{f'(x)}{\varphi'(x)}$ to the form $\frac{1 - \cos^3 x}{3 \sin^2 x \cdot \cos^3 x}$ and, noting that $\lim\limits_{x \to 0} (\cos^3 x) = 1$, to seek $\lim\limits_{x \to 0} \frac{1 - \cos^3 x}{3 \sin^2 x}$. By l'Hospital's rule, this limit is equal to

$$\lim_{x \to 0} \frac{3 \cos^2 x \cdot \sin x}{6 \sin x \cdot \cos x} = \lim_{x \to 0} \frac{1}{2} \cos x = \frac{1}{2}$$

From the very start we can replace $\sin^3 x$ with the equivalent infinitesimal x^3. Then

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x} = \lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos^3 x}{3x^2 \cdot \cos^3 x} =$$

$$= \lim_{x \to 0} \frac{1 - \cos^3 x}{3x^2}$$

Using the l'Hospital rule again, we get

$$\lim_{x \to 0} \frac{3 \cos^2 x \cdot \sin x}{6x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2}$$

Note 2. It may happen that the ratio $\frac{f'(x)}{\varphi'(x)}$ does not tend to any limit as $x \to a$ (or as $x \to \infty$). In such cases, the ratio $\frac{f(x)}{\varphi(x)}$ may likewise have no limit, but on the other hand it may have one. Thus, if

$f(x)=x+\sin x$ and $\varphi(x)=x$, then the ratio $\frac{f'(x)}{\varphi'(x)}=1+\cos x$ has no limit as $x\to\infty$. However, the ratio

$$\frac{f(x)}{\varphi(x)}=\frac{x+\sin x}{x}=1+\frac{\sin x}{x}$$

approaches unity as $x\to\infty$.

268. Evaluating the Indeterminate Form $\frac{\infty}{\infty}$

L'Hospital's rule (Sec. 267) also holds true for the ratio $\frac{f(x)}{\varphi(x)}$ of two functions which are infinitely great as $x\to a$ (or as $x\to\infty$).

Example 1. Evaluate $\lim\limits_{x\to\infty}\frac{\ln x}{x^2}$.

The functions $f(x)=\ln x$ and $\varphi(x)=x^2$ become infinite as $x\to\infty$. The ratio $\frac{f'(x)}{\varphi'(x)}=\frac{\frac{1}{x}}{2x}$ tends to the limit 0 as $x\to\infty$. $\frac{\ln x}{x^2}$ tends to the same limit.

Note. If $f(x)$ and $\varphi(x)$ have infinite limits as $x\to a$, then the limits of $f'(x)$ and $\varphi'(x)$ (if they exist) are also infinite, and l'Hospital's rule is useful only when the expression $\frac{f'(x)}{\varphi'(x)}$ can be reduced to a more convenient form in simpler fashion than the expression $\frac{f(x)}{\varphi(x)}$.

Example 2. Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\frac{\tan 3x}{\tan x}$.

The functions $\tan 3x$ and $\tan x$ and also their derivatives $\frac{3}{\cos^2 3x}$ and $\frac{1}{\cos^2 x}$ become infinite as $x\to\frac{\pi}{2}$. Representing the ratio of the derivatives in the form $3\left(\frac{\cos x}{\cos 3x}\right)^2$, we seek $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{\cos 3x}$ (now the numerator and denominator are infinitely small). Applying the rule of Sec. 267, we get $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{\cos 3x}=\lim\limits_{x\to\frac{\pi}{2}}\frac{-\sin x}{-3\sin 3x}=-\frac{1}{3}$. Consequently,

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan 3x}{\tan x} = 3 \cdot \left(-\frac{1}{3}\right)^2 = \frac{1}{3}$$

But the original expression is more readily transformed to a convenient form. Namely, $\frac{\tan 3x}{\tan x} = \frac{\cot x}{\cot 3x}$ so that

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan 3x}{\tan x} = \lim_{x \to \frac{\pi}{2}} \frac{\cot x}{\cot 3x} = \lim_{x \to \frac{\pi}{2}} \frac{\sin^2 3x}{3 \sin^2 x} = \frac{1}{3}$$

269. Other Indeterminate Expressions

I. *Indeterminate form* $0 \cdot \infty$, i.e. the product $f(x)\,\varphi(x)$ where $f(x) \to 0$ and $\varphi(x) \to \infty$. This expression may be reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

$$f(x)\,\varphi(x) = f(x) : \frac{1}{\varphi(x)} = \varphi(x) : \frac{1}{f(x)}$$

and then the l'Hospital rule can be employed.

Example 1. Find $\lim\limits_{x \to 0} x \cot \frac{x}{2}$.

We transform $x \cot \frac{x}{2} = x : \tan \frac{x}{2}$ and find

$$\lim_{x \to 0} x \cot \frac{x}{2} = \lim_{x \to 0} \left[1 : \frac{1}{2 \cos^2 \frac{x}{2}} \right] = 2$$

Example 2. Find $\lim\limits_{x \to 0} x^4 \ln x$.

We have

$$\lim_{x \to 0} x^4 \ln x = \lim_{x \to 0} \left[\ln x : \frac{1}{x^4} \right] = \lim_{x \to 0} \left[\frac{1}{x} : \frac{-4}{x^5} \right] = 0$$

II. *Indeterminate form* $\infty - \infty$, i.e. the difference of two functions each of which has the limit $+\infty$ (or the limit $-\infty$). This expression is also reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 3. Evaluate $\lim\limits_{x \to 0} \left[\frac{1}{x} - \frac{2}{x(e^x + 1)} \right]$.

Reduce the fractions to a common denominator; the desired quantity is $\lim\limits_{x \to 0} \frac{e^x - 1}{x(e^x + 1)}$, i.e. we have the indeterminate form $\frac{0}{0}$. Since $\lim\limits_{x\to 0}(e^x+1)=2$, it follows that

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{2}{x(e^x+1)}\right]=\frac{1}{2}\lim_{x\to 0}\frac{e^x-1}{x}=\frac{1}{2}\lim_{x\to 0}\frac{e^x}{1}=\frac{1}{2}$$

III. *Indeterminate forms* 0^0, ∞^0, 1^∞, i.e. functions of the form $f(x)^{\varphi(x)}$, where $\lim f(x)=0$ and $\lim\varphi(x)=0$ or $\lim f(x)=\infty$, $\lim\varphi(x)=0$ or $\lim f(x)=1$, $\lim\varphi(x)=\infty$.

Here we first seek the limit of the logarithm of the given function. In all three cases, we obtain an indeterminate form like $0\cdot\infty$.

Example 4. Evaluate $\lim\limits_{x\to 0} x^x$ (indeterminate form 0^0).

Assuming $y=x^x$, we have $\ln y=x\ln x$. Further,

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\left(\ln x:\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{1}{x}:-\frac{1}{x^2}\right)=0$$

Whence

$$\lim_{x\to 0} y=1$$

Example 5. Evaluate $\lim\limits_{x\to\infty}(1+2x)^{\frac{1}{x}}$ (indeterminate form ∞^0).

Assuming $y=(1+2x)^{\frac{1}{x}}$, we have $\ln y=\frac{1}{x}\ln(1+2x)$.

Further,

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(1+2x)}{x}=\lim_{x\to\infty}\frac{2}{1+2x}=0$$

Hence, $\lim\limits_{x\to\infty} y=1$.

Example 6. Evaluate $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$ (indeterminate form 1^∞).

We have

$$\lim_{x\to\frac{\pi}{4}}\ln y=\lim_{x\to\frac{\pi}{4}}\tan 2x\ln\tan x=\lim_{x\to\frac{\pi}{2}}\frac{\ln\tan x}{\cot 2x}=$$

$$=\lim_{x\to\frac{\pi}{4}}\left(\frac{1}{\sin x\cos x}:-\frac{2}{\sin^2 2x}\right)=-1$$

Hence

$$\lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x} = e^{-1}$$

270. Taylor's Formula (Historical Background) [1]

1. **Newton and infinite series.** In order to find the derivative of a given function and, mainly, to solve the inverse problem, Newton replaced the given function by an *infinite power series*, i.e. the expression

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots \qquad (1)$$

with the number of terms increasing without bound. The coefficients a_0, a_1, a_2, ... were taken so that expression (1) yielded more exact values of the function as the number of terms was increased. Thus, Newton replaced the function $\frac{1}{1+x}$ by the expression $1-x+x^2-x^3+\ldots+(-1)^nx^n+\ldots$ and wrote:[2]

$$\frac{1}{1+x} = 1-x+x^2-x^3+\ldots \qquad (2)$$

If $|x| < 1$, then the terms 1, $-x$, x^2, ... form an infinite decreasing geometric progression, and the sum is equal to $\frac{1}{1+x}$. But if $|x| \geqslant 1$, then the sum $1-x+x^2-x^3+\ldots$ $\ldots+(-1)^nx^n$, as $n \longrightarrow \infty$, does not tend to $\frac{1}{1+x}$. Taking into account this circumstance, Newton always confined himself to sufficiently small values of x.

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[1] The present section serves as an introduction to Secs. 271 and 272; the latter may be read independently.

[2] The expansion (2) is obtained if to the quotient $1:(1+x)$ we apply the rule for dividing polynomials arranged in increasing powers. Prior to Newton, formula (2) was used by N. Mercator (in 1665) in the computation of logarithms $\left[\text{the derivative of } \ln(1+x) \text{ is equal to } \frac{1}{1+x}\right]$. Mercator confined the infinite series expansion to this single case. In the hands of Newton it became a general method.

In expanding functions in infinite series, Newton made use of a variety of devices. Thus, Newton took the formula

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \dots \quad (3)$$

which had earlier been established by Pascal[1] for positive integral m, and applied it to fractional and negative values of m. Then the number of terms increases without bound. For $m = -1$ we get formula (2), for $m = -2$ we have[2]

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots \quad (4)$$

In order to find the derivative of $\frac{1}{1+x}$, Newton differentiated the expression (2) termwise.[3] A comparison with (4) shows that

$$\left[\frac{1}{1+x}\right]' = -\frac{1}{(1+x)^2} \quad (5)$$

2. **Taylor's series.** In 1715 Taylor,[4] using a complicated and extremely nonrigorous method, found a general form of expression (1) for the given function $f(x)$. In present-day notation, the result is of the form

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \quad (6)$$

Thus, if $f(x) = \frac{1}{1+x}$, then $f^{(n)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$. Hence, $f(0) = 1$ and $\frac{f^{(n)}(0)}{n!} = (-1)^n$ so that formula (6) yields the expansion (2). If $f(x) = \frac{1}{(1+x)^2}$, we get the expansion (4).

---

[1] Blaise Pascal (1623-1662), celebrated French philosopher, mathematician and physicist.

[2] Realizing that his derivations were not rigorous, Newton verified them by means of examples. Thus, performing termwise multiplication of $(1-x+x^2-x^3+\dots)\times(1-x+x^2-x^3+\dots)$, he found $1-2x+3x^2-4x^3+\dots$ and in this way checked formula (4).

[3] Newton did not know that the theorem on the derivative of a sum might prove invalid for a boundlessly increasing number of terms. Incidentally, for a series like (1) this theorem (given sufficiently small values of x) holds true, so there were no mistakes.

[4] Brook Taylor (1685-1731), English mathematician, pupil of Newton.

3. **Maclaurin's derivation.** Thirty years later, Maclaurin[1] gave the following simple derivation of Taylor's formula. He considered the equality

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots \tag{7}$$

and, desiring to determine the coefficients $a_0, a_1, a_2, \dots$, he found by successive differentiation

$$\left.\begin{aligned} f'(x)&=a_1+2a_2x+3a_3x^2+4a_4x^3+\dots,\\ f''(x)&=\qquad 2a_2+2\cdot 3a_3x+3\cdot 4a_4x^2+\dots,\\ f'''(x)&=\qquad\qquad 2\cdot 3a_3+2\cdot 3\cdot 4a_4x+\dots,\\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{8}$$

Putting $x=0$ into (7) and (8), he obtained successively[2]

$$a_0=f(0),\ a_1=f'(0),\ a_2=\frac{f''(0)}{1\cdot 2},\ a_3=\frac{f'''(0)}{1\cdot 2\cdot 3},\ \text{etc.} \tag{9}$$

4. **Taylor's series in the general form.** The following formula is derived in the same manner:

$$f(x)=f(a)+\frac{f'(a)}{1}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots \tag{10}$$

It gives the expansion of the function in powers of $(x-a)$. This formula was also known to Taylor; actually, it adds nothing new to (6).

Thus, for the function $f(x)=\ln x$, for $a=1$, formula (10) yields

$$\ln x=\frac{x-1}{1}-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\dots \tag{11}$$

But if we take the function $f(x)=\ln(1+x)$, then by formula (6) we find

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots \tag{12}$$

Putting $1+x=z$, we obtain the formula

$$\ln z=\frac{z-1}{1}-\frac{(z-1)^2}{2}+\frac{(z-1)^3}{3}-\frac{(z-1)^4}{4}+\dots \tag{13}$$

which differs from (11) solely in notation.

---

[1] Maclaurin, Colin (1698-1746), English mathematician; the power series (6) is now (without sufficient grounds) known as Maclaurin's series.

[2] If one proves the validity of the termwise differentiation of series (7), then Maclaurin's derivation flawlessly proves the following theorem: if $f(x)$ is expanded in the series (7), then the coefficients $a_0, a_1, a_2, \dots$ are expressed by formulas (9). However, there are functions which cannot be expanded in the series (7) [although their derivatives $f'(0), f''(0), \dots$ exist]. An instance of such a function is given in the last footnote of this section.

5. **Remainder of Taylor's series.** The functions which were known in the 18th century permit expansion in the Taylor series (10) (for any values of a, except those for which the function or one of its derivatives becomes infinite). Proceeding from their restricted experience, the mathematicians of the 18th century did not doubt that any continuous function could be expanded in a Taylor series. However, the need was felt for a precise estimate of the error which formula (10) yields if it is terminated at the term $\frac{f^{(n)}(a)}{n!}(x-a)^n$.

In 1799, Lagrange derived for the "remainder of the Taylor series", i.e. for the difference R_n,

$$R_n=f(x)-\left[f(a)+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n\right] \tag{14}$$

the following expression:

$$R_n=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{(n+1)} \tag{15}$$

Here, ξ is some number between a and x.

The proof of Lagrange presumed the expansibility of the function $f(x)$ in a Taylor series.1) A quarter of a century later Cauchy proved formula (15) without that assumption; he also gave an alternative expression for the remainder. It became possible, from the expression of the remainder, to judge the expansibility of the function in a Taylor series: if $\lim\limits_{n\to\infty} R_n=0$, then the function $f(x)$ can be expanded in a Taylor series, otherwise it cannot. Cauchy gave the first example of a function2) which, though it possesses all derivatives at a point $x=a$, cannot be expanded in the series (10) in powers of $x-a$ (these functions are of no practical value).

---

1) Lagrange even proved that such an expansion is possible for any continuous function, but the proof was unsatisfactory.

2) This function is given by the formula $f(x)=e^{-\frac{1}{x^2}}$ under the additional condition $f(0)=0$ (for $x=0$ the formula becomes meaningless). The function $f(x)$ has derivatives of any order at $x=0$. They are all zero at this point so the right side of (10) is identically zero However, $f(x)$ does not vanish anywhere except at $x=0$.

271. Taylor's Formula[1]

Theorem. If a function $f(x)$ has derivatives up to the $(n+1)$th order inclusive[2] in a closed interval (a, b) then

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots$$
$$\ldots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1} \qquad (1)$$

where ξ is some number between a and b.

Formula (1) is called Taylor's formula.

The last term $\frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{(n+1)}$ is called *Lagrange's form of the remainder*[3] and yields a precise expression for the difference R_n between $f(b)$ and expression

$$f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(b-a)^n$$

("Taylor's polynomial"):

$$R_n = f(b) - \left[f(a) + \ldots + \frac{f^{(n)}(a)}{n!}(b-a)^n\right] =$$
$$= \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1} \qquad (2)$$

Taylor's formula establishes that Eq. (1), in which ξ is taken as the unknown, has at least one solution[4] between a and b (cf. Sec. 264).

When a is regarded as a constant and b as a variable, then x is written in place of b:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n +$$
$$+ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \qquad (3)$$

For $a=0$ we obtain the so-called[5] "Maclaurin formula"

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1} \qquad (4)$$

---

[1] It is advisable to read Sec. 270 first.

[2] The $(n+1)$th derivative may not exist at the end-points of the interval; the main thing is that the nth derivative be continuous not only at interior points but at the end-points of the interval as well.

[3] Unlike other forms of the remainder.

[4] For fixed values of a and b, the quantity ξ varies, as a rule, with n.

[5] Cf. Sec. 270, Item 3.

Example. Apply formula (4), for $n=2$, to the function $f(x)=\frac{1}{1+x}$. We have

$$f'(x)=\frac{-1}{(1+x)^2},\quad f''(x)=\frac{2}{(1+x)^3},\quad f'''(x)=\frac{-6}{(1+x)^4}$$

Hence

$$f(0)=1,\quad \frac{f'(0)}{1!}=-1,\quad \frac{f''(0)}{2!}=+1,\quad \frac{f'''(\xi)}{3!}=-\frac{1}{(1+\xi)^4}$$

Formula (4) takes the form

$$\frac{1}{1+x}=1-x+x^2-\frac{x^3}{(1+\xi)^4} \tag{5}$$

Here, ξ lies between zero and x. It is important to note that the formula holds true only when $x>-1$. In this case, the condition of the theorem is fulfilled: the function $\frac{1}{1+x}$ has all derivatives in the closed interval $(0, x)$.

Solving (5) for ξ, we find

$$\xi_1=\sqrt[4]{1+x}-1,\quad \xi_2=-\sqrt[4]{1+x}-1 \tag{6}$$

It is easy to verify that for $x>-1$ the first root ξ_1 indeed lies between zero and x.

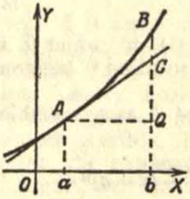


Fig. 258

Now if $x\leqslant-1$, then the condition of the theorem is not fulfilled, because the function $\frac{1}{1+x}$ does not have derivatives at the point -1, and this point either lies inside the interval $(0, x)$ (if $x<-1$) or coincides with its endpoint (if $x=-1$).

Formula (5) becomes incorrect: for $x=-1$, the left side is meaningless, for $x<-1$, Eq. (5) has imaginary roots.

Note. For $n=0$, Taylor's formula (2) [in which we have to write $f(a)$ in place of $f^{(0)}(a)$] yields the formula of finite increments (Sec. 265)

$$f(b)-f(a)=f'(\xi)(b-a) \tag{7}$$

For $n=1$ we get

$$f(b)-f(a)-f'(a)(b-a)=\frac{f''(\xi)}{2!}(b-a)^2 \tag{8}$$

or, in other notations,

$$[f(x+\Delta x)-f(x)]-f'(x)\Delta x=\frac{f''(\xi)}{2!}\Delta x^2 \tag{8a}$$

This formula yields an expression for the difference between the increment in the function and its differential (segment CB in Fig. 258).

If the second derivative $f''(x)$ is continuous for the value of x under consideration, the difference between the increment in the function and its differential is of second order with respect to Δx [when $f''(x) \neq 0$] or of higher order [when $f''(x)=0$]. Cf. Sec. 230.

272. Taylor's Formula for Computing the Values of a Function

Taylor's formula frequently permits computing the values of a function to any degree of accuracy.

Let the following values

$$f(a),\quad f'(a),\quad f''(a),\quad f'''(a),\ \ldots$$

of the function $f(x)$ and its successive derivatives at the "initial" point $x=a$ be known. It is required to find the value of the function $f(x)$ for a different value of x.

In many cases it is sufficient for this purpose to compute the value of Taylor's polynomial

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n \quad (1)$$

taking two, three, or more terms, depending on the required degree of accuracy. Of course, in doing so, we allow for a certain error R_n, which is equal to

$$R_n=f(x)-\left[f(a)+\frac{f'(a)}{1!}(x-a)+\right.$$
$$\left.+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n\right] \quad (2)$$

But it frequently happens that the error R_n diminishes without bound (in absolute value) with increasing number of terms (i. e. $\lim\limits_{n\to\infty} R_n=0$). Then Taylor's polynomial can yield the desired value of $f(x)$ to any degree of accuracy.

The number of terms that ensure the requisite degree of accuracy is essentially dependent on the distance $|x-a|$ between the initial point a and the point x. The greater $|x-a|$, the more terms one has to take (see Example 1). Also, we often find that the approach of R_n to zero not only slows down with increasing distance $|x-a|$, but even ceases altogether as the increase continues (see Example 2). Then the polynomial (1) can be used to compute $f(x)$ only over limited distances from the initial point.

Thus, we have to be able to answer the following questions: is the polynomial (1) suitable for computing $f(x)$ at a given distance $|x-a|$ from the initial point a, and if it is, then how many terms have to be taken to attain the required accuracy? It is also important to know whether for all distances the error R_n tends to zero with increasing number of terms, and if not so for every distance, then where its boundary lies.

Answers to these questions are obtained by applying a number of artifices. One of them [1] is based on the theorem of Sec. 271, which permits representing the error R_n in the form [2]

$$R_n=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \tag{3}$$

Here, the number ξ is unknown; the only thing we know is that ξ lies between a and x. But even this suffices to evaluate the error R_n and answer the foregoing questions.

Example 1. Let $f(x)=e^x$. All the derivatives of this function are equal to e^x. We know the value of e^x at the point $x=0$ (namely $e^0=1$). We will take this point as the initial one. The conditions of the theorem of Sec. 271 are fulfilled for all values of x. In Taylor's polynomial (1) we must put

$$a=0, \quad f(a)=f'(a)=\ldots=f^{(n)}(a)=1 \tag{4}$$

and then it assumes the form

$$1+\frac{1}{1!}x+\frac{1}{2!}x^2+\ldots+\frac{1}{n!}x^n \tag{5}$$

Substituting for the value e^x the value of the polynomial (5), we allow for a certain error R_n, which is

$$R_n=e^x-\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\right] \tag{6}$$

Since $f^{(n+1)}(x)=e^x$, the error R_n, according to formula (3), may be given as

$$R_n=\frac{e^{\xi}}{(n+1)!}x^{n+1} \tag{7}$$

---

[1] This device is not the best, and at times is totally useless. Other devices are given below (Sec. 401).

[2] It is assumed that the function $f(x)$ satisfies the conditions of the theorem of Sec. 271, which is the case in numerous instances of practical importance.

The number ξ lies somewhere between zero and x (it depends both on x and on n). Hence, e^{ξ} lies between $e^0 = 1$ and e^x. This is sufficient for evaluating the error.

For example, let it be required to compute the value of e^x for $x = \frac{1}{2}$, i. e. to extract the square root of the number e.

Since e lies between 2 and 3, it follows that $e^{\frac{1}{2}}$ is less than 2 and so e^{ξ} is most definitely less than 2. From (7) it follows that $|R_n| < \frac{2}{(n+1)!}\left(\frac{1}{2}\right)^{n+1}$, i.e.

$$|R_n| < \frac{1}{(n+1)!\,2^n} \tag{8}$$

With increasing n, the quantity $\frac{1}{(n+1)!\,2^n}$ (limiting error) tends to zero, and the error R_n all the more so tends to zero. Hence, the polynomial (5), which now takes on the value

$$1 + \frac{1}{1!\,2} + \frac{2}{2!\,2^2} + \frac{1}{3!\,2^3} + \ldots + \frac{1}{n!\,2^n} \tag{9}$$

is suitable for computing $\sqrt{e}$ to any degree of accuracy.

Now let us find out how many terms the sum (9) must have in order to ensure four-decimal-place accuracy (up to $\pm 0.5 \cdot 10^{-4}$). To do this we compute the limiting error $\frac{1}{(n+1)!\,2^n}$ for $n = 1, 2, 3$, and so on:[1]

$$\frac{1}{2!\,2} = \frac{1}{4},$$

$$\frac{1}{3!\,2^2} = \frac{1}{2!\,2} : 6 = \frac{1}{24},$$

$$\frac{1}{4!\,2^3} = \frac{1}{3!\,2^2} : 8 = \frac{1}{192},$$

$$\frac{1}{5!\,2^4} = \frac{1}{4!\,2^3} : 10 = \frac{1}{1920},$$

$$\frac{1}{6!\,2^5} = \frac{1}{5!\,2^4} : 12 = \frac{1}{23040}$$

We can stop here because $\frac{1}{23040} < 0.5 \cdot 10^{-4}$

---

[1] Beginning with the second row of the computation that follows we resort to a consecutive division by the even numbers 6, 8, 10, ... proceeding from the identity

$$\frac{1}{(n+1)!\,2^n} = \frac{1}{2(n+1)} \cdot \frac{1}{n!\,2^{n-1}}$$

Thus, to ensure an accuracy of $0.5 \cdot 10^{-4}$ it is sufficient for the sum (9) to have six terms. We obtain[1]

$$\begin{aligned} &1 &&= 1.00000, \\ &\frac{1}{1!\,2} &&= 0.50000, \\ &\frac{1}{2!\,2^2} = \frac{1}{1!\,2} : 4 &&= 0.12500, \\ &\frac{1}{3!\,2^3} = \frac{1}{2!\,2^2} : 6 &&= 0.02083, \\ &\frac{1}{4!\,2^4} = \frac{1}{3!\,2^3} : 8 &&= 0.00260, \\ &\frac{1}{5!\,2^5} = \frac{1}{4!\,2^4} : 10 &&= \underline{0.00026} \\ & && \ 1.64869 \end{aligned}$$

We finally obtain

$$\sqrt{e} = 1.6487$$

We thus find that to ensure an accuracy up to $\pm 0.5 \cdot 10^{-8}$, the sum (9) must have 10 terms because

$$|R_9| < \frac{1}{10!\,2^9} \approx 0.55 \cdot 10^{-9} < 0.5 \cdot 10^{-8}$$

The computation yields

$$\sqrt{e} = 1 + \frac{1}{1!\,2} + \frac{1}{2!\,2^2} + \ldots + \frac{1}{9!\,2^9} = 1.64872127$$

Taking 15 terms, it is possible to compute $e^{\frac{1}{2}}$ to within $0.5 \cdot 10^{-16}$, etc. The accuracy of the result increases rapidly with increasing number of terms.

The accuracy increases more slowly if we compute e^x for larger values of $|x|$, say for $x = 1$, or for $x = -1$.

Suppose we take $x = 1$. Then the polynomial (5) takes the form

$$1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \tag{10}$$

and yields an approximate value of the number e. The error R_n, by (7), is

$$R_n = \frac{e^{\xi}}{(n+1)!} \tag{11}$$

---

[1] Each term is computed to the fifth decimal in order to avoid an accumulation of errors.

The number e^{ξ} now lies between e^0 and e^1, i.e. between 1 and e, and since $e < 3$, it follows that

$$|R_n| < \frac{3}{(n+1)!} \tag{12}$$

As before, the error approaches zero with increasing n. But now one has to take 9 terms in place of 6 in order to ensure an accuracy up to $0.5\cdot 10^{-4}$ because the limiting error $\frac{3}{(n+1)!}$ becomes less than $0.5\cdot 10^{-4}$ only for $n=8$. The computation yields

$$e \approx 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{8!}=2.7183$$

If we want to obtain an accuracy up to $0.5\cdot 10^{-8}$, we have to take 13 terms (in place of 10); and the computation yields

$$e \approx 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{12!}=2.71828183$$

Taking 15 terms, we can compute e to within only $0.5\cdot 10^{-10}$ (instead of $0.5\cdot 10^{-16}$ as in the computation of $\sqrt{e}$).

Now take $x=-1$. The polynomial (5) takes the form

$$1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}$$

and yields an approximate value of the number e^{-1} $\left(\text{or } \frac{1}{e}\right)$.

By (7), the error R_n is

$$R_n=(-1)^{n+1}\frac{e^{\xi}}{(n+1)!}$$

The number ξ lies between minus unity and zero; hence, $e^{\xi} < e^0$, i.e. $e^{\xi} < 1$. Consequently

$$|R_n| < \frac{1}{(n+1)!}$$

Here the limiting error is less than in the preceding case by a factor of three For this reason, the number of terms needed to ensure the required accuracy may be reduced, but not by more than unity. Thus, an accuracy of up to $0.5\cdot 10^{-10}$ is now ensured by 14, not 15 terms, which is not essential as far as the actual computations are concerned.

If instead of $x=\pm 1$ we take values of x still greater in absolute value, then the error of the approximate equality

$$e^x \approx 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!} \tag{13}$$

will tend to zero more slowly still. However, using formula (7) and reasoning as above, we are convinced that the error R_n will tend to zero for *any* value of x.

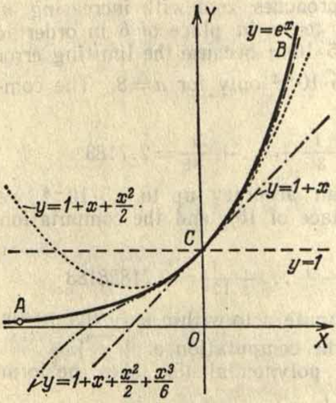


Fig. 259

Fig. 259 depicts the graph ACB of the function $y=e^x$ and the curves of its Taylor polynomials

$$y=1,\ y=1+x,$$
$$y=1+x+\frac{x^2}{2},\ y=1+x+$$
$$+\frac{x^2}{2}+\frac{x^3}{6}$$

Example 2. Let

$$f(x)=\ln(1+x)$$

As in Example 1, take the point $x=0$ as the initial point. The conditions of the theorem of Sec. 271 are fulfilled only for $x>-1$ [for $x\leqslant -1$, the function $\ln(1+x)$ becomes meaningless]. The consecutive derivatives are expressed as follows:

$$f'(x)=\frac{1}{1+x},\quad f''(x)=-\frac{1}{(1+x)^2},\quad f'''(x)=\frac{1\cdot 2}{(1+x)^3},$$
$$f^{\mathrm{IV}}(x)=\frac{1\cdot 2\cdot 3}{(1+x)^4},\ \ldots,\ f^{(n)}(x)=(-)^{n-1}\frac{(n-1)!}{(1+x)^n}$$

so that (Sec. 256, Example 3) we will have

$$f(0)=0,\quad \frac{f'(0)}{1!}=1,\quad \frac{f''(0)}{2!}=-\frac{1}{2},$$
$$\frac{f'''(0)}{3!}=\frac{1}{3},\ \ldots,\ \frac{f^{(n)}(0)}{n!}=(-1)^{n-1}\frac{1}{n}$$

The Taylor polynomial (1) yields the approximate equality

$$\ln(1+x)\approx x-\frac{1}{2}x^2+\frac{1}{3}x^3-\ldots+\frac{(-1)^{n-1}}{n}x^n \tag{14}$$

Since $f^{(n+1)}[\ln(1+x)]=\frac{(-1)^n\, n!}{(1+x)^{n+1}}$, the error R_n of (14) may, by formula (3), be represented in the form

$$R_n = \frac{(-1)^n}{n+1}\left(\frac{x}{1+\xi}\right)^{n+1} \tag{15}$$

where ξ lies somewhere between zero and x.

Let us, for example, compute the value of $\ln(1+x)$ for $x=-0.1$. We obtain the approximate equality

$$\ln 0.9 \approx -0.1 - \frac{1}{2}\cdot 0.1^2 - \frac{1}{3}\cdot 0.1^3 - \ldots - \frac{1}{n}\cdot 0.1^n \tag{16}$$

Its error is

$$R_n = -\frac{1}{n+1}\left(\frac{0.1}{1+\xi}\right)^{n+1}$$

Since ξ lies between zero and -0.1, it follows that $1+\xi > 0.9$. Hence $|R_n| < \frac{1}{n+1}\left(\frac{0.1}{0.9}\right)^{n+1}$ or

$$|R_n| < \frac{1}{n+1}\left(\frac{1}{9}\right)^{n+1} \tag{17}$$

The limiting error obviously approaches zero with increasing n, i.e. formula (16) is capable of yielding ln 0.9 to any degree of accuracy. Thus, to ensure an accuracy up to $0.5\cdot 10^{-4}$ we have to take $n=4$, and we get

$$\ln 0.9 \approx -\left[0.1 + \frac{1}{2}\cdot 0.01 + \frac{1}{3}\cdot 0.001 + \frac{1}{4}\cdot 0.0001\right] \approx -0.1054$$

In the same way we can convince ourselves that formula (14) always holds if $-\frac{1}{2} \leqslant x \leqslant 1$.[1] But as $|x|$ increases, the error R_n tends to zero more slowly. This approach is weakest of all when $x=1$. Then formula (14) yields

$$\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \ldots + (-1)^{n-1}\frac{1}{n}$$

For example, to ensure an accuracy up to $0.5\cdot 10^{-4}$, we have to take 19999 terms.

And if x is just the slightest bit more than unity, the error does not tend to zero at all; on the contrary, $|R_n|$ increases without bound with increasing n.

---

[1] It also holds for all x between -1 and $-\frac{1}{2}$, but expression (15) does not convince us of this fact

Fig. 260 depicts the graphs of the function $y=\ln(1+x)$ (the curve ACB) and of the first three Taylor polynomials.

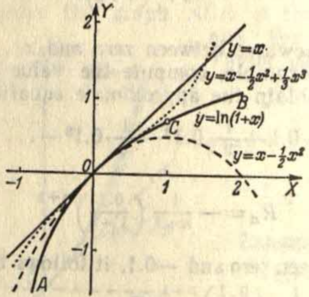


Fig. 260

273. Increase and Decrease of a Function

Definition 1. A function $f(x)$ is called an *increasing function at a point* $x=a$ if, *in a sufficiently small neighbourhood*, values of x greater than a are associated with values of $f(x)$ greater than $f(a)$, and smaller values correspond to smaller values.

A function $f(x)$ is called a *decreasing function at a point* $x=a$ if, *in a sufficiently small neighbourhood* of this point, values of x greater than a are associated with values of $f(x)$ smaller than $f(a)$, and smaller values are associated with greater values.

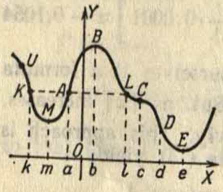


Fig. 261

Example 1. The function depicted in Fig. 261 increases at the point $x=a$ because to the right of A the points of the curve lie above A and to the left, below A. Here we consider only those points of the curve whose ordinates are sufficiently close to the ordinate aA; in the given instance, these are the points which do not go beyond the limits of the arc KL. Outside this arc the relation no longer holds: point C lies to the right of A but below it, U lies to the left, but above it.

The same function is decreasing at the point $x=d$ because in a sufficiently close neighbourhood of D the points of the curve to the right lie below D, those to the left lie above D.

The function is also decreasing at the point $x=c$.

At the points $x=b$, $x=e$, $x=m$ the function is neither increasing nor decreasing (at $x=b$ it has a maximum, at $x=e$ and $x=m$ a minimum; Sec. 275).

Definition 2. A function is called *increasing in an interval* (a, b) if it is increasing at every point within the interval (but not necessarily at the end-points).

A function *decreasing in an interval* (a, b) is similarly defined.

Example 2. The function shown in Fig. 261 is decreasing in the interval (l, d) because it is decreasing at every point within the interval (and at its end-points as well). The same function is also decreasing in the interval (b, e) because it is decreasing at all interior points of the interval (but at the end-points b and e the function is not decreasing). In the interval (m, b) the function is increasing; in the interval (a, d) it is neither an increasing nor a decreasing function. Now if we split up the interval into two parts: (a, b) and (b, d), then in the former the function is increasing and in the latter it is decreasing.

If the function is increasing in the interval (a, b), then in that interval a greater value of the argument is always associated with a greater value of the function; conversely, if in the interval (a, b) a greater value of the argument is always associated with a greater value of the function, then the function is increasing in (a, b).[1]

If the function is decreasing in the interval (a, b), then a greater value of the argument is always associated with a smaller value of the function, and vice versa.

Geometrically, in those intervals in which the function is increasing its curve (rightward motion) rises; in intervals where the function is decreasing, the curve drops (cf. Example 2).

Definition 3. A function which in a given interval is increasing or decreasing is called *monotonic* (in that interval).

---

[1] This property is often taken as a definition of a function increasing in an interval. A function decreasing in an interval is similarly defined.

274. Tests for the Increase and Decrease of a Function at a Point

Sufficiency test. If the derivative $f'(x)$ is positive at a point $x=a$, then the function $f(x)$ at this point is increasing, if it is negative, then the function is decreasing.

Geometrically, if the slope of the tangent MT (Fig. 262) is positive, then near M the curve lies above the point M to the right and below it to the left; if the slope is negative (Fig. 263), then near M the curve lies below M to the right and above M to the left.

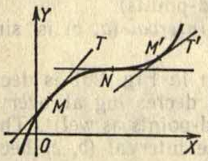


Fig. 262

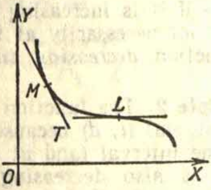


Fig. 263

Note. If $f'(a)=0$, then for $x=a$ the function may be increasing (point N in Fig. 262); it may be decreasing too (point L in Fig. 263). But as a rule, the function will not (for $x=a$) be either decreasing or increasing (points B and C in Fig. 264). Ways of distinguishing these cases are indicated in Secs. 278 and 279.

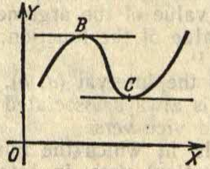


Fig. 264

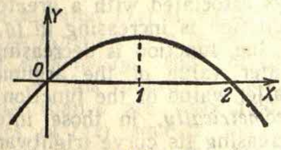


Fig. 265

Example 1. The function $y=x-\frac{1}{2}x^2$ (Fig. 265) is increasing at the point $x=0$, because $y'=1-x=1>0$. The same function is decreasing at the point $x=2$ where $y'=-1<0$.

At $x=1$, where $y'=0$, the function is neither decreasing nor increasing.

Necessity test. If the function $f(x)$ is increasing at the point $x=a$, then its derivative[1] at this point is positive (as at point M in Fig. 262) or is equal to zero (as at point N in Fig. 262):

$$f'(a) \geqslant 0$$

Similarly for a decreasing function; its derivative is negative or zero at the point $x=a$:

$$f'(a) \leqslant 0$$

Example 2. The function $y=x^3$ (Fig. 266) is increasing at every point. Its derivative $y'=3x^2$ is positive everywhere except at the point $x=0$, where $y'=0$.

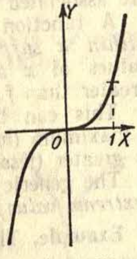


Fig. 266

274a. Tests for the Increase and Decrease of a Function in an Interval

Sufficiency test. If the derivative function $f'(x)$ in an interval (a, b) is everywhere positive, then the function $f(x)$ in this interval is increasing; if $f'(x)$ is everywhere negative, then $f(x)$ is decreasing (cf. Sec. 274).

Note. The test (criterion) also holds true when the derivative takes on zero values in the interval (a, b) so long as $f(x)$ does not identically become zero throughout the interval (a, b) or in some interval (a', b') comprising a part of (a, b) [the function $f(x)$ would be a constant on such an interval].

Example. The function $y=x-\frac{1}{2}x^2$ (Fig. 265) is increasing in the interval $(0, 1)$ because the derivative $y'=1-x$ takes on a zero value only at the point $x=1$, whereas at the remaining points of the interval $(0, 1)$ it is positive. The same function is decreasing in the interval $(1, 2)$ because here the derivative y' is everywhere negative except at the point $x=1$, where $y'=0$.

Necessity test. If the function $f(x)$ is increasing in the interval (a, b), then the derivative[2] $f'(x)$ is positive or zero in that interval

$$f'(x) \geqslant 0 \text{ for } a \leqslant x \leqslant b$$

The same holds true for a decreasing function:

$$f'(x) \leqslant 0 \text{ for } a \leqslant x \leqslant b$$

---

[1] It is assumed that $f(x)$ is differentiable at this point.

[2] It is assumed that the function is differentiable in the interval (a, b).

275. Maxima and Minima

Definition. We say that a function $f(x)$ has a *maximum at a point* $x=a$ if, *in a sufficiently close neighbourhood of the point*, all values of x (both greater and smaller than a) are associated with values of $f(x)$ smaller than $f(a)$.

A function $f(x)$ has a *minimum at a point* $x=a$ if, *within a sufficiently close neighbourhood of the point*, all values of x are associated with values of $f(x)$ which are greater than $f(a)$.

This can be stated more succinctly: a function $f(x)$ has a maximum (minimum) at a point $x=a$ if the value of $f(a)$ is greater (less) than all neighbouring values.

The generic term for maximum and minimum is *extremum (extreme value)*.

Example. The function $f(x)=\frac{1}{3}x^3-x^2+\frac{1}{3}$ (Fig. 267) has a maximum at the point $x=0$ $\left[\text{the point } A\left(0, \frac{1}{3}\right) \text{ is higher than all neighbouring points}\right]$ and a minimum at the point $x=2$ [the point B $(2, -1)$ is lower than all neighbouring points].

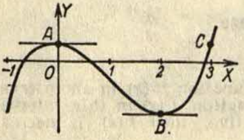


Fig. 267

Note. In ordinary language, the expressions "maximum" and "greatest quantity" are synonymous. In analysis, the term "maximum" has a narrower meaning: the maximum of a function need not be its greatest value. Thus, the function $f(x)=\frac{1}{3}x^3-x^2+\frac{1}{3}$ (see Fig. 267), if considered, say, in the interval $(-1, 4)$ has a maximum at $x=0$ because *near* this point [namely, in the interval $(-1, 3)$] all the values of x are associated with values of $f(x)$ which are smaller than $f(0)$, i.e. than $\frac{1}{3}$ (in that interval the graph is located below the point A). Still, the maximum $f(0)$ is not the greatest value of the function in the interval $(-1, 4)$ because for $x>3$ we have

$$f(x)>\frac{1}{3}$$

(the graph to the right of C is located above the point A). However in the given interval, finding the greatest value of the function is closely associated with finding its maxima (see Sec. 280).

The same remark applies to minima as well.

276. Necessary Condition for a Maximum and a Minimum

Theorem. If a function $f(x)$ has an extremum (a maximum or a minimum) at a point $x=a$, then the derivative at this point is either zero, infinite or does not exist.

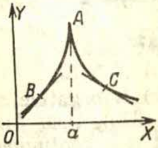


Fig. 268

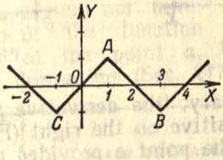


Fig. 269

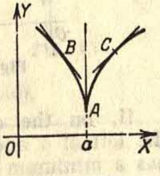


Fig. 270

Geometrically, if a graph has a maximum ordinate at a point A, then the tangent at this point is either horizontal (Fig. 267), vertical (Fig. 268) or does not exist (Fig. 269). The same applies to the minimum ordinate (point B in Fig. 267, point A in Fig. 270, points B and C in Fig. 269).

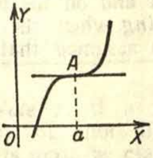


Fig. 271

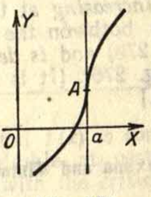


Fig. 272

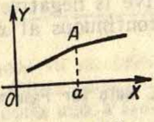


Fig. 273

Note. The condition for an extremum as given in the theorem is necessary but *not sufficient*, that is, the derivative at the point $x=a$ can vanish (Fig. 271), become infinite (Fig. 272) or not exist (Fig. 273) without the function having an extremum at that point.

277. The First Sufficient Condition for a Maximum and a Minimum

Theorem. If, sufficiently close to a point $x=a$, the derivative $f'(x)$ is positive on the left of a and negative on the right of a (Fig. 274), then at the point itself, $x=a$, the function $f(x)$ has a maximum provided that $f(x)$ is continuous here.1)

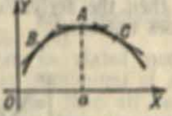


Fig. 274

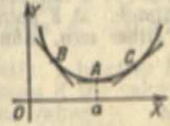


Fig. 275

If, on the contrary, the derivative $f'(x)$ is negative on the left of a and positive on the right (Fig. 275), then $f(x)$ has a minimum at the point a provided that it is continuous here.2)

The theorem states that when $f(x)$ passes from increasing values to decreasing values, it has a maximum; when it passes from decreasing to increasing values, it has a minimum.

Note. According to the theorem, the test for an extremum of a function $f(x)$ is the *change of sign* of the derivative $f'(x)$ when the argument passes through the value $x=a$ under consideration.

Now, if in passing through $x=a$ the derivative *retains* its sign, then $f(x)$ *is increasing* at the point $x=a$ when the derivative is positive both on the right and on the left of $x=a$ (Figs. 271, 272, 273) and *is decreasing* when the derivative is negative (Fig. 276). [It is again assumed that $f(x)$ is continuous at $x=a$.]

278. Rule for Finding Maxima and Minima

Let a function $f(x)$ be differentiable in an interval (a, b). In order to find all its maxima and minima in the interval, it is necessary to:

(1) *Solve the equation* $f'(x)=0$ (the roots of this equation are called the *critical* values of the argument; among them

---

1) However, $f(x)$ need not be differentiable at $x=a$ (see Fig. 268).
2) However, $f(x)$ need not be differentiable at $x=a$ (see Fig. 270).

we have to seek the values of x which yield an extremum of the function $f(x)$; see Sec. 276).

(2) *Investigate, for every critical value $x=a$, to see whether the derivative $f'(x)$ changes sign when the argument passes through this value.* If $f'(x)$ passes *from positive values to negative values* (when going from $x<a$ to $x>a$), then we have a *maximum* (Sec. 277). If it goes *from negative values to positive values*, then we have a *minimum*.

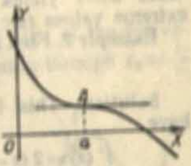
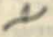
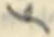


Fig. 276

But if $f'(x)$ preserves its sign, then there is neither maximum nor minimum: for $f'(x)>0$, the function $f(x)$ is increasing at the point a, for $f'(x)<0$, it is decreasing (Sec. 277, Note).

Sign of Derivative		Shape of Graph Near Point a	
for $x<a$	for $x>a$		
+	−		maximum
−	+		minimum
+	+		increase
−	−		decrease

Note 1. If a function $f(x)$ is continuous in an interval (a, b), but not differentiable at certain points, then these points must be classed with the critical points and a similar investigation must be carried out.

Note 2. The maxima and minima of a continuous function follow one another in alternation.

Example 1. Find all the maxima and minima of the function $f(x)=x-\frac{1}{2}x^2$.

Solution. This function is everywhere differentiable (i. e. it has a finite derivative everywhere) $f'(x)=1-x$.

(1) Solve the equation $1-x=0$. It has a unique root $x=1$.

(2) The derivative $f'(x)=1-x$ changes sign as the argument passes through the value $x=1$. Namely, for $x<1$ the derivative is positive, for $x>1$, it is negative. Hence, the critical value $x=1$ yields a maximum. The function has no other extreme values (see Fig. 265).

Example 2. Find all the maxima and minima of the function

$$f(x)=(x-1)^2(x+1)^3 \tag{1}$$

Solution. This function is everywhere differentiable. We have

$$f'(x)=2(x-1)(x+1)^3+3(x-1)^2(x+1)^2=$$
$$=(x-1)(x+1)^2(5x-1)$$

(1) Solve the equation $f'(x)=0$. Its roots (in increasing order) are

$$x_1=-1,\ x_2=\frac{1}{5},\ x_3=1 \tag{2}$$

(2) Representing the derivative in the form

$$f'(x)=5(x+1)^2\left(x-\frac{1}{5}\right)(x-1) \tag{3}$$

we investigate each of the critical values.

(a) For $x<-1$, all three binomials of formula (3) are negative, so that to the left of $x=-1$ we have

$$f'(x)=5(-)^2(-)(-)=+ \tag{4}$$

Let the argument pass through the value $x_1=-1$, but suppose it has not yet reached the next critical value, $x_2=\frac{1}{5}$. Then the binomial $x+1$ is positive, the two other binomials of (3) remain negative, and we have

$$f'(x)=5(+)^2(-)(-)=+ \tag{5}$$

Comparing (4) and (5) we see that the derivative does not change sign (remains positive) when passing through the critical value $x_1=-1$. Hence there is no extremum at the point $x=-1$; here the function $f(x)$ is increasing (Fig. 277).

(b) Investigate the nearest larger critical value $x_2=\frac{1}{5}$. Sufficiently close on the left $\left(\text{i.e. between } x_1=-1 \text{ and } x_2=\frac{1}{5}\right)$ the derivative is positive by virtue of (5). Sufficiently close on the right $\left(\text{between } x_1 = \frac{1}{5} \text{ and } x_2 = +1\right)$ the second factor is positive and we have

$$f'(x) = 5(+)^2(+)(-) = - \tag{6}$$

Comparing (5) and (6) we see that the sign of the derivative changes from plus to minus when passing through $x_2 = \frac{1}{5}$ [the function $f(x)$ passes from increasing values to decreasing values]. Hence the function has a maximum at the point $x = \frac{1}{5}$; it is equal to

$$f\left(\frac{1}{5}\right) = \left(\frac{1}{5} - 1\right)^2\left(\frac{1}{5} + 1\right)^3 \approx 1.1$$

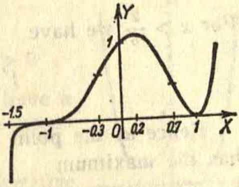


Fig. 277

(c) Investigate the last critical value, $x_3 = 1$. Sufficiently close on the left, the derivative is negative by virtue of (6). To the right of $x = 1$ we have

$$f'(x) = \frac{1}{5}(+)^2(+)(+) = + \tag{7}$$

When passing through $x = 1$, the derivative changes from minus to plus [the function $f(x)$ passes from decreasing to increasing values]. Hence at $x = 1$ the function has a minimum, which is equal to

$$f(1) = (1-1)^2(1+1)^3 = 0$$

Example 3. Find all the extrema of the function

$$f(x) = (x-1)\sqrt[3]{x^2}$$

Solution. The given function is differentiable for all positive and negative values of x and we have

$$f'(x) = \sqrt[3]{x^2} + \frac{2(x-1)}{3\sqrt[3]{x}} = \frac{5}{3}\,\frac{x - \frac{2}{5}}{\sqrt[3]{x}}$$

At the point $x = 0$ the function $f(x)$ is not differentiable (its derivative is infinite). Therefore (see Note 1) we have two critical values: $x_1=0$ and $x_2=\frac{2}{5}$. For $x<0$, we have

$$f'(x)=\frac{5}{3}\frac{(-)}{\sqrt[3]{-}}=+$$

For $0<x<\frac{2}{5}$ we have

$$f'(x)=\frac{5}{3}\frac{(-)}{\sqrt[3]{+}}=-$$

For $x>\frac{2}{5}$ we have

$$f'(x)=\frac{5}{3}\frac{(+)}{\sqrt[3]{+}}=+$$

Hence at the point $x=0$ the function $f(x)=(x-1)\sqrt[3]{x^2}$ has the maximum

$$f(0)=0$$

and at the point $x=\frac{2}{5}$, the minimum

$$f\left(\frac{2}{5}\right)=-\frac{3}{5}\sqrt[3]{\frac{4}{25}}\approx-0.33$$

279. The Second Sufficient Condition for a Maximum and a Minimum

When it is difficult to distinguish the sign of the derivative near critical points (Sec. 278), one can use the following sufficient condition for an extremum.

Theorem 1. Let the first derivative $f'(x)$ vanish at the point $x=a$; if the second derivative $f''(a)$ is then negative, the function $f(x)$ has a maximum at $x=a$, if it is positive, then the function has a minimum. For the case $f''(a)=0$, see Theorem 2.

The second condition is related to the first in the following manner. We consider $f''(x)$ as a derivative of $f'(x)$. The relation $f''(a)<0$ means (Sec. 274) that $f'(x)$ is decreasing at the point $x=a$. And since $f'(a)=0$, it follows that $f'(x)$ is positive for $x<a$ and negative for $x>a$. Hence (Sec. 277), $f(x)$ has a maximum at $x=a$. The situation is similar for the case $f''(a)>0$.

Example 1. Find the maxima and minima of the function

$$f(x)=\frac{1}{2}x^4-x^2+1$$

Solution. Solving the equation

$$f'(x)=2x^3-2x=0$$

we obtain the critical values

$$x_1=-1,\ x_2=0,\ x_3=1$$

Substituting them into the expression of the second derivative

$$f''(x)=6x^2-2=2(3x^2-1)$$

we find that

$$f''(-1)>0,\quad f''(0)<0,\quad f''(1)>0$$

Hence at $x=-1$ and $x=1$ we have a minimum, and at $x=0$ a maximum (Fig. 278).

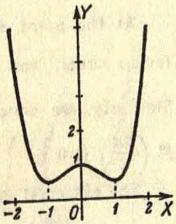


Fig. 278

It may happen that the second derivative vanishes together with the first; this can also happen with regard to a number of subsequent derivatives. In such cases one can make use of the following generalization of Theorem 1.

Theorem 2. If at the point $x=a$, where the first derivative is zero, the closest nonzero derivative is of even order, $2k$, then the function $f(x)$ has, at $x=a$, a maximum when $f^{(2k)}(a)<0$, and a minimum when $f^{(2k)}(a)>0$.

Now if the closest nonzero derivative is of odd order, $2k+1$, then the function $f(x)$ does not have an extremum at the point a; it is increasing when $f^{(2k+1)}(a)>0$ and is decreasing when $f^{(2k+1)}(a)<0$.

Note. Theoretically, it is not precluded that at a point $x=a$ *all the derivatives* of the function $f(x)$ (which is not a constant) are equal to zero. [1] However, this case is of no practical significance.

Example 2. Find the maxima and minima of the function

$$f(x)=\sin 3x-3\sin x$$

Solution. We have

$$f'(x)=3\cos 3x-3\cos x$$

Solving the equation

$$3\cos 3x-3\cos x=0$$

we find

$$x=k\frac{\pi}{2}$$

where k is any integer.

Since this function has a period 2π, it is sufficient to investigate four roots:

$$x_1=0,\ x_2=\frac{\pi}{2},\ x_3=\pi,\ x_4=\frac{3\pi}{2}$$

---

[1] Such, for instance, is the function considered in the last footnote of Sec. 270 (p. 350).

Take the second derivative

$$f''(x) = -9 \sin 3x + 3 \sin x$$

Substituting the critical values x_1, x_2, x_3, x_4, we find

$$f''(0) = 0, \quad f''\left(\frac{\pi}{2}\right) = 12,$$

$$f''(\pi) = 0, \quad f''\left(\frac{3\pi}{2}\right) = -12$$

At the point $x_2 = \frac{\pi}{2}$, the nearest nonzero derivative is of second (even) order, and $f''\left(\frac{\pi}{2}\right) > 0$. Hence, it has a minimum at $x = \frac{\pi}{2}$. Similarly, we conclude that at $x = \frac{3\pi}{2}$ it has a maximum $\left[\text{because } f''\left(\frac{3\pi}{2}\right) < 0\right]$

The extremal values will be

$$f\left(\frac{\pi}{2}\right) = \sin 3\frac{\pi}{2} - 3 \sin \frac{\pi}{2} = -1 - 3 = -4 \text{ (minimum)}$$

$$f\left(\frac{3\pi}{2}\right) = \sin \frac{9\pi}{2} - 3 \sin \frac{3\pi}{2} = 1 - (-3) = 4 \text{ (maximum)}$$

To investigate the critical values $x_1 = 0$ and $x_3 = \pi$, let us find the third derivative

$$f'''(x) = -27 \cos 3x + 3 \cos x$$

We have

$$f'''(0) = -24, \quad f'''(\pi) = +24$$

At the point $x = 0$ the nearest nonzero derivative is of third (odd) order, and $f'''(0) < 0$. Hence, at $x = 0$ there is no extremum. Here, the function $f(x)$ is decreasing. Similarly, we conclude that at $x = \pi$ as well there is no extremum; but here the function $f(x)$ is increasing [because $f'''(\pi) > 0$].

280. Finding Greatest and Least Values of a Function

1. Suppose that by the conditions of the problem the argument of a continuous function $f(x)$ varies in an infinite interval, say in the interval $(a, +\infty)$. Then it may happen that there is no greatest value of the function $f(x)$; see Fig. 279*a* where $f(x)$ increases without bound as $x \to +\infty$. But if $f(x)$ has a greatest value, then this value is definitely one of the extrema of the function; see Fig. 279*b*, where the greatest value of the function is $f(c)$.

Now let it be given that the argument x varies in a *closed* interval (a, b). Then $f(x)$ definitely assumes a greatest value

(Sec. 221). However, this value may not belong to the extrema, for it may be attained at one of the end-points of the interval (at point $x=b$[1] in Fig. 279*c*).

The same goes for the least value.

2. Let it be required to find the greatest (or least) value of a geometric or physical quantity which obeys definite conditions (see examples below). Then it is necessary to represent the quantity as a function of some argument. From the conditions of the problem we determine the range of the argument. Then we find all the critical values of the argument lying within this interval and compute the appropriate values of the function, and also the values of the function at the end-points of the interval. From the values thus found we choose the greatest (least).

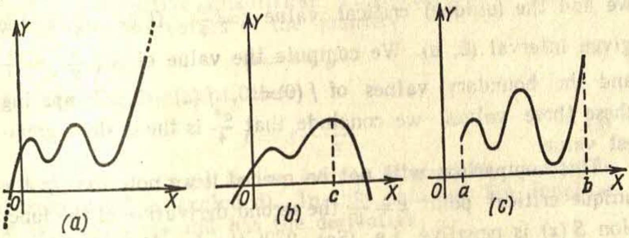


Fig. 279

Note 1. It often happens that the argument may be chosen in a variety of ways; a lucky choice can simplify the solution. Allowance for the peculiarities of the problem can also simplify the solution.

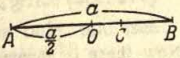


Fig. 280

For instance, if within a given interval there is only one critical value of the argument and, on the basis of some test (see Secs. 277, 279), it should yield a maximum (minimum), then even without a comparison with the boundary values of the function we are justified in concluding that this maximum (minimum) is the desired greatest (least) value.

Example 1. In Fig. 280, the segment $AB=a$ is divided into two parts by C; on AC and CB construct the rectangle $ACBD$. Find the greatest value of its area S.

---

[1] If the end-point $x=b$ is not considered, then over the remaining *open* interval the function $f(x)$ will not have a greatest value.

Solution. For the argument x we take the length AC; then

$$CB = a - x \text{ and } S = x(a - x)$$

The argument x of the continuous function S varies in the interval $(0, a)$.

From the equation

$$\frac{dS}{dx} = a - 2x = 0$$

we find the (unique) critical value $x = \frac{a}{2}$. It belongs to the given interval $(0, a)$. We compute the value of $S\left(\frac{a}{2}\right) = \frac{a^2}{4}$ and the boundary values of $f(0) = 0$, $f(a) = 0$. Comparing these three values, we conclude that $\frac{a^2}{4}$ is the desired greatest value.

This comparison will not be needed if we note that in the unique critical point $x = \frac{a}{2}$ the second derivative of the function $S(x)$ is negative, i.e. (Sec. 279) the function $S(x)$ has a maximum there.

The variable rectangle $ACBD$ always has one and the same perimeter $(2a)$. Hence, *of all rectangles of a given perimeter the square has the greatest area.*

Note 2. Most convenient of all is to take for the argument the distance z from the point C to the midpoint O of the segment AB (see Fig. 280). Then

$$AC = AO + OC = \frac{a}{2} + z, \quad CB = OB - OC = \frac{a}{2} - z$$

and

$$S = \left(\frac{a}{2} + z\right)\left(\frac{a}{2} - z\right) = \left(\frac{a}{2}\right)^2 - z^2$$

Now there is no need to seek an extremum because $\left(\frac{a}{2}\right)^2 - z^2$ quite obviously does not exceed $\left(\frac{a}{2}\right)^2$.

Example 2. Under the conditions of Example 1 find the least value of the area S.

Solution. Taking $x = AC$ for the argument, we compare the unique extremum $\left(\frac{a^2}{4}\right)$ of the function $S = x(a - x)$ with its value $(S = 0)$ at the end-points of the interval $x = 0$ and $x = a$. We find that zero is the least value of S [*in the closed interval* $(0, a)$].

However, for $x = 0$ and $x = a$ we do not have a rectangle in the proper sense of the word (it degenerates into the line segment AB). If we consider only "real" rectangles, then the end-points of the interval

$(0, a)$ ought to be excluded and then S does not have a least value [in the open interval $(0, a)$].

Example 3. Find the least and the greatest values of the semiperimeter p of a rectangle having a given area S.

Solution. Denote the sides of the rectangle by x, y. It is given that

$$xy = S \tag{1}$$

(x and y are positive quantities). It is required to find the least and greatest values of the quantity

$$p = x + y \tag{2}$$

Take x for the argument; then

$$p = x + \frac{S}{x} \tag{3}$$

The argument x varies in the infinite interval $(0, +\infty)$ (the end-point $x=0$ is excluded). In this interval the function $p(x)$ is continuous and has the derivative

$$\frac{dp}{dx} = 1 - \frac{S}{x^2} \tag{4}$$

From the equation

$$1 - \frac{S}{x^2} = 0 \tag{5}$$

we find the unique (in this interval) critical value

$$x = \sqrt{S}$$

From (4) it is seen that for $0 < x < \sqrt{S}$ the derivative $\frac{dp}{dx}$ is negative and for $x > \sqrt{S}$ it is positive. Hence (Sec. 277) it has a minimum. Since this minimum is the only one (see Note 1) it is the least value of the semiperimeter:[1]

$$p_{min} = \sqrt{S} + \frac{S}{\sqrt{S}} = 2\sqrt{S} \tag{6}$$

---

[1] The problem may be solved without finding the extremum. Equalities (2) and (1) yield $p^2 = (x+y)^2 = (x-y)^2 + 4xy = (x-y)^2 + 4S$. Since $4S$ is a constant and the least value of $(x-y)^2$ is zero (when $x=y$), it follows that the least value of p^2 is $4S$; hence, the least value of p is $2\sqrt{S}$.

This method is simpler (in that it does not require higher mathematics) and is shorter. But it is based on guesswork, and in that sense it is more difficult than the general method given above.

i. e., *of all rectangles of a given area S the square* ($x=\sqrt{S}$, $y=\sqrt{S}$) *has the smallest semiperimeter.*

The quantity p does not have a greatest value since the given interval $(0, +\infty)$ is open.

Example 4. Find the least amount of tin to be used in making a cylindrical tin can with a volume of two litres ($V=2l$, the extra material for seams is not taken into account).

Solution. Let the surface of the can be S, the radius of the base r, the height h. It is required to find the least value of the quantity

$$S=2\pi rh+2\pi r^2 \tag{7}$$

provided that

$$\pi r^2 h=V \tag{8}$$

For the argument it is convenient to take r. From (7) and (8) we find

$$S=2\left(\frac{V}{r}+\pi r^2\right) \tag{9}$$

where the argument varies in the interval $(0, \infty)$. From the meaning of the problem it is clear that the quantity S reaches a least value somewhere inside this interval. It is therefore sufficient to consider the values of the function at the critical points.

Solve the equation

$$\frac{dS}{dr}=2\left(-\frac{V}{r^2}+2\pi r\right)=0 \tag{10}$$

Its sole root

$$r=\sqrt[3]{\frac{V}{2\pi}} \tag{11}$$

corresponds to the least value of S. From (8) and (11) we get $h=\frac{V}{\pi r^2}=\sqrt[3]{\frac{4V}{\pi}}=2r$, that is, *the height of the can must be equal to the diameter of the base*. The least amount of tin required to make the can is then

$$S_{min}=2\pi(rh+r^2)=6\pi r^2=3\sqrt[3]{2\pi V^2}\approx 879\text{ cm}^2$$

Example 5. (Descartes' paradox). In 1638 Descartes received (through M. Mersenne) a letter of Fermat, where Fermat gave without proof a rule which he had discovered for finding extrema. Translated into modern language, the Fermat rule reduces to finding the values which make the derivative $f'(x)$ of the function $f(x)$ under consideration vanish.

In a return letter Descartes described the following example which he believed proved the Fermat rule to be erroneous.

Let there be given a circle

$$x^2+y^2=r^2 \tag{12}$$

(Fig. 281) and a point $A(-a, 0)$ distinct from the centre (that is, $a \neq 0$). It is required to find on the circle (12) a point that is closest to A. The square of the distance of an arbitrary point $M(x, y)$ from the point A is given as

$$AM^2=(x+a)^2+y^2 \tag{13}$$

Now if M lies on the circle (12), then

$$y^2=r^2-x^2$$

so that

$$AM^2=(x+a)^2+r^2-x^2$$

In order to find the value of x which minimizes the quantity AM^2, Descartes followed Fermat's rule and obtained the absurd equality $2a=0$.

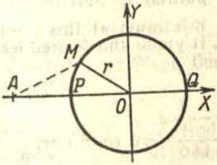


Fig. 281

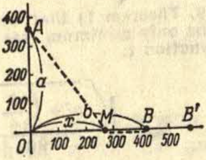


Fig. 282

Yet, geometrically, it is clear that the desired point exists and coincides with the point $P(-r, 0)$. From this Descartes concluded that the test for the minimum was incorrect. Actually, the point P $(x=-r)$ is not revealed for a different reason: the least value of AM^2 corresponding to it *is not a minimum*. Indeed, x varies only in the interval $(-r, +r)$. The function at hand assumes its least value at the end-point of the interval.

Example 6. A group of swimmers strike out from boat A (Fig. 282) to a point B on the shore. The conditions of the competition allow part of the distance to be covered by land. The boat stands opposite pier O at a distance $OA=a=360$ metres; the point of destination B is at a distance of $OB=b=420$ metres from the pier. What is the best result that can be obtained by a participant if he covers 90 metres per minute swimming and 150 metres per minute running?

Solution. Let the swimmer land at point M, a distance $OM=x$ from the pier. It will suffice to consider the variation of x in the interval $(0, b)$.[1]

The time t (in minutes) spent in covering the distance AMB is

$$t=\frac{\sqrt{a^2+x^2}}{90}+\frac{b-x}{150} \tag{14}$$

---

[1] It is senseless for a swimmer to get to the shore outside OB because it takes longer to swim to B' than to B and he would have to run the distance between B' and B.

Here, $a=360$, $b=420$. It is required to find the least value of the function t in the interval $(0, b)$.

We have

$$\frac{dt}{dx}=\frac{x}{90\sqrt{a^2+x^2}}-\frac{1}{150} \tag{15}$$

Solving the equation

$$\frac{x}{90\sqrt{a^2+x^2}}-\frac{1}{150}=0 \tag{16}$$

we find the sole critical value $x=\frac{3}{4}a=270$ metres. This value lies in the interval $(0, b)$ under consideration. Since the second derivative

$$\frac{d^2t}{dx^2}=\frac{1}{90}\frac{d}{dx}\left(\frac{x}{\sqrt{x^2+a^2}}\right)=\frac{a^2}{90\sqrt{(a^2+x^2)^3}}$$

at the point $x=\frac{3}{4}a$ (as at all other points) is positive, it follows (Sec. 279, Theorem 1) that we have a minimum at this point. Since this is the only minimum (see Note 1), it yields the desired least value of the function t:

$$t_{min}=\frac{\sqrt{a^2+\left(\frac{3}{4}a\right)^2}}{90}+\frac{b-\frac{3}{4}a}{150}=6 \text{ (minutes)}$$

The path of the swimmer is shown in Fig. 282 by the dashed line.

Example 6a. Solve the same problem as in Example 6 but with $b=420$ metres changed to $b=225$ metres (Fig. 283).

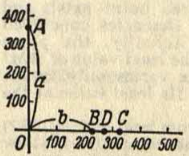


Fig. 283

Solution. It suffices to consider the variation of x in the interval $(0, 225)$. Since the root $x=270$ of Eq. (16) lies beyond this interval, the function t now has no minimum inside the interval. The least value is assumed at the end-point $x=b=225$. Here,

$$t=\frac{\sqrt{a^2+b^2}}{90}=4 \text{ min } 43 \text{ sec}$$

The swimmer has to swim directly to the finishing point.

Note 3. When solving this problem, we considered, as common sense suggests, the variation of the argument x of the function

$$t=\frac{\sqrt{a^2+x^2}}{90}+\frac{b-x}{150} \tag{14}$$

(where $a=360$, $b=225$) only in the interval $(0, 225)$.

But we could have extended the range of the argument and considered, say, the interval $(0, 325)$. Then, reasoning as in Example 6, we would have found that the function (14) has a minimum at $x=270$ [because this point lies in the interval of interest, is the sole critical value of the function (14), and minimizes the function].

From this it would seem possible to conclude that the swimmer ought to swim to point D, which is at a greater distance than the finishing point B, but this is manifestly absurd.

The mistake stems from the fact that the function (14) expresses the dependence of t on x *only* over the interval OB, whereas on the segment BC the dependence is expressed by the formula

$$t=\frac{\sqrt{a^2+x^2}}{90}+\frac{x-b}{150} \qquad (14')$$

(see schematic graph in Fig. 284).

When $x=b$ both formulas (14) and (14′) yield the same value, so that the function $t(x)$ is continuous only at $x=b$, but the derivative $\frac{dt}{dx}$ does not exist at $x=b$. For this reason, the point $x=b$ is now a critical point of the function $t(x)$ (cf. Sec. 278, Note 1). There are no other critical points in the interval (0, 325).

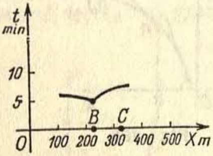


Fig. 284

281. The Convexity of Plane Curves. Point of Inflection

A plane curve L is called *convex at a point* M (Fig. 285) if in a sufficiently small neighbourhood of M the curve L lies on one side of the tangent MT (the *direction of concavity* of L). The opposite side is called the *direction of convexity.*

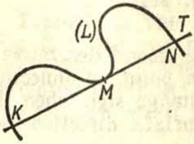


Fig. 285

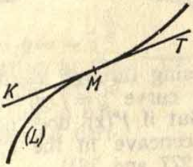


Fig. 286

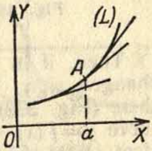


Fig. 287

If the curve L near the point M lies on both sides of the tangent MT (Fig. 286), then M is called a *point of inflection* of the curve L.

When passing through a point of inflection, convexity turns to concavity and vice versa.

Let L be given by the equation $y=f(x)$. If the *derivative* $f'(x)$ increases at the point $x=a$, then L is concave up (Fig. 287), if it decreases, then it is concave down (Fig. 288).

Now if the derivative $f'(x)$ has an extremum at $x=a$ (Figs. 289, 290), then the curve L has a point of inflection there.

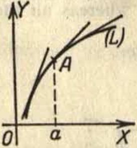


Fig 288

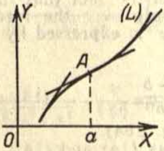


Fig. 289

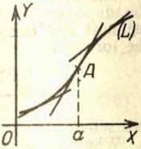


Fig. 290

282. Direction of Concavity

1. If the second derivative $f''(x)$ at a point $x=a$ is positive, then the curve $y=f(x)$ is concave up, if the derivative is negative, then it is concave down (schematic figure 291).

Explanation. If $f''(a)>0$, then $f'(x)$ is increasing at $x=a$ (Sec. 274); hence (Sec. 281) the concavity is up. The reasoning is similar for the case $f''(a)<0$.

2. Let the second derivative $f''(x)$ be zero, infinite or nonexistent altogether at the point $x=a$.

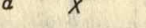


Fig. 291

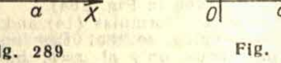


Fig. 292

Then, if in passing through $x=a$ the second derivative1) changes sign, the curve $y=f(x)$ has a point of inflection there (Fig. 292). But if $f''(x)$ does not change sign, then the curve $y=f(x)$ is concave in the appropriate direction (see Item 1) (cf. Secs. 277 and 281)

Example 1. The curve

$$y=3x^4-4x^3$$

(Fig. 293) at point $A\left(-\frac{1}{3}, \frac{5}{27}\right)$ is concave up, but at point $B\left(\frac{1}{3}, -\frac{1}{9}\right)$ it is concave down because the second derivative

$$y''=36x^2-24x=12x(3x-2)$$

---

1) It is assumed that it exists *near* point *a*.

is positive for $x=-\frac{1}{3}$ [both factors $12x$ and $(3x-2)$ are negative] and negative for $x=\frac{1}{3}$.

At point $O\,(0, 0)$, where $y''=0$, we have an inflection because when passing through $x=0$ the second derivative changes sign from plus (for $x<0$) to minus (for $x>0$). To the left of O the curve is concave up and to the right it is concave down.

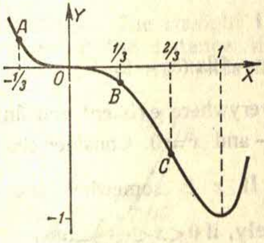


Fig. 293

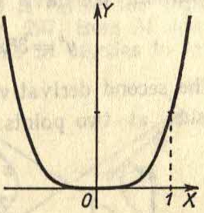


Fig 294

Example 2. The curve $y=x^4$ (Fig. 294) at the point $O\,(0, 0)$, where $y''=0$, is concave up because when passing through $x=0$ the function $y''=12x^2$ preserves the plus sign.

Example 3. The curve $y=-x^{\frac{1}{3}}$ (Fig. 295) has an inflection at the point $O\,(0, 0)$ where the second derivative is infinite, because in passing through $x=0$ the second derivative $y''=+\frac{2}{9}x^{-\frac{5}{3}}$ changes sign from minus to plus. To the left of O the curve is concave down and to the right it is concave up.

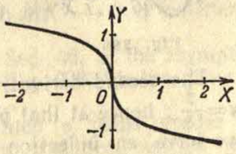


Fig. 295

283. Rule for Finding Points of Inflection

In order to find all the points of inflection of a curve $y=f\,(x)$, it is necessary to test all those values of x for which the second derivative $f''\,(x)$ is zero, infinite or nonexistent (inflections are possible only at such points; Sec. 282).

If in passing through one of these values, the second derivative changes sign, then the curve has a point of inflection at that point. If there is no change, there is no inflection (Sec. 282, Item 2).

Example 1. Find the points of inflection of the curve $y=3x^4-4x^3$.

Solution. We have

$$y''=36x^2-24x=12x(3x-2)$$

The second derivative is everywhere existent and finite; it vanishes at two points: $x=\frac{2}{3}$ and $x=0$. Consider the point $x=\frac{2}{3}$. If x is somewhat less than $\frac{2}{3}$ $\left(\text{namely, if } 0<x<\frac{2}{3}\right)$, then

$$y''=12(+)(-)=-$$

if x is somewhat greater than $\frac{2}{3}$ $\left(\text{in the given case, any number may be taken which is greater than } \frac{2}{3}\right)$, then

$$y''=12(+)(+)=+$$

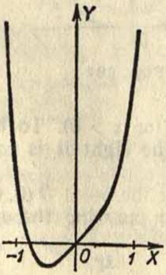


Fig. 296

The second derivative changes sign when passing through $x=\frac{2}{3}$; hence at that point of the graph (point C in Fig. 293) we have an inflection. At $x=0$ there is also an inflection (Sec. 282, Example 1).

Example 2. Find the points of inflection of the curve

$$y=x+2x^4$$

Solution. We have $y''=24x^2$

The second derivative is everywhere finite and vanishes only at $x=0$. When passing through $x=0$, the second derivative preserves the plus sign, as it does everywhere. Hence there is no inflection either here or at any other points. The curve is concave up (Fig. 296).

284. Asymptotes

Let point M (Fig. 297) be in motion in some direction along the curve L from a position M_0. If in such motion the distance M_0M (reckoned along a straight line) increases without bound, than we say that the point M *recedes to infinity*.

Definition. The straight line AB is called the *asymptote* of curve L if the distance MK (Fig. 297) from M (on L) to the straight line AB tends to zero as M recedes to infinity.

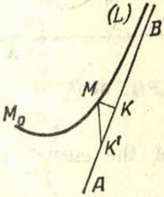


Fig. 297

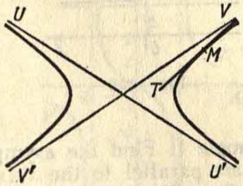


Fig. 298

Note 1. The distance from M to AB may be measured over any *constant* direction MK' and not only along the perpendicular because if $MK \longrightarrow 0$, then $MK' \longrightarrow 0$ as well, and vice versa.

Note 2. The definition, given in Sec. 46, of the asymptotes of a hyperbola ($U'U$ and $V'V$ in Fig. 298) fits the general definition given here.

Note 3. Not every line along which a point recedes to infinity possesses an asymptote. For instance, neither the parabola nor the spiral of Archimedes has asymptotes.

285. Finding Asymptotes Parallel to the Coordinate Axes

1. **Asymptotes parallel to the axis of abscissas.** To find the horizontal asymptotes of a curve $y=f(x)$, seek the limits of $f(x)$ as $x \longrightarrow +\infty$ and as $x \longrightarrow -\infty$.

If $\lim\limits_{x \to \infty} f(x)=b$, then the straight line $y=b$ is an asymptote (in the case of infinite recession to the right; Fig. 299).

If $\lim\limits_{x \to -\infty} f(x) = b'$, then the straight line $y = b'$ is an asymptote (in the case of recession to infinity on the left; Fig. 300).

If $f(x)$ does not have a finite limit either as $x \longrightarrow +\infty$ or as $x \longrightarrow -\infty$, then the curve $y = f(x)$ has no asymptotes parallel to the x-axis.

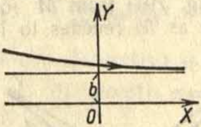


Fig. 299

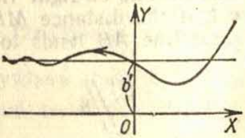


Fig. 300

Example 1. Find the asymptotes of the curve $y = 1 + e^x$ which are parallel to the x-axis.

Solution The function $1 + e^x$, as $x \longrightarrow +\infty$, does not have a finite limit $[\lim\limits_{x \to +\infty} (1 + e^x) = +\infty]$ and tends to unity as $x \longrightarrow -\infty$ Therefore, the straight line $y = 1$ is an asymptote in the case of recession leftwards (Fig. 301).

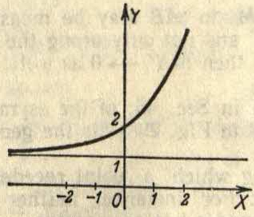


Fig. 301

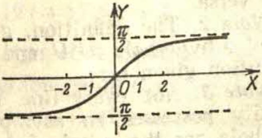


Fig. 302

Example 2. Find the horizontal asymptotes of the curve $y = \arctan x$.

Solution. We have

$$\lim_{x \to +\infty} \arctan x = \frac{\pi}{2}, \quad \lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$$

The straight lines $y = \frac{\pi}{2}$, $y = -\frac{\pi}{2}$ are asymptotes (Fig. 302).

2. Asymptotes parallel to the axis of ordinates. To find the vertical asymptotes of a curve $y=f(x)$, it is necessary to find those values $x_1, x_2, x_3, \ldots$ of the argument x, where $f(x)$ has an infinite limit (one-sided or two-sided). The straight lines $x=x_1$, $x=x_2$, $x=x_3$, $\ldots$ will be the asymptotes. If $f(x)$ does not have an infinite limit for any value of x, then there are no vertical asymptotes.

Example 3. Let us consider the curve $y=\ln x$ (Fig. 303). The function $\ln x$ has an infinite limit on the right as $x \longrightarrow 0$

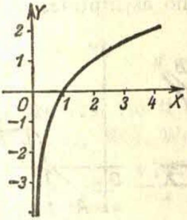


Fig. 303

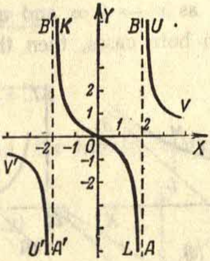


Fig. 304

$\left(\lim\limits_{x\to+0} \ln x=-\infty\right)$. The straight line $x=0$ (axis of ordinates) serves as asymptote in the case of recession to infinity downwards.

Example 4. Find the vertical asymptotes of the curve

$$y=\frac{2x}{x^2-4}$$

Solution. The function $\frac{2x}{x^2-4}$ has an infinite limit as $x \longrightarrow 2$ and $x \longrightarrow -2$.

Hence the straight lines

$$x=2 \text{ and } x=-2$$

(AB and $A'B'$ in Fig. 304) are asymptotes. The straight line AB serves as asymptote to two branches, UV and KL. Along the first, the recession to infinity is upwards, along the second, downwards $\left(\text{because } \lim\limits_{x\to2+0} \frac{2x}{x^2-4}=+\infty \text{ and } \lim\limits_{x\to2-0} \frac{2x}{x^2-4}= =-\infty\right)$. Similarly for the straight line $A'B'$.

Note that the straight line $x=0$ serves as horizontal asymptote (for the branches UV and $U'V'$) (cf. Item 1).

286. Finding Asymptotes Not Parallel to the Axis of Ordinates[1]

To find the asymptotes of a curve $y=f(x)$ which are not parallel to the y-axis, it is necessary first to seek the $\lim \frac{f(x)}{x}$ as $x \longrightarrow +\infty$ and as $x \longrightarrow -\infty$. If there is no finite limit in both cases, then there are no asymptotes.

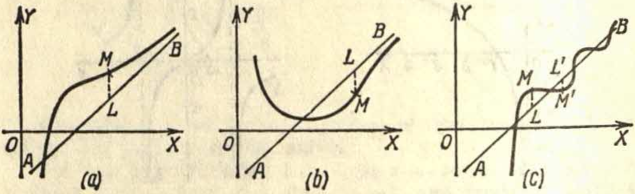


Fig. 305

But if $\lim\limits_{x\to+\infty} \frac{f(x)}{x}=c$, then one must seek $\lim\limits_{x\to+\infty} [f(x)-cx]$. If this limit is equal to d, then the straight line $y=cx+d$ is an *asymptote in the case of recession to infinity on the right*. Similarly, if $\lim\limits_{x\to-\infty} \frac{f(x)}{x}=c'$ and $\lim\limits_{x\to-\infty} [f(x)-c'x]=d'$, then the straight line $y=c'x+d'$ is an *asymptote in the case of recession leftwards*.

If the quantity $f(x)-cx$ [or $f(x)-c'x$] has no finite limit as $x \longrightarrow +\infty$ [as $x \longrightarrow -\infty$], then there is no corresponding asymptote.

The expression $f(x)-(cx+d)$ gives the vertical deviation LM (Fig. 305) of the given curve from its asymptote AB, the equation of which is $y=cx+d$.

If, as $x\to+\infty$, this expression does not change the plus sign from some instant onwards, then the point M approaches the asymptote AB from above (Fig. 305a), if the minus sign, then from below (Fig. 305b).

---

[1] The method described here reveals, for example, horizontal asymptotes (if they exist). But if we are interested only in horizontal asymptotes, then it is simpler to use the method of Sec. 285, Item 1. The method given here does not reveal vertical asymptotes.

If the sign changes, then the point M oscillates about the asymptote (Fig. 305c).

The same goes for the asymptote $y=c'x+d'$.

Example 1. Find the asymptotes of the hyperbola

$$\frac{x^2}{9}-\frac{y^2}{4}=1 \qquad (1)$$

Solution. Eq. (1) is associated with two single-valued functions:

$$y=\frac{2}{3}\sqrt{x^2-9} \qquad (2)$$

and

$$y=-\frac{2}{3}\sqrt{x^2-9} \qquad (3)$$

Consider the first (to it correspond the infinite branches AN and $A'K'$, Fig. 306). We have

$$\lim_{x\to+\infty}\frac{y}{x}=\frac{2}{3}\lim_{x\to+\infty}\frac{\sqrt{x^2-9}}{x}=\frac{2}{3}\ (=c)$$

Further,

$$\lim_{x\to+\infty}(y-cx)=$$

$$=\lim_{x\to+\infty}\left(\frac{2}{3}\sqrt{x^2-9}-\frac{2}{3}x\right)=0\ (=d)$$

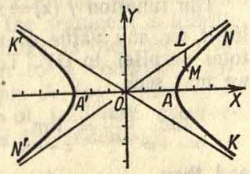


Fig. 306

Consequently, the straight line $y=\frac{2}{3}x$ is an asymptote to the branch AN.

The expression $y-(cx+d)=\frac{2}{3}\sqrt{x^2-9}-\frac{2}{3}x$ preserves the minus sign as $x\to+\infty$. Therefore the branch AN approaches the asymptote from below.

Then we find

$$\lim_{x\to-\infty}\frac{y}{x}=-\frac{2}{3}\ (=c'),$$

$$\lim_{x\to-\infty}(y-c'x)=\lim_{x\to-\infty}\left(\frac{2}{3}\sqrt{x^2-9}+\frac{2}{3}x\right)=0\ (=d')$$

Thus, the straight line $y=-\frac{2}{3}x$ is an asymptote of the branch $A'K'$.

The expression $\frac{2}{3}\sqrt{x^2-9}+\frac{2}{3}x$ preserves the minus sign as $x\to-\infty$. Therefore, branch $A'K'$ also approaches the asymptote from below.

Investigating the function $y=-\frac{2}{3}\sqrt{x^2-9}$ in this fashion (to it correspond the branches AK and $A'N'$), we find that the straight line $y=\frac{-2}{3}x$ is an asymptote to the branch AK, and the straight line $y=\frac{2}{3}x$ is an asymptote to the branch $A'N'$.

Each of the branches AK, $A'N'$ approaches its asymptote from above.

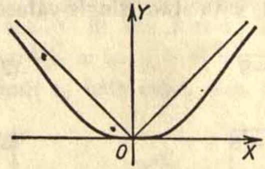


Fig. 307

Example 2. Find all the asymptotes of the curve

$$y=x\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

The function $f(x)=x\frac{e^x-e^{-x}}{e^x+e^{-x}}$ does not have an infinite limit for any value of x. Consequently, there are no asymptotes parallel to OY. To find asymptotes not parallel to OY, we first seek

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to+\infty}\frac{1-e^{-2x}}{1+e^{-2x}}=1\ (=c)$$

and then

$$\lim_{x\to+\infty}[f(x)-cx]=\lim_{x\to+\infty}\frac{-2xe^{-x}}{e^x+e^{-x}}=-\lim_{x\to+\infty}\frac{2x}{e^{2x}+1}=0\quad(=d)$$

Consequently, the straight line $y=x$ is the asymptote of the right infinite branch. Computing the same limits as $x\to-\infty$, we find $c'=-1$, $d'=0$, i.e. the left infinite branch has the asymptote $y=-x$ (Fig. 307).

287. Construction of Graphs (Examples)

The graph of a function given by the formula $y=f(x)$ is constructed by plotting points which are then connected by a smooth curve. However, if the points are taken haphazardly, one can easily make mistakes. 1)

---

1) Thus, if we construct the graph of the function $y=\frac{1}{2}(x+2)^2\times\times(x-1)^3$ (see Fig. 308 below) by plotting the points F, B, L, K (which correspond to the values of the argument -2.5, -0.8, 0, 1.5), the graph will be completely wrong on the segment FB.

In order to draw the graph with extreme accuracy when only a few points are employed, it is useful first to determine its characteristic features. To do this, one has to:

1. Establish in what region the function $f(x)$ is defined and whether it has any discontinuities. Take into account the sign of $f(x)$ on the right and on the left for every infinite discontinuity; we obtain the vertical asymptotes of the graph (Sec. 285).
2. Find the first and second derivatives $f'(x)$, $f''(x)$, and also determine whether there are any points where $f'(x)$ or $f''(x)$ is nonexistent.
3. Find all extrema of the function $f(x)$ (Secs. 278 and 279); we obtain the highest points of crests and the lowest points of troughs.
4. Find all points of inflection (Sec. 283) and the inclination of the tangent line at these points.
5. Establish the existence of horizontal and inclined asymptotes (Sec. 286) if the range of the argument is infinite.

It is useful to tabulate these findings as they are obtained (see examples). Transferring them to a coordinate grid yields a general picture of the graph. A few intermediate points will suffice to yield a curve of sufficient accuracy.

Example 1. Construct the graph of the function[1]

$$f(x)=\frac{1}{2}(x+2)^2(x-1)^3$$

1. The function is defined and continuous everywhere, there are no vertical asymptotes.
2. We find

$$f'(x)=\frac{1}{2}(x+2)(x-1)^2(5x+4),$$

$$f''(x)=(x-1)(10x^2+16x+1)$$

Both derivatives are finite and exist at all points.

3. To find extrema, solve the equation $f'(x)=0$. We find the critical values

$$x_1=-2,\ x_2=-0.8,\ x_3=1$$

Enter these values in the table and also enter the corresponding values of the function

$$f(x_1)=0,\quad f(x_2)\approx-4.20,\quad f(x_3)=0$$

---

[1] An advisable procedure is to compile a table while reading the examples.

Put zeros in the y' column.

It is convenient here to use the second derivative to examine for extrema, and so we postpone the investigation till Item 4.

4. To find the points of inflection, solve the equation $f''(x)=0$, which yields the earlier found value $x_3=1$ and, besides,

$$x_4=-1.5, \quad x_5=-0.07$$

Enter these values and also the corresponding values of the function and its first derivative:

$$f(x_4)=-2.0, \quad f(x_5)=-2.3,$$
$$f'(x_4)=-5.5, \quad f'(x_5)=4.0$$

Put zeros in the y'' column.

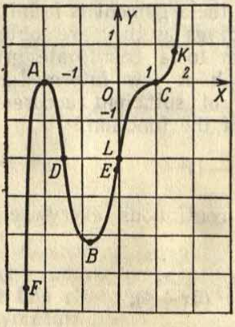


Fig. 308

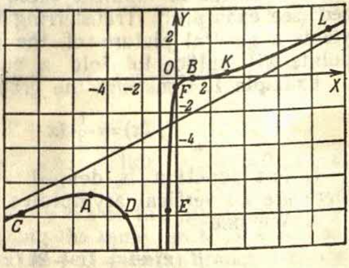


Fig. 309

Determine the sign of $f''(x)$ prior to and after transition through each of the values

$$x=x_3, \quad x=x_4, \quad x=x_5$$

and enter them in the appropriate places of the table. For example, in the third row of the y'' column the entry $-0+$ signifies that $f''(x)$ changes sign from minus to plus as it passes through $x=x_3$ from left to right. Since the second derivative changes sign at each of the points x_3, x_4, x_5, we have an inflection at each of the three points.

Now determine the signs of $f''(x)$ at the critical points $x_1 = -2$ and $x_2 = -0.8$:

$$f''(-2) < 0, \quad f''(-0.8) > 0$$

In the first row of the y'' column put a minus sign, and in the second a plus sign. We have a maximum for $x = x_1$ and a minimum for $x = x_2$.

5. There are no horizontal or inclined asymptotes because $\lim \frac{y}{x} = \infty$.

In Fig. 308 we plot the points we have found (A, B, C, D, E) and indicate the directions of the tangents. Adding another three points, $x_6 = -2.5$, $x_7 = 0$, $x_8 = 1.5$ (F, L, K), we obtain a rather exact graph of the function.

Number of Point	x	y	y'	y''	Extremum, Inflection	Point Labels
1	−2	0	0	−	maximum	*A*
2	−0.8	−4.2	0	+	minimum	*B*
3	1	0	0	−0+	inflection	*C*
4	−1.5	−2.0	−5.5	−0+	inflection	*D*
5	−0.07	−2.3	4.0	+0−	inflection	*E*
6	−2.5	5.4	26			*F*
7	0	−2	4			*L*
8	1.5	0.8	5			*K*

Example 2. Construct the graph of the function $y = \frac{1}{2}\frac{(x-1)^3}{(x+1)^2}$.

1. The function is defined and is continuous everywhere except at $x = -1$ where it has an infinite discontinuity. The function has a minus sign both on the right and on the left of the point of discontinuity (in the column y we write $-\infty$). We obtain the asymptote $x = -1$. Both infinite branches are directed downwards (Fig. 309).

2. We find

$$y' = \frac{1}{2}\frac{(x-1)^2(x+5)}{(x+1)^3}, \quad y'' = 12\frac{x-1}{(x+1)^4}$$

Both derivatives exist at all points except at the point of discontinuity.

3. The equation $f'(x) = 0$ has two roots:

$$x_1 = -5, \quad x_2 = 1$$

The corresponding values of y are

$$y_1 = -6.75, \quad y_2 = 0$$

From the sign of $f'(x)$ near the critical points (see table below) we see that there is a maximum at the point $x=-5$ and there is no extremum at $x=1$.

4. The equation $y''(x)=0$ has a unique root $x_2=1$; the sign of the second derivative (see table) indicates an inflection there.

5. We seek inclined asymptotes; we have, as $x\to+\infty$ and as $x\to-\infty$,

$$\lim\frac{y}{x}=\frac{1}{2},\quad \lim\left(y-\frac{1}{2}x\right)=-\frac{5}{2}$$

Hence, the straight line $y=\frac{1}{2}x-\frac{5}{2}$ serves as asymptote for two infinite branches.

The right branch lies above the asymptote, the left below, since the expression $y-\left(\frac{1}{2}x-\frac{5}{2}\right)$ preserves the plus sign as $x\to+\infty$ and the minus sign as $x\to-\infty$. Incidentally, this is evident from the drawing too when the points C, D, E, F, K, L are labelled.

Number of Point	x	y	y'	y''	Extremum, Inflection, Discontinuities	Point Labels
1	-1	$-\infty$			discontinuity	
2	-5	-6.75	$+0-$		maximum	A
3	1	0	$+0+$	$-0+$	inflection	B
4	-9	-7.81				C
5	-3	-8.00				D
6	-0.5	-6.75				E
7	0	-0.50				F
8	3	0.25				K
9	9	2.56				L

288. Solution of Equations. General Remarks

Algebraic equations of first and second degree are solved by the familiar formulas of algebra. For equations of third and fourth degree, the formulas are complicated, and the general equation of the fifth degree or a higher degree is not solvable in terms of radicals. However, both algebraic and nonalgebraic equations can be solved to the required accuracy if rough approximations are first found, which are then gradually refined.

A rough solution may be found graphically by one of the following methods.

First method. To solve an equation $f(x)=0$ construct a graph of $y=f(x)$ (see Sec. 287) and read off the abscissas of those points where the graph intercepts the x-axis.

Example 1. Solve the equation $x^3-9x^2+24x-18=0$.

Construct (Fig. 310) the graph of $y=x^3-9x^2+24x-18$; take the abscissas $x_1=1.3$, $x_2=3$, $x_3=4.7$. Substitution will show that the second root is exact, the first and third are approximate.

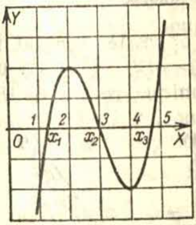


Fig. 310

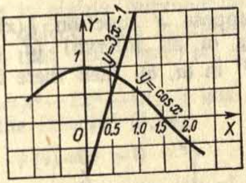


Fig. 311

Second method. The equation $f(x)=0$ may be given in the form $f_1(x)=f_2(x)$, where one of the functions $f_1(x)$, $f_2(x)$ is arbitrary. The arbitrariness is utilized so as to be able to construct the graphs of $y=f_1(x)$ and $y=f_2(x)$ in as simple a manner as possible. Find the points of intersection of the graphs. Reading off their abscissas, we get approximate values of the roots of the equation $f(x)=0$.

Example 2. Solve the equation $3x-\cos x-1=0$.

Give the equation in the following form:

$$3x-1=\cos x$$

Construct, as shown in Fig. 311, the graphs of the functions $y=3x-1$ and $y=\cos x$. They intersect in one point. Taking the abscissa, we get the approximate root $x_1=0.6$.

Secs. 289 to 291 indicate three methods for refining roots. They require that the desired root $\bar{x}$ be *isolated*, i.e. that some interval (a, b) (*interval of isolation*) be known to contain $\bar{x}$ and not to contain any other roots of the equation. The end-points a, b are themselves approximate values of the root (in defect and in excess). They may be found graphically by one of the above-indicated methods. The smaller the interval (a, b), the better.

Example 3. Isolate the roots of the equation $x^3-9x^2+24x-18=0$.

From the graph (Fig. 310), if it is a rough sketch, we read off the interval of isolation (1, 1.5) for the least root. In a more exact construction we get a smaller interval, say (1.2, 1.4). For the largest root we get the interval (4.6, 4.8). The root $x=3$ does not need to be isolated since it is exact.

Note. There are special methods of solving algebraic equations. Lobachevsky's method is worthy of particular mention: it permits, by means of algebraic operations on the coefficients of the equation, finding all roots, including imaginary ones, to any degree of accuracy. Lobachevsky's method does not require separation of roots.

289. Solution of Equations. Method of Chords

Suppose a function $f(x)$ has opposite signs at the endpoints of an interval (a, b) (Fig. 312). If $f'(x)$ preserves sign[1] in (a, b), then there is a unique root $\bar{x}$ of the equation $f(x)=0$ within the interval [if $f'(x)$ changes sign, then there is also a root, but it may not be the only one].

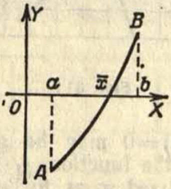


Fig. 312

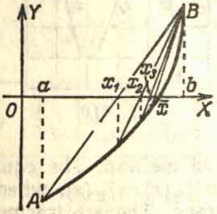


Fig. 313

For the first approximation of the root $\bar{x}$ take the point $x=x_1$ where the chord AB (Fig. 313) intersects the x-axis:

$$x_1=a-\frac{(b-a)f(a)}{f(b)-f(a)} \tag{1}$$

or, what is the same,[2]

$$x_1=b-\frac{(b-a)f(b)}{f(b)-f(a)} \tag{2}$$

---

[1] This means that on AB the curve of the graph is everywhere up or everywhere down.

[2] In symmetric form $x_1=\frac{af(b)-bf(a)}{f(b)-f(a)}$, but formulas (1) and (2) are computationally more convenient.

Then compute $f(x_1)$ and take that one of the intervals (a, x_1), (x_1, b) at the end-points of which $f(x)$ has opposite signs [the interval (x_1, b) in Fig. 313]. The required root lies in this interval. Applying a formula similar to (1), we get the second approximation x_2. Continuing the process, we obtain a sequence $x_1, x_2, \ldots, x_n, \ldots$; the limit of this sequence is the required root $\bar{x}$.

The following is a practical procedure for determining the degree of approximation. Let it be required to obtain an accuracy up to 0.01. We then stop at the approximation x_n which differs from the preceding one by less than 0.01. Incidentally, it may be (though this is highly improbable) that the accuracy will prove to be in defect. The guarantee will be complete if we are convinced that $f(x_n)$ and $f(x_n \pm 0.01)$ have opposite signs.

Example. The function $f(x)=x^3-2x^2-4x-7$ has opposite signs at the end-points of the interval (3, 4):

$$f(3)=-10<0, \quad f(4)=9>0$$

The derivative $f'(x)=3x^2-4x-4$ preserves the plus sign over the interval (3, 4). Hence, within (3, 4) there is one root of the equation

$$x^3-2x^2-4x-7=0$$

Let us find it to within 0.01. Formula (1) yields

$$x_1=3-\frac{1\cdot(-10)}{9-(-10)}=3+\frac{10}{19}\approx 3.53$$

We now compute

$$f(3.53)\approx -2.05$$

Of the two intervals (3, 3.53) and (3.53, 4) we choose the second because the signs of $f(x)$ are opposite at the end-points.

We find the second approximation:

$$x_2=3.53-\frac{0.47\cdot f(3.53)}{f(4)-f(3.53)}\approx 3.53+\frac{0.47\cdot 2.05}{11.05}=3.62$$

The value of

$$f(3.62)=-0.24$$

is negative, and so we take the interval (3.62, 4) and find

$$x_3\approx 3.62+\frac{0.38\cdot 0.24}{9.24}=3.63$$

and

$$f(3.63)=-0.04$$

As the computation proceeds we should expect that x_4 will differ from x_3 by less than 0.01 and that x_3 yields the desired approximation. Since to obtain a complete guarantee, we will compute $f(3.64)$ anyway, we will not determine x_4 and straight off find

$$f(3.64)=0.17$$

The signs of $f(3.63)$ and $f(3.64)$ are opposite, and so x_3 is the desired approximation.

Note. The method of chords, like all methods of successive approximation, "does not fear errors". an error in an intermediate computation will automatically be rectified in the next step. But the final computation must be carried out with extreme care. To avoid errors in rounding off operations, it is useful to retain extra digits.

290. Solution of Equations. Method of Tangents

At the end-points of the interval (a, b) let a function $f(x)$ have opposite signs (Figs. 314 and 315), and let the derivatives $f'(x)$, $f''(x)$ preserve sign in the interval (a, b).[1] To find the root $\bar{x}$ which lies inside the interval (a, b) (Sec. 289), do as follows.

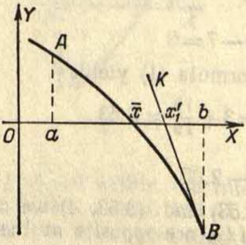


Fig. 314

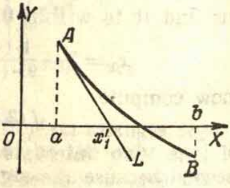


Fig. 315

At the end of the arc AB where the signs of $f(x)$ and $f''(x)$ are the same,[2] draw a tangent (BK in Fig. 314, AL in Fig. 315). For the first approximation of the desired root, take the point $x=x_1'$ [1] where the tangent crosses the x-axis. If the tangent is taken at the point $x=b$, then

---

[1] That is, on segment AB the curve of the graph is always up or always down and everywhere concave up or concave down.

[2] At the upper end if AB is concave up, and the lower end if AB is concave down.

$$x_1'=b-\frac{f(b)}{f'(b)} \qquad (1)$$

but if it is taken at $x=a$, then

$$x_1'=a-\frac{f(a)}{f'(a)} \qquad (2)$$

In both cases the second approximation is found by the formula

$$x_2'=x_1'-\frac{f(x_1')}{f'(x_1')} \qquad (3)$$

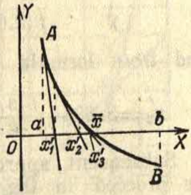
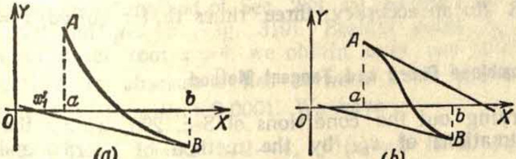


Fig. 316

Continuing the process we find, in succession, x_1', x_2', x_3', ... (Fig. 316). The sequence has as its limit the required root $\bar{x}$. The degree of approximation may be determined in the same way as in the method of chords.

Note 1. If a tangent is drawn at the end of the arc where $f(x)$ and $f''(x)$ have opposite signs, then x_1 may go beyond the interval (a, b) and thus worsen the approximations (Fig. 317*a*).



(*a*) (*b*)

Fig. 317

Note 2. If $f''(x)$ changes sign in (a, b), then the tangents at both ends of the arc can cross the x-axis outside the interval (Fig. 317*b*).

Example. Compute to within 0.01 the root of the equation

$$f(x)=x^3-2x^2-4x-7=0$$

which lies (see Example, Sec. 289) in the interval (3, 4).

---

[1] The labels x_1', x_2', ... are used to distinguish approximations obtained by the method of tangents from the approximations x_1, x_2, ... obtained by the method of chords.

Solution. We have

$$f(3)=-10,\quad f(4)=9,$$
$$f'(x)=3x^2-4x-4,\quad f''(x)=6x-4$$

Both derivatives preserve the plus sign in the interval (3, 4). And so we take that end of the interval where $f(x)>0$, i.e. $b=4$. From formula (1) we find the first approximation:

$$x_1'=4-\frac{f(4)}{f'(4)}=4-\frac{9}{28}\approx 3.68$$

Then we find

$$f(3.68)=1.03,\quad f'(3.68)=21.9$$

and from formula (3) we obtain the second approximation:

$$x_2'=3.68-\frac{f(3.68)}{f'(3.68)}=3.68-0.047=3.633 \quad \text{(in excess)}$$

Subsequent approximations will be less and less, but as we proceed in the computations it may be foreseen that further refinements of the root will not affect the hundreds digit. We therefore confine our computations to $f(3.633)$ and $f(3.630)$. This yields

$$f(3.633)=0.020,\ f(3.630)=-0.042$$

so that (to an accuracy three times that required) $\bar{x}=3.63$.

291. Combined Chord and Tangent Method

Carrying out the conditions of Sec. 290, we see that the approximations of x_n (by the method of chords) and the approximations of x_n' (by the method of tangents) approach the root $\bar{x}$ from opposite directions (the former from the direction of concavity, the latter from the direction of convexity of the graph; see Fig. 318). A joint application of the two methods yields, at once, excessive and defective approximations, and the degree of accuracy estimated directly.

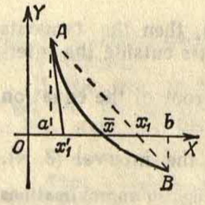


Fig. 318

Let a be that end of the interval (a, b) where the signs of $f(x)$ and

$f''(x)$ are the same. Then by formulas (1), Sec. 289, and (2), Sec. 290, we find[1]

$$x_1 = a - \frac{(b-a)\,f(a)}{f(b)-f(a)}, \quad x_1' = a - \frac{f(a)}{f'(a)} \tag{1}$$

The required root lies between x_1 and x_1'. Here $f(x_1')$ has the same sign as $f''(x_1')$ (see Fig. 318). Hence we can again use formulas (1) of this section by substituting x_1 for a and x_1' for b. This yields the second approximations:

$$x_2 = x_1' - \frac{(x_1 - x_1')\,f(x_1')}{f(x_1) - f(x_1')},$$

$$x_2' = x_1' - \frac{f(x_1')}{f'(x_1')}$$

Use the same formulas for computing x_3, substituting in them x_2 for x_1 and x_2' for x_1' and so on. Continuing this process we find $\bar{x}$ to the desired degree of accuracy.

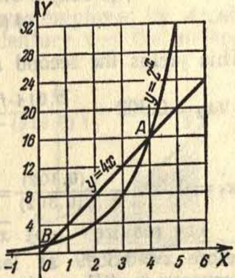


Fig. 319

Example. Solve the equation $2^x = 4x$.

By the second method of Sec. 288, we construct the graphs of $y = 2^x$ and $y = 4x$ (Fig. 319). Besides point A, which yields the exact root $x = 4$, we obtain only one point B of intersection. Its abscissa $\bar{x}$ lies between $a = 0$ and $b = 0.5$.

Compute $\bar{x}$ to within 0.0001. We have

$$f(x) = 2^x - 4x, \quad f'(x) = 2^x \ln 2 - 4, \quad f''(x) = 2^x \ln^2 2, \quad f(0) = 1,$$
$$f(0.5) = -0.586$$

In the interval (0, 0.5) the first derivative preserves the minus sign,[2] the second derivative, the plus sign. To compute x_1', take the end-point $a = 0$ because the signs of $f(x)$

---

[1] For the case when the signs of $f(x)$ and $f''(x)$ are the same at the end-point b, the second formula is replaced by the formula $x_1' = b - \frac{f(b)}{f'(b)}$.

[2] From the figure it is clear that in the interval (0, 0.5) the inclination of $y = 2^x$ is less than that of the graph $y = 4x$.

and $f''(x)$ there are the same. We find

$$x_1 = a - \frac{(b-a)\,f(a)}{f(b)-f(a)} = \frac{0.5\cdot 1}{0.586+1} \approx 0.316 \text{ (in excess)}$$

$$x_1' = a - \frac{f(a)}{f'(a)} = -\frac{1}{\ln 2 - 4} = -\frac{1}{0.69315-4} \approx 0.302 \text{ (in defect)}$$

Using five-place tables of logarithms, we obtain

$$f(0.302) = 0.0249,\ f'(0.302) = -3.14544,$$
$$f(0.316) = -0.0191$$

This yields the second approximations:

$$x_2 = 0.302 - \frac{0.014\cdot f(0.302)}{f(0.316) - f(0.302)} = 0.302 + 0.0079 = 0.3099$$
$$\text{(in excess)}$$

$$x_1' = 0.302 - \frac{f(0.302)}{f'(0.302)} = 0.302 + 0.0079 = 0.3099 \quad \text{(in defect)}$$

The required root $\bar{x}$ lies in the interval (x_2', x_2) and therefore $\bar{x} = 0.3099$ at least to within $0.5\cdot 10^{-4}$. Actually the accuracy is still greater (using seven-place tables of logarithms, we obtain, for the same values of x_1, x_1', the following boundaries of $\bar{x}$: 0.30990 and 0.30991).

INTEGRAL CALCULUS

292. Introductory Remarks

1. **Historical background.** The integral calculus developed out of the need to create a general method for finding areas, volumes, and centres of gravity.

In embryo, a method of this sort was employed by *Archimedes*. However, only in the 17th century was the method systematized in the works of Cavalieri,[1] Torricelli,[1] Fermat, Pascal and other scholars. In 1659 Barrow[2] established a connection between the problem of finding an area and that of finding a tangent. In the seventies of the 17th century, Newton and Leibniz abstracted this relationship from the above-mentioned particular geometrical problems, thus establishing the relationship between integral and differential calculus (see Item 3 below).

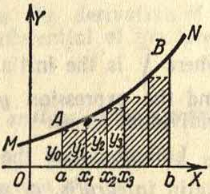


Fig. 320

This relationship was utilized by Newton, Leibniz and their pupils to develop the techniques of integration. In the main, the methods of integration reached their present state in the works of L. Euler. Refinements were introduced in the works of M. V Ostrogradsky[3] and P. L. Chebyshev.[4]

2. **The concept of the integral.** Let a curve *MN* (Fig. 320) be described by the equation

$$y=f(x)$$

and let it be necessary to find the area F of the curvilinear trapezoid $aABb$.

Divide the segment ab into n parts ax_1, x_1x_2, $\ldots$, $x_{n-1}b$ (equal or unequal) and construct the step-like figure shown hatched in Fig. 320. Its area is equal to

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[1] Bonaventura Cavalieri (1591-1647) and Evangelista Torricelli (1608-1647), Italian scholars, pupils of Galileo.

[2] Isaac Barrow (1630-1677). English mathematician, pupil of Newton.

[3] M. V. Ostrogradsky (1801-1861), celebrated Russian mathematician.

[4] P L. Chebyshev (1821-1894), great Russian mathematician, trail blazer in many fields of science.

$$F_n = y_0(x_1 - a) + y_1(x_2 - x_1) + \ldots + y_{n-1}(b - x_{n-1}) \tag{1}$$

If we introduce the notations

$$x_1 - a = dx_0,\ x_2 - x_1 = dx_1,\ \ldots,\ b - x_{n-1} = dx_{n-1} \tag{2}$$

then formula (1) becomes

$$F_n = y_0 dx_0 + y_1 dx_1 + \ldots + y_{n-1} dx_{n-1} \tag{3}$$

The required area is the limit of the sum (3) as n goes to infinity. For this limit, Leibniz introduced the symbol

$$\int y\,dx \tag{4}$$

where $\int$ is the initial letter of the word "summa" (or sum) and the expression $y\,dx$ indicates the typical form of the individual terms.[1]

Leibniz called the expression $\int y\,dx$ the *integral* (from the Latin *integralis*, or whole).[2]

Fourier [3] refined the notation of Leibniz and gave it the form

$$\int_a^b y\,dx \tag{5}$$

where the initial and terminal values of x are shown.

3. **Relationship between integration and differentiation.** Let us consider a a constant and v a variable. Accordingly, we change b to $\bar{x}$. Then the integral

$$\int_a^{\bar{x}} f(x)\,dx$$

which is the area $aABb$ for a fixed ordinate aA and a moving ordinate bB, will be a function of $\bar{x}$. It appears that the differential of this function is equal to $f(\bar{x})\,d\bar{x}$:[1]

---

[1] The concept of a limit had not yet crystallized, and Leibniz spoke of the sum of an infinite number of terms.

[2] This name was suggested by John Bernoulli, one of Leibniz' pupils, in order to be able to distinguish the "sum of an infinite number of terms" from an ordinary sum.

[3] Fourier, J. B. J. (1768-1830), French mathematician and physicist, the founder of the mathematical theory of heat.

$$d\int_{a}^{\bar{x}} f(x)\,dx = f(\bar{x})\,d\bar{x} \qquad (6)$$

4. Principal problem of integral calculus. Thus, the evaluation of the integral (5) reduces to *finding the function from the given expression of its differential*. The fundamental task of integral calculus is to find this function.

293. Antiderivative

Definition. Let a function $f(x)$ be the derivative of a function $F(x)$, that is, $f(x)\,dx$ is the differential of the function $F(x)$:

$$f(x)\,dx = dF(x)$$

Then the function $F(x)$ is called the *antiderivative (primitive)* of the function $f(x)$.

Example 1. The function $3x^2$ is the derivative of x^3, i.e. $3x^2\,dx$ is the differential of the function x^3:

$$3x^2\,dx = d(x^3)$$

By definition, the function x^3 is the antiderivative of the function $3x^2$.

Example 2. The expression $3x^2\,dx$ is the differential of the function x^3+7:

$$3x^2\,dx = d(x^3+7)$$

Hence the function x^3+7 (like the function x^3 too) is an antiderivative of the functions $3x^2$.

Any continuous function $f(x)$ *has an infinity of antiderivatives*. If $F(x)$ is one of them, then any other one may be given by the expression $F(x)+C$, where C is an arbitrary constant.

Example 3. The function $3x^2$ has an infinite number of antiderivatives. One of them (see Example 1) is x^3, any

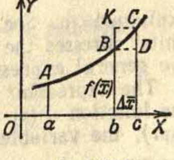

Fig. 321

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[1] This is evident from Fig. 321. The increment ΔF of the area $aABb$ is the area $bBCc$, which may be represented in the form of a sum of the area $bBDc$ + the area BDC. Here, the first term is equal to $bB\cdot bc = f(\bar{x})\,\Delta\bar{x}$, and the second is of higher order than Δx (it is less than area $BDCK = \Delta x\cdot\Delta y$). Hence, (Sec. 228), $f(\bar{x})\,\Delta\bar{x}$ is the differential of the area F.

other one is given by the expression x^3+C, where C is a constant. For $C=7$ we get the antiderivative x^3+7 (Example 2), for $C=0$ we again have the antiderivative x^3.

Example 4. One of the antiderivative functions of $3x^2$ is x^3+7. Any other one is given by the expression x^3+7+C. For $C=-7$, we obtain the antiderivative x^3.

Caution. Any antiderivative of the function $3x^2$ may be represented either in the form x^3+C or x^3+7+C. But these expressions *cannot be equated* because the constants C are not the same. For example, the first expression yields the antiderivative x^3+10 for $C=10$, whereas the second yields it for $C=3$.

If, contrary to this warning, we equate x^3+C and x^3+7+C, we obtain the absurd equality $0=7$. However, we can write

$$x^3+C=x^3+7+C_1$$

where C and C_1 are constants. They are connected by the relation

$$C=C_1+7$$

294. Indefinite Integral

The *indefinite integral* of a given expression $f(x)\,dx$ [or of a given function $f(x)$] is the most general form of its antiderivative.

The indefinite integral of the expression $f(x)\,dx$ is denoted by

$$\int f(x)\,dx$$

A constant term is implied here.

The origin of the symbol $\int$ and the name "integral" is explained in Sec. 292, Items 2 and 3. The word "indefinite" stresses the fact that an arbitrary constant enters into the general expression of the antiderivative.[1]

The expression $f(x)\,dx$ is called the *integrand expression*, the function $f(x)$ is called the *integrand (integrand function)*, the variable x is the *variable of integration*. Finding the indefinite integral of a given function is called *integration.*[1]

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[1] In contrast to the indefinite integral, the limit of the sum $y_0\,dx_0+y_1\,dx_1+\ldots+y_{n-1}\,dx_{n-1}$ (Sec. 292, Item 2) is called the definite integral. The indefinite integral is a *function*. The definite integral is a *number*.

Example 1. The most general form of the antiderivative of the expression $2x\,dx$ is x^2+C. This function is the indefinite integral of the expression $2x\,dx$:

$$\int 2x\,dx = x^2 + C \tag{1}$$

We can also write

$$\int 2x\,dx = x^2 - 5 + C_1 \tag{2}$$

The difference in the designations of the constants (C and C_1) emphasizes that they are not the same ($C = C_1 - 5$; cf. Sec. 293, Caution).

Example 2. Find the indefinite integral of the expression $\cos x\,dx$.

Solution. The function $\cos x$ is the derivative of $\sin x$. Therefore

$$\int \cos x\,dx = \sin x + C$$

Example 3. Find the indefinite integral of the expression $\frac{dx}{x}$.

Solution. The function $\frac{1}{x}$ is discontinuous at $x=0$. We will first consider the positive values of x. Since $d \ln x = \frac{dx}{x}$, it follows that

$$\int \frac{dx}{x} = \ln x + C \tag{3}$$

Since $d \ln 3x = \frac{dx}{x}$, we can write

$$\int \frac{dx}{x} = \ln 3x + C_1 \tag{4}$$

The constants C and C_1 are connected by the relation

$$C = \ln 3 + C_1$$

Similarly, we can write

$$\int \frac{dx}{x} = \ln \frac{x}{7} + C_2 \tag{5}$$

etc. The function $\ln x$ is not defined for negative values of x, and formulas (3), (4) and (5) are unsuitable. On the other hand, the function $\ln(-x)$ is defined: its differential is also equal to $\frac{dx}{x}$. Now we have

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[1] Finding the definite integral is also termed integration.

$$\int \frac{dx}{x} = \ln(-x) + C \tag{6}$$

and, similarly,

$$\int \frac{dx}{x} = \ln(-2x) + C_1, \quad \int \frac{dx}{x} = \ln\left(-\frac{x}{5}\right) + C_2$$

and so forth. Formulas (3) and (6) may be combined:

$$\int \frac{dx}{x} = \ln|x| + C \tag{7}$$

Formula (7) is suitable for any values of x except $x=0$ (cf. Sec. 295, Example 3).

295. Geometrical Interpretation of Integration

Let $f(x)$ be a given continuous function, and $F(x)$ one of its antiderivatives. If we construct a graph PQ of the function $y=F(x)$ (Fig. 322), the slope of the tangent MT will be expressed by the given function $f(x)$.

Let $F_1(x)$ be another antiderivative of the same function $f(x)$. Then the slopes of the tangents MT and M_1T_1 (the points of tangency M, M_1 have the same abscissa x) are the same, i.e. MT is parallel to M_1T_1.

The graph of the antiderivative $F(x)$ is called the *integral curve of the function* $f(x)$ [or of the equation $dy=f(x)\,dx$]. The tangents to the two integral curves at the appropriate points are parallel. At the same time, the two integral curves are separated (vertically) by a constant distance C (MM_1 in Fig. 322) so that it is easy to construct other integral curves if we have one integral curve.

Through each point there passes a unique integral curve.

Integral curves are constructed (in approximate fashion) as follows. Through a number of points (see, for example, Fig. 323) densely populating some portion of a plane, draw short segments (or arrows) indicating the directions of the tangent lines.

This gives us a "direction field" (or tangent field). Then draw freehand a smooth curve so that it touches the arrows at a number of points. The result is one integral curve. Others can be constructed in the same fashion.

Example 1. Find the integral curves of the equation

$$dy = dx$$

In this example, the function $f(x)$ is the constant 1. The slope of all arrows is equal to unity, i.e. the inclination of the tangent line is everywhere equal to 45°. The integral curves (Fig. 323) are parallel straight lines. The equation of each one of them is $y = \int dx$, i.e. $y = x + C$. The quantity C, which is constant for each curve, varies from one straight line to another.

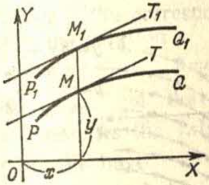

Fig. 322

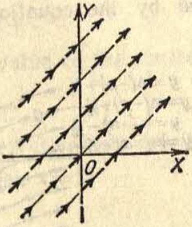

Fig. 323

Example 2. Find the integral curves of the function $\frac{1}{2}x$ $\left(\text{that is, of the equation } dy = \frac{1}{2}x\,dx\right)$.

In Fig. 324, along the y-axis ($x=0$) take horizontal arrows $\left(\frac{1}{2}x = 0\right)$; along the ordinate $x=1$ take arrows with slope $\frac{1}{2}x = \frac{1}{2}$, etc. Drawing the integral curves freehand, we obtain "parallel" parabolas $\left(y = \int \frac{1}{2}x\,dx = \frac{1}{4}x^2 + C\right)$.

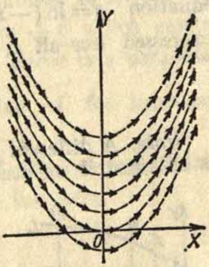

Fig. 324

Example 3. Fig. 325 depicts the integral curves of the function $\frac{1}{x}$. Not one of them crosses the y-axis since for

$x=0$ the antiderivatives are not defined $\left(\text{the function } \frac{1}{x}\right.$ is discontinuous at $\left.x=0\right)$. For this reason, only those integral curves are equidistant from one another which lie on one side of the axis of ordinates. Those on the right are described by the equation $y=\ln x+C$, on the left, by the equation $y=\ln(-x)+C$. The indefinite integral $\int \frac{dx}{x}$ is expressed (for all x, except $x=0$) by the formula

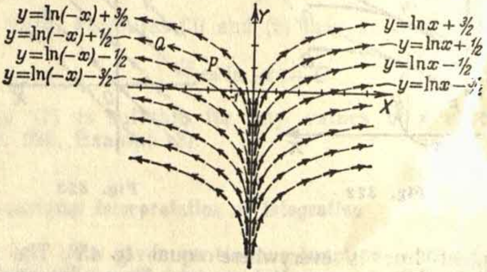

Fig. 325

$$\int \frac{dx}{x}=\ln|x|+C$$

Note. A different geometrical interpretation of integration is obtained if we draw the graph KL (Fig. 326) of the given function $f(x)$. Let the arc KL lie entirely above the x-axis. Draw two ordinates aA and mM. Consider the left-hand ordinate aA as fixed and the right-hand one mM as moving. The area $aAMm$ will be one of the antiderivatives of the function $f(x)$ of the argument $x=Om$ (cf. Sec. 292, Item 2). Taking, in place of aA, the fixed ordinate bB, we obtain another antiderivative, the area $bBMm$. These two antiderivatives differ by the constant quantity $C=$ area $aABb$.

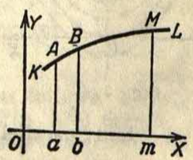

Fig. 326

296. Computing the Integration Constant from Initial Data

Of the multitude of antiderivatives of a given function $f(x)$, only one can assume the given value b for a given value of the argument $x=a$. If the indefinite integral

$$\int f(x)\,dx=F(x)+C$$

is known, then the corresponding value of the constant C is found from the relation

$$b=F(a)+C$$

Example 1. Find that antiderivative of the function $\frac{1}{2}x$ which assumes the value 3 when $x=2$.

Solution. We have

$$\int \frac{1}{2}x\,dx=\frac{1}{4}x^2+C \tag{1}$$

We find the constant C from the relation $3=\frac{1}{4}\cdot 2^2+C$. It is $C=2$. Substituting into (1) we obtain the desired antiderivative function

$$y=\frac{1}{4}x^2+2 \tag{2}$$

Geometrically the problem may be formulated as follows: to find the integral curve of the function $\frac{1}{2}x$ which passes through the point (2, 3). The required curve is a parabola (Fig. 324).

Example 2. Find that antiderivative of the function $\frac{1}{x}$ which assumes the value $\frac{1}{2}$ for $x=-1$.

Solution. For negative x, the indefinite integral of the function $\frac{1}{x}$ (Sec. 294, Example 3) is of the form

$$\int \frac{dx}{x}=\ln(-x)+C \tag{3}$$

It is given that

$$\frac{1}{2}=\ln 1+C \tag{4}$$

whence

$$C=\frac{1}{2}$$

The required function in $\ln(-x)+\frac{1}{2}$. To it corresponds the integral curve PQ in Fig. 325.

297. Properties of the Indefinite Integral

1. The sign of the differential in front of the sign of the integral cancels the latter:

$$d\int f(x)\,dx = f(x)\,dx \tag{1}$$

(by the definition of the indefinite integral).

To put it otherwise: the derivative of an indefinite integral is equal to the integrand:

$$\frac{d}{dx}\int f(x)\,dx = f(x) \tag{2}$$

Example.

$$d\int 2x\,dx = d\,(x^2+C) = 2x\,dx, \tag{1a}$$

$$\frac{d}{dx}\int 2x\,dx = 2x$$

2. The sign of the integral in front of the sign of the differential cancels the latter, but introduces an arbitrary additive constant.

Example.

$$\int d\sin x = \sin x + C \tag{3}$$

3. A constant factor may be taken outside the sign of the integral:

$$\int a\,f(x)\,dx = a\int f(x)\,dx \tag{4}$$

Example.

$$\int 6x\,dx = 6\int x\,dx = 6\left(\frac{1}{2}x^2 + C\right) = 3x^2 + 6C = 3x^2 + C_1$$

where $C_1 = 6C$.

4. The integral of an algebraic sum is equal to the sum of the integrals of the summands. For three summands:

$$\int [f_1(x) + f_2(x) - f_3(x)]\,dx = \int f_1(x)\,dx + \\ + \int f_2(x)\,dx - \int f_3(x)\,dx \tag{5}$$

Similarly for any other (fixed) number of terms.

Example.

$$\int (5x^2-2x+4)\,dx=\int 5x^2\,dx-\int 2x\,dx+\int 4\,dx=$$
$$=\left(\frac{5}{3}x^3+C_1\right)-(x^2+C_2)+(4x+C_3)=$$
$$=\frac{5}{3}x^3-x^2+4x+C$$

where

$$C=C_1-C_2+C_3$$

Note. There is no need, in intermediate computations, to write out the constant term for every integral. It suffices to adjoin it after all integrations have been performed.

298. Table of Integrals

If inverted, every formula of differentiation becomes a corresponding formula of integration. Thus, from the formula

$$d\ln(x+\sqrt{a^2+x^2})=\frac{dx}{\sqrt{a^2+x^2}} \tag{1}$$

we obtain the formula[1]

$$\int \frac{dx}{\sqrt{a^2+x^2}}=\ln(x+\sqrt{a^2+x^2})+C \tag{2}$$

Of the following ten formulas the first nine are obtained by inverting the basic formulas of differentiation; the tenth coincides with (2). Its derivation is given in Example 1, Sec. 312.

$$\text{I. } \int x^n\,dx=\frac{x^{n+1}}{n+1}+C \quad (n\neq-1)$$

$$\text{II. } \int \frac{dx}{x}=\ln|x|+C\ ^{2)}$$

$$\text{III. } \int e^x\,dx=e^x+C$$

$$\text{IIIa. } \int a^x\,dx=\frac{a^x}{\ln a}+C$$

---

[1] The quantity $x+\sqrt{a^2+x^2}$ is positive for any x and, for this reason, in (2) we do not write $\ln|x+\sqrt{a^2+x^2}|$.

[2] Cf. Sec. 294, Example 3.

$$\text{IV.} \quad \int \sin x\, dx = -\cos x + C$$

$$\text{V.} \quad \int \cos x\, dx = \sin x + C$$

$$\text{VI.} \quad \int \frac{dx}{\sin^2 x} = -\cot x + C$$

$$\text{VII.} \quad \int \frac{dx}{\cos^2 x} = \tan x + C$$

$$\text{VIII.} \quad \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\text{VIIIa.} \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C.$$

$$\text{IX.} \quad \int \frac{dx}{1+x^2} = \arctan x + C$$

$$\text{IXa.} \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\text{X.} \quad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C$$

These formulas should be memorized (in each of the three pairs of formulas III, VIII, IX, only one need be remembered, the one labelled "a").

In formula IXa, unlike VIIIa, the arc sign is preceded by the factor $\frac{1}{a}$. This is connected with the *dimensions* of the expression $\frac{dx}{a^2+x^2}$: in the numerator, dx is the first power, in the denominator we have second powers in a and x. The dimensionality is equal to -1; the right-hand side has the same dimensionality because of the factor $\frac{1}{a}$.

In formula VIIIa the expression $\frac{dx}{\sqrt{a^2-x^2}}$ is of zero dimensionality, like the right-hand side.

Note 1. Formulas I to X are best learnt gradually, as they occur in exercises. It is useful also to know five more formulas.[1]

$$\text{XI.} \quad \int \tan x\, dx = -\ln |\cos x| + C$$

---

[1] Like the formulas given in the Appendix, pp. 846-853 they are all derived from formulas I-X in accord with the rules given in Secs. 300-302.

$$\text{XII.}\quad \int \cot x\,dx = \ln|\sin x| + C$$

$$\text{XIII.}\quad \int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right| + C$$

$$\text{XIV.}\quad \int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$$

$$\text{XV.}\quad \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Note 2. The integrals XIII and XIV can also be expressed as follows:

$$\text{XIIIa.}\quad \int \frac{dx}{\sin x} = \ln|\csc x - \cot x| + C$$

$$\text{XIVa.}\quad \int \frac{dx}{\cos x} = \ln|\sec x - \tan x| + C$$

In this latter form, it is easier to see their mutual relationship, but for calculations they are not so convenient as those given earlier.

299. Direct Integration

Taking advantage of Properties 3 and 4, Sec. 297, it is possible in a number of cases to reduce integration to the tabular formulas of Sec. 298.

Example 1.

$$\int (3\sqrt{x} - 4x)\,dx = 3\int x^{1/2}\,dx - 4\int x\,dx =$$

$$= 3\,\frac{x^{3/2}}{\frac{3}{2}} - 4\,\frac{x^2}{2} + C = 2x\sqrt{x} - 2x^2 + C$$

In the first transformation we employed the properties of Sec. 297, in the second, the standard (tabular) formula I. The constant C appears when the integral signs are removed.

Example 2.

$$\int (2\sin t - 3\cos t)\,dt = 2\int \sin t\,dt - 3\int \cos t\,dt =$$

$$= -2\cos t - 3\sin t + C$$

(using formulas IV and V).

Example 3.

$$\int \frac{\sin^3\varphi + 1}{\sin^2\varphi}\,d\varphi = \int \sin\varphi\,d\varphi + \int \frac{d\varphi}{\sin^2\varphi} = -\cos\varphi - \cot\varphi + C$$

(using formulas IV and VI).

Example 4.

$$\int (x^2+1)^4 x^3 dx = \int (x^{11}+4x^9+6x^7+4x^5+x^3)\, dx =$$
$$= \frac{1}{12}x^{12} + \frac{2}{5}x^{10} + \frac{3}{4}x^8 + \frac{2}{3}x^6 + \frac{1}{4}x^4 + C$$

300. Integration by Substitution (Change of Variable)

In place of x we can introduce into the integrand expression $f(x)\,dx$ an auxiliary variable z connected with x by a certain relation.[1] Let the transformed expression be $f_1(z)\,dz$;[2] then $\int f(x)\,dx = \int f_1(z)\,dz$. If the integral $\int f_1(z)\,dz$ is of tabular form or reduces to such more easily than the original one, then the transformation achieves its aim.

There is no general answer to the question of how to choose a good substitution (cf. Sec. 309); rules for important particular cases are given below in connection with examples.

Example 1. $\int \sqrt{2x-1}\, dx$.

No suitable tabular integral is available, but by formula I it is possible to compute the integral $\int \sqrt{x}\, dx$, which is similar to the given one. So let us introduce an auxiliary variable z connected with x by the relation

$$2x-1=z \tag{1}$$

Differentiating (1), we get

$$2dx=dz \tag{2}$$

The expression under the integral sign $\sqrt{2x-1}\,dx$ is transformed to $\sqrt{z}\,\frac{dz}{2}$ by means of (1) and (2), and we get

$$\int \sqrt{2x-1}\, dx = \int \sqrt{z}\,\frac{dz}{2} = \frac{1}{2}\cdot\frac{z^{3/2}}{\frac{3}{2}} + C = \frac{1}{3}z^{3/2} + C \tag{3}$$

---

[1] It is assumed that the function $x=\varphi(z)$ which expresses this relation has a continuous derivative.

[2] We have $f_1(z)\,dz = f[\varphi(z)]\,\varphi'(z)\,dz$.

Returning to the variable x, we obtain

$$\int \sqrt{2x-1}\,dx = \frac{1}{3}(2x-1)^{3/2} + C$$

Checking by differentiation, we get

$$d\left[\frac{1}{3}(2x-1)^{3/2} + C\right] = \frac{1}{3} \cdot \frac{3}{2}(2x-1)^{1/2}\, d(2x-1) = \sqrt{2x-1}\,dx$$

Here, the function $2x-1$ is again used as an auxiliary function (cf. Sec. 237).

Note 1. In simple cases there is no need to introduce a new letter. In Example 1, for instance, where we took the auxiliary function $2x-1$, we see by inspection that its differential is $d(2x-1) = 2dx$. Introduce the factor 2 in the expression under the integral sign. To compensate, put $\frac{1}{2}$ in front of the integral sign. We thus have

$$\frac{1}{2}\int \sqrt{2x-1}\,2dx = \frac{1}{2}\int (2x-1)^{1/2}\,d(2x-1) = \frac{1}{2}\,\frac{(2x-1)^{3/2}}{\frac{3}{2}} + C$$

Rule 1. If the integrand (as in Example 1) is of the form $f(ax+b)$, the substitution $ax+b=z$ may prove useful.

Example 2. $\int \frac{dx}{(8-3x)^2}$.

Introducing the auxiliary function $8-3x=z$, we find $dx = -\frac{dz}{3}$ and

$$\int \frac{dx}{(8-3x)^2} = \int -\frac{dz}{3z^2} = \frac{1}{3z} + C = \frac{1}{3(8-3x)} + C$$

Example 3. $\int \frac{dx}{6x-7}$.

We take $6x-7$ as the auxiliary function. Without introducing a literal notation for it (see Note 1), we find (by means of II)

$$\int \frac{dx}{6x-7} = \frac{1}{6}\int \frac{6dx}{6x-7} = \frac{1}{6}\int \frac{d(6x-7)}{6x-7} = \frac{1}{6}\ln|6x-7| + C$$

Example 4. $\int e^{3x}\,dx$ (auxiliary function $3x$).

$$\int e^{3x}\,dx = \frac{1}{3}\int e^{3x}\,d(3x) = \frac{1}{3}e^{3x} + C$$

Example 5.

$$\int \cos\frac{x+1}{3}\,dx=3\int \cos\frac{x+1}{3}\,d\left(\frac{x+1}{3}\right)=3\sin\frac{x+1}{3}+C$$

Rule 2. Let the expression under the integral sign be split up into two factors and let it be easy to recognize in one of them the differential of some function $\varphi(x)$. It may happen that after the substitution $\varphi(x)=z$ the second factor becomes a function of z that we are able to integrate. Then the substitution will be useful.

Example 6. $\int \frac{2x\,dx}{1+x^2}$

Break up the integrand expression into factors $\frac{1}{1+x^2}$ and $2x\,dx$. The factor $2x\,dx$ is the differential of the function $1+x^2$ in the denominator of the other factor. After the substitution $1+x^2=z$, the factor $\frac{1}{1+x^2}$ will take the form $\frac{1}{z}$. We can integrate this function. The computation may be performed as follows:

$$\int \frac{2x\,dx}{1+x^2}=\int \frac{d\,(1+x^2)}{1+x^2}=\ln(1+x^2)+C$$

Note 2. The external similarity of this integral to the standard form $\int \frac{dx}{1+x^2}$ in the table of integrals is deceiving. The presence of the factor $2x$ in the numerator changes the form of the antiderivative essentially.

Example 7. $\int \sin x\cos^3 x\,dx$.

Split the integrand expression into the factors $\cos^3 x$ and $\sin x\,dx=-d\cos x$. The substitution $\cos x=z$ transforms $\cos^3 x$ into the function z^3, which we can integrate. The computation is performed as follows:

$$\int \sin x\cos^3 x\,dx=-\int \cos^3 x\,d\cos x=-\frac{\cos^4 x}{4}+C$$

Example 8. $\int \frac{x\,dx}{\sqrt{a^2-x^2}}$.

The similarity to the standard integral VIIIa is deceiving. Introduce the auxiliary function $a^2-x^2=z$. We then have $-2x\,dx=dz$, i.e. $x\,dx=-\frac{dz}{2}$ The integral takes the form

$$\int -\frac{dz}{2\sqrt{z}}=-\sqrt{z}+C.$$

The computation is performed as follows:

$$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\frac{1}{2}\int \frac{-2x\,dx}{\sqrt{a^2-x^2}} = -\frac{1}{2}\int \frac{d\,(a^2-x^2)}{\sqrt{a^2-x^2}} =$$
$$= -\sqrt{a^2-x^2}+C$$

Example 9. $\int \frac{5x\,dx}{\sqrt{a^4-x^4}}$.

The auxiliary function x^2 is introduced to yield

$$\int \frac{5x\,dx}{\sqrt{a^4-x^4}} = \frac{5}{2}\int \frac{2x\,dx}{\sqrt{a^4-x^4}} = \frac{5}{2}\int \frac{d\,(x^2)}{\sqrt{(a^2)^2-(x^2)^2}} = \frac{5}{2}\arcsin\frac{x^2}{a^2}+C$$

Example 10. $\int \frac{\ln^2 x\,dx}{x} = \int \ln^2 x\,d\ln x = \frac{1}{3}\ln^3 x + C.$

It is not always easy to distinguish a good substitution from an unlucky one. This is seen from Examples 11 and 12.

Example 11. $\int (x^2+1)^4 x^3\,dx.$

Here the substitution $x^2+1=z$ is good. The integrand expression is broken up into the factors $x\,dx=\frac{1}{2}\,dz$ and $(x^2+1)^4 x^2 = z^4(z-1)$. This yields

$$\int (x^2+1)^4 x^3\,dx = \frac{1}{2}\int z^4(z-1)\,dz = \frac{1}{2}\int z^5\,dz - \frac{1}{2}\int z^4\,dz =$$
$$= \frac{1}{12}(x^2+1)^6 - \frac{1}{10}(x^2+1)^5 + C$$

(cf. Example 4, Sec. 299, where the same integral was found without substitution).

Example 12. $\int (x^2+1)^4 x^2\,dx.$

Here, the substitution $x^2+1=z$ is not good. It yields an integral, $\frac{1}{2}\int z^4\sqrt{z-1}\,dz$, which is more involved than the original one. The given integral is best evaluated directly, as in Example 4, Sec. 299. This yields $\frac{1}{11}x^{11}+\frac{4}{9}x^9+\frac{6}{7}x^7+$ $+\frac{4}{5}x^5+\frac{1}{3}x^3+C.$

Example 13. $\int \frac{dx}{(1+x^2)\arctan x} = \ln|\arctan x| + C$ (the auxiliary function is $\arctan x$).

Example 14. $\int \frac{y^2\,dy}{\sqrt{1-y^6}}=\frac{1}{3}\arcsin y^3+C$ (the auxiliary function is y^3).

Example 15. $\int \frac{u^3\,du}{\sqrt{1-u^4}}=-\frac{1}{2}\sqrt{1-u^4}+C$ (the auxiliary function is $1-u^4$).

Example 16. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx=\ln(e^x+e^{-x})+C$ (the auxiliary function is e^x+e^{-x}).

Example 17. $\int \frac{\sin x\,dx}{\cos^4 x}=\frac{1}{3\cos^3 x}+C$ (the auxiliary function is $\cos x$).

Example 18. $\int \frac{\sin^3 x\,dx}{\cos^5 x}=\frac{1}{4}\tan^4 x+C$ $\left(\text{the auxiliary function is } \tan x\text{; the integrand expression is } \frac{\tan^3 x\,dx}{\cos^2 x}\right)$

301. Integration by Parts

Any expression under the integral sign may be represented in infinitely many ways in the form $u\,dv$ (u and v are functions of the variable of integration).

Integration by parts is the reduction of a given integral $\int u\,dv$ to the integral $\int v\,du$ by means of the formula

$$\int u\,dv=uv-\int v\,du \tag{1}$$

This device suits the purpose if $\int v\,du$ is evaluated more easily than $\int u\,dv$ (Examples 1 to 4) or if one of these integrals can be expressed in terms of the other (Example 5).

Example 1. $\int e^x x\,dx$.

We represent the integrand in the form $x(e^x\,dx)=x\,de^x$. Here the role of u is played by x, the role of v, by the function e^x. Using formula (1) we have

$$\int x\,de^x=xe^x-\int e^x\,dx$$

The integral $\int e^x\,dx$ is standard (of tabular form). The computation is performed as follows:

$$\int e^x x\,dx=\int x\,de^x=xe^x-\int e^x\,dx=xe^x-e^x+C$$

Note 1. If the integrand is given in the form $e^x d\left(\frac{1}{2}x^2\right)$, i.e. if we take $u=e^x$, $v=\frac{1}{2}x^2$, then by formula (1) we get

$$\int e^x d\left(\frac{1}{2}x^2\right)=\frac{1}{2}x^2e^x-\int\frac{1}{2}x^2e^x\,dx$$

The integral $\int\frac{1}{2}x^2e^x\,dx$ is no easier than the original one.

The expression $e^x x\,dx$ may be given in the form $u\,dv$ in an infinity of ways by taking an arbitrary function for v. Thus, if we take $v=x^4$, then $dv=4x^3\,dx$ Then $e^x x\,dx=\frac{e^x}{4x^2}(4x^3\,dx)$ or $u=\frac{e^x}{4x^2}$. But formula (1) again leads to an integral which is more complicated than the original one.

Before integrating by parts it is necessary first to make a guess as to what the choice of the function v will yield.

Example 2. $\int x\ln x\,dx$.

Here it is well to represent the integrand function in the form $\ln x\,d\left(\frac{1}{2}x^2\right)$. Formula (1) $\left(\text{for } u=\ln x,\ v=\frac{1}{2}x^2\right)$ yields

$$\int\ln x\,d\left(\frac{1}{2}x^2\right)=\ln x\left(\frac{1}{2}x^2\right)-\int\frac{1}{2}x^2\,d\ln x=$$
$$=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x^2\frac{dx}{x}$$

The integral $\int\frac{1}{2}x^2\frac{dx}{x}=\frac{1}{2}x\,dx$ is equal to $\frac{1}{4}x^2+C$ so that

$$\int x\ln x\,dx=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$$

Example 3. $\int x\sin x\,dx$.

We have

$$\int x \sin x\, dx = \int x\, d(-\cos x) = -x \cos x - \int (-\cos x)\, dx =$$
$$= -x \cos x + \sin x + C$$

Example 4. $\int x^2 \cos x\, dx$.

We have

$$\int x^2 \cos x\, dx = \int x^2\, d \sin x = x^2 \sin x - 2 \int x \sin x\, dx$$

To the integral obtained we again apply integration by parts (see Example 3). This finally yields

$$\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

Example 5. $\int e^x \cos x\, dx$.

Represent the integrand as $e^x d \sin x$:

$$\int e^x \cos x\, dx = e^x \sin x - \int \sin x\, e^x\, dx + C_1 \qquad (2)$$

This integral is not simpler than the original one, but it can be expressed in terms of the original one. To do this, integrate it by parts once again:

$$-\int \sin x\, e^x\, dx = \int e^x\, d \cos x = e^x \cos x - \int \cos x\, e^x\, dx + C_2 \quad (3)$$

Substituting (3) into (2), we obtain the equation

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx + C_1 + C_2 \quad (4)$$

from which we find the unknown $\int e^x \cos x\, dx$:

$$\int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$$

where C denotes $\frac{C_1 + C_2}{2}$.

Note 2. We can represent the integrand as $\cos x\, de^x$. Then in the second integration as well we will have to represent the new expression $e^x \sin x\, dx$ in the form $\sin x\, de^x$ (and not in the form $e^x\, d \cos x$), otherwise the equation for determining $\int e^x \cos x\, dx$ will become an identity.

302. Integration of Some Trigonometric Expressions

Rule 1. For evaluating integrals of the form

$$\int \cos^{2n+1} x\,dx, \quad \int \sin^{2n+1} x\,dx \tag{1}$$

(where n is an integer) it is convenient to introduce the auxiliary function $\sin x$ in the first case and $\cos x$ in the second.

Example 1.

$$\int \cos^3 x\,dx = \int (1-\sin^2 x)\,d\sin x = \sin x - \frac{1}{3}\sin^3 x + C$$

Example 2.

$$\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx = -\int (1-\cos^2 x)^2\, a \cos x =$$

$$= -\int (1 - 2\cos^2 x + \cos^4 x)\,d\cos x = -\cos x + \frac{2}{3}\cos^3 x - $$

$$-\frac{1}{5}\cos^5 x + C$$

For even powers of $\sin x$ or $\cos x$, Rule 1 does not achieve our aim (see Rule 2).

Rule 2. For evaluating integrals of the type

$$\int \cos^{2n} x\,dx, \quad \int \sin^{2n} x\,dx \tag{2}$$

it is convenient to use the formulas

$$\cos^2 x = \frac{1+\cos 2x}{2}, \tag{3}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \tag{4}$$

and introduce the auxiliary function $\cos 2x$.

Example 3.

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Example 4.

$$\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2x\,dx +$$

$$+\frac{1}{4}\int \cos^2 2x\,dx$$

The first two integrals can be evaluated at once; again apply formula (3) to the third, rewriting it in the form

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

This yields

$$\int \cos^4 x\,dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\,dx =$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$$

It remains only to collect terms.

Rule 3. In evaluating integrals of the form

$$\int \cos^m x \sin^n x\,dx \tag{5}$$

where at least one of the numbers m, n is odd, it is convenient to introduce the auxiliary function $\cos x$ (if m is odd) or $\sin x$ (if n is odd) and proceed as in Examples 1 and 2.

Example 5. $\int \cos^6 x \sin^5 x\,dx$.

Here we have an odd power of the sine. Represent the integrand as

$$\cos^6 x \sin^4 x\,d(-\cos x) = -\cos^6 x\,(1-\cos^2 x)^2\,d\cos x$$

We get

$$\int \cos^6 x \sin^5 x\,dx = -\int \cos^6 x\,d\cos x + 2\int \cos^8 x\,d\cos x -$$

$$-\int \cos^{10} x\,d\cos x = -\frac{1}{7}\cos^7 x + \frac{2}{9}\cos^9 x - \frac{1}{11}\cos^{11} x + C$$

When both numbers m, n are even, Rule 3 does not achieve our aim (see Rule 4).

Rule 4. In evaluating integrals of form (5), where m and n are even numbers, it is convenient to use the formulas

$$\cos^2 x = \frac{1+\cos 2x}{2}, \tag{3}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}, \tag{4}$$

$$\sin x \cos x = \frac{\sin 2x}{2} \tag{6}$$

Example 6. $\int \cos^4 x \sin^2 x\, dx$.

Representing the integrand as

$$(\cos x \sin x)^2 \cos^2 x\, dx$$

and applying (6) and (3), we get

$$\int \cos^4 x \sin^2 x\, dx = \frac{1}{8}\int \sin^2 2x\,(1+\cos 2x)\, dx =$$

$$= \frac{1}{8}\int \sin^2 2x\, dx + \frac{1}{8}\int \sin^2 2x \cos 2x\, dx$$

Transform the first summand by formula (4) and rewrite as

$$\sin^2 2x = \frac{1-\cos 4x}{2}$$

Compute the second summand in terms of the auxiliary function $\sin 2x$. We obtain

$$\int \cos^4 x \sin^2 x\, dx = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$$

Rule 5. In evaluating integrals of the form

$$\int \sin mx \cos nx\, dx, \tag{7}$$

$$\int \sin mx \sin nx\, dx, \tag{8}$$

$$\int \cos mx \cos nx\, dx \tag{9}$$

it is convenient to take advantage of the transformations

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \tag{7'}$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \tag{8'}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x] \tag{9'}$$

Example 7.

$$\int \sin 5x \cos 3x\, dx = \frac{1}{2}\int [\sin(5-3)x + \sin(5+3)x]\, dx =$$

$$= -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C$$

Rule 6. In evaluating integrals of the form

$$\int \tan^n x\,dx, \quad \int \cot^n x\,dx$$

(n is an integer greater than 1) it is convenient to separate out the factor $\tan^2 x$ (or $\cot^2 x$).

Example 8. $\int \tan^5 x\,dx$.

Taking out the factor $\tan^2 x = \sec^2 x - 1 = \frac{1}{\cos^2 x} - 1$, we get

$$\int \tan^5 x\,dx = \int \tan^3 x \frac{dx}{\cos^2 x} - \int \tan^3 x\,dx$$

The first integral is equal to $\frac{1}{4}\tan^4 x$. The second is computed by the same procedure:

$$\int \tan^3 x\,dx = \int \tan x \frac{dx}{\cos^2 x} - \int \tan x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x$$

Finally,

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$$

303. Trigonometric Substitutions

For integrands containing the radicals

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2}, \quad \sqrt{x^2 - a^2}$$

(and also the squares of these radicals, $a^2 - x^2$, $x^2 \pm a^2$), it is often convenient to use the following substitutions:

for the case $\sqrt{a^2 - x^2}$, the substitution $x = a\sin t$,
for the case $\sqrt{x^2 + a^2}$, the substitution $x = a\tan t$,
for the case $\sqrt{x^2 - a^2}$, the substitution $x = a\sec t$.

Example 1. $\int \sqrt{a^2 - x^2}\,dx$.

Putting $x = a\sin t$, we get1)

$$\sqrt{a^2 - x^2} = a\cos t, \quad dx = a\cos t\,dt \tag{1}$$

---

1) The radical sign is taken under the assumption that $-\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$.

Hence

$$\int \sqrt{a^2-x^2}\,dx = a^2 \int \cos^2 t\,dt = \frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right)+C \qquad (2)$$

see (3), Sec. 302. Returning to the variable x, we find

$$t=\arcsin\frac{x}{a},\quad \frac{1}{2}\sin 2t=\sin t\cos t=\frac{x\sqrt{a^2-x^2}}{a^2} \qquad (3)$$

Finally we have

$$\int \sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$$

Example 2. $\int \frac{dx}{(x^2+a^2)^2}$.

Putting $x=a\tan t$, we obtain

$$x^2+a^2=a^2(\tan^2 t+1)=\frac{a^2}{\cos^2 t},\quad dx=\frac{a\,dt}{\cos^2 t}$$

Consequently

$$\int \frac{dx}{(x^2+a^2)^2}=\frac{1}{a^3}\int \cos^2 t\,dt=\frac{1}{2a^3}\left(t+\frac{1}{2}\sin 2t\right)+C$$

Returning to the variable x, we find

$$t=\arctan\frac{x}{a},\quad \frac{1}{2}\sin 2t=\sin t\cos t=\frac{ax}{a^2+x^2}$$

Finally

$$\int \frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^3}\left(\operatorname{arc\,tan}\frac{x}{a}+\frac{ax}{a^2+x^2}\right)+C$$

Example 3. $\int \frac{dx}{x\sqrt{x^2-a^2}}$.

Putting $x=a\sec t$, we obtain[1]

$$\sqrt{x^2-a^2}=a\tan t,\quad dx=a\tan t\sec t\,dt$$

Hence

$$\int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\int dt=\frac{1}{a}t+C=\frac{1}{a}\operatorname{arcsec}\frac{x}{a}+C=$$

$$=\frac{1}{a}\arccos\frac{a}{x}+C$$

---

[1] The radical sign is taken under the assumption that $-\frac{\pi}{2}\leqslant t\leqslant\frac{\pi}{2}$.

304. Rational Functions

An *integral rational function* of an argument x is a function given by the polynomial

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \tag{1}$$

A *fractional rational function* is a ratio of integral rational functions:

$$\frac{b_0x^m + b_1x^{m-1} + \ldots + b_{m-1}x + b_m}{a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n} \tag{2}$$

If the degree of the numerator is less than that of the denominator, the fraction (2) is called *proper*, otherwise it is *improper*.

Examples. The function $\frac{0.3x^2 + \sqrt{2}x}{\sqrt{3}}$ is an integral rational function. The functions $\frac{2x^2-1}{4x^3-5}$, $\frac{3x^2+\pi}{x^2+4}$ are fractional rational functions. The first fraction is proper, the second, improper. The function $\frac{2\sqrt{x}}{x-1}$ is irrational.

304a. Taking out the Integral Part

It is possible, by means of division with a remainder, to *take out the integral part* of an improper fraction; i. e. an improper fraction may be represented in the form of a sum of an integral rational function and a proper fraction. It may happen that the division is exact; then the improper fraction is an integral function.

Example 1. After taking out the integral part, the improper fraction $\frac{4x^3-16x}{15x^2-3}$ becomes $\frac{4}{15}x - \frac{15\frac{1}{5}x}{15x^2-3}$ $\left(\frac{4}{15}x\right.$ is the quotient and $-15\frac{1}{5}x$ is the remainder after dividing the numerator by the denominator$\left.\right)$.

Example 2. $\frac{1+x^5-x^6}{1-x} = x^5 + \frac{1}{1-x}$.

This result is obtained by dividing $-x^6+x^5+1$ by $-x+1$, or, more concisely, as follows:

$$\frac{1+x^5-x^6}{1-x} = \frac{1}{1-x} + \frac{x^5(1-x)}{1-x} = \frac{1}{1-x} + x^5$$

Example 3. Taking out the integral part of the fraction $\frac{x^3-x^2}{x-1}$, we obtain an integral rational function x^2 (the division is exact).

305. Techniques for Integrating Rational Fractions

When integrating an improper rational fraction, first take out the integral part (Sec. 304a).

Example 1.

$$\int \frac{1+x^5-x^6}{1-x}\,dx=\int \left(x^5+\frac{1}{1-x}\right) dx=\frac{x^6}{6}-\ln|1-x|+C$$ [1]

(cf. Sec. 304a, Example 2).

Since the integral part may be integrated directly, the integration of any fractional rational function reduces to the integration of a proper fraction. For this there is a general method (Sec. 307), which, however, often involves arduous computations. It is therefore useful, wherever possible, to take advantage of peculiarities of the integrand.

If the numerator of the integrand is equal to the differential of the denominator (or differs from it by a constant factor), then the denominator should be taken as the auxiliary function.

Example 2.

$$\int \frac{(2x^3+6x^2+7x+3)\,dx}{x^4+4x^3+7x^2+6x+2}=\frac{1}{2}\int \frac{d\,(x^4+4x^3+7x^2+6x+2)}{x^4+4x^3+7x^2+6x+2}=$$

$$=\frac{1}{2}\ln(x^4+4x^3+7x^2+6x+2)+C$$

The technique is similar when in the numerator we have the differential of some polynomial, and in the denominator we have a power of the same polynomial.

Example 3.

$$\int \frac{(3x^2+1)\,dx}{x^2(x^2+1)^2}=\int \frac{d\,(x^3+x)}{(x^3+x)^2}=-\frac{1}{x^3+x}+C$$

If the numerator and denominator have a common factor, it is often useful to cancel it.

Example 4. $\int \frac{(x^2-x-2)\,dx}{x^3+x^2+x+1}$.

---

[1] We could use the substitution $1-x=z$ without first taking out the integral part, but the computation would be longer.

Here the fraction can be simplified by cancelling out $x+1$. This yields

$$\int \frac{(x-2)\,dx}{x^2+1} = \frac{1}{2}\ln(x^2+1) - 2\arctan x + C$$

Note 1. It is sometimes senseless to reduce a fraction. For instance, in Example 2, the fraction may be given in the form

$$\frac{(x+1)(2x^2+4x+3)}{(x+1)^2(x^2+2x+2)}$$

and $x+1$ may be cancelled. But it is more difficult to evaluate the integral

$$\int \frac{(2x^2+4x+3)\,dx}{(x+1)(x^2+2x+2)}$$

than the original one, to say nothing of the fact that factoring is another rather considerable difficulty.

Note 2. The general method of integrating rational fractions consists in decomposing the given fraction into a sum of so-called *partial fractions*. These fractions are defined in Sec. 306 and ways of integrating them are given. Partial fraction decomposition is explained in Sec. 307.

306. Integration of Partial Rational Fractions

Partial rational fractions are fractions that reduce to the following two types:

I. $\frac{A}{(x-a)^n}$ (n a natural number)

II. $\frac{Mx+N}{(x^2+px+q)^n}$ (n a natural number)

where x^2+px+q cannot be factored into real linear factors $\left[\text{i. e., } q-\left(\frac{p}{2}\right)^2>0\right]$; if x^2+px+q can be factored into real linear factors $\left[\text{i. e., } q-\left(\frac{p}{2}\right)^2\leqslant 0\right]$, then fraction II is not considered a partial fraction.

The fractions $\frac{5}{x+2}$, $\frac{\sqrt{3}}{(x-\sqrt{2})^3}$ are partial of the first type, the fractions $\frac{0\cdot 2}{x^2+1}$, $\frac{7x-1}{x^2+2}$, $\frac{5(x+4)}{x^2+\sqrt{3}}$ are partial of the second type. The fractions $\frac{1}{x^2-1}$, $\frac{3x-2}{(x^2-\sqrt{3})^3}$ are not partial because the expressions x^2-1, $x^2-\sqrt{3}$ can be factored into real linear factors.

The fraction $\frac{3}{2x-9}$ is a partial fraction since it can be put in the form $\frac{\frac{3}{2}}{x-\frac{9}{2}}$. The fraction $\frac{18x-3}{(x^2+x+1)^2}$ is partial because it is of type II.

(A) *Partial fractions of the first type* are integrated by the formulas

$$\int \frac{A\,dx}{(x-a)^n} = -\frac{1}{n-1}\,\frac{A}{(x-a)^{n-1}} + C \quad (n > 1), \tag{1}$$

$$\int \frac{A\,dx}{x-a} = A \ln|x-a| + C \tag{2}$$

(B) *Partial fractions of the second type in the case* $n=1$ are integrated completely by the substitution

$$x+\frac{p}{2}=z$$

which reduces the denominator

$$x^2+px+q=\left(x+\frac{p}{2}\right)^2+q-\left(\frac{p}{2}\right)^2$$

to the form z^2+k^2 $\left[\text{where } k^2=q-\left(\frac{p}{2}\right)^2\right]$.

Example 1.

$$\int \frac{3x-5}{x^2-8x+25}\,dx \left[p=-8, \quad q=25; \quad q-\left(\frac{p}{2}\right)^2=9\right]$$

The substitution

$$x-4=z$$

transforms the integral to

$$\int \frac{3z+7}{z^2+9}\,dz = 3\int \frac{z\,dz}{z^2+9} + 7\int \frac{dz}{z^2+9} = \frac{3}{2}\ln(z^2+9) +$$

$$+\frac{7}{3}\arctan\frac{z}{3}+C$$

Returning to the argument x, we obtain

$$\int \frac{3x-5}{x^2-8x+25}\,dx = \frac{3}{2}\ln(x^2-8x+25)+\frac{7}{3}\arctan\frac{x-4}{3}+C$$

The formula (which need not be memorized) is of the form

$$\int \frac{Mx+N}{x^2+px+q}\,dx=\frac{M}{2}\ln(x^2+px+q)+$$
$$+\frac{2N-Mp}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}}+C$$

(C) *Partial fractions of the second type for the case* $n>1$ are integrated by the same substitution

$$x+\frac{p}{2}=z$$

which transforms the integral $\int \frac{Mx+N}{(x^2+px+q)^n}\,dx$ to

$$\int \frac{Mz+L}{(z^2+k^2)^n}\,dz \tag{3}$$

$\left[\text{where } L=\frac{2N-Mp}{2},\ k^2=q-\left(\frac{p}{2}\right)^2\right].$

The first term $\int \frac{Mz\,dz}{(z^2+k^2)^n}$ is integrated directly via the auxiliary function z^2+k^2

$$\int \frac{Mz\,dz}{(z^2+k^2)^n}=-\frac{M}{2}\,\frac{1}{(n-1)(z^2+k^2)^{n-1}}+C \tag{4}$$

The second term $L\int \frac{dz}{(z^2+k^2)^n}$ is evaluated by a trigonometric substitution (Sec. 303, Example 2) or by the *reduction formula*[1]

$$\int \frac{dz}{(z^2+k^2)^n}=\frac{1}{2(n-1)k^2}\left[\frac{z}{(z^2+k^2)^{n-1}}+(2n-3)\int \frac{dz}{(z^2+k^2)^{n-1}}\right] \tag{5}$$

(it can be checked by differentiation). It reduces the integral $\int \frac{dz}{(z^2+k^2)^n}$ to an integral of the same type but with the exponent n in the denominator diminished by unity. Repeating the procedure we finally arrive at the integral

$$\int \frac{dz}{z^2+k^2}=\frac{1}{k}\arctan\frac{z}{k}+C$$

---

[1] A reduction formula is any formula which expresses some quantity, dependent on the number n $\left[\text{in our case } \int \frac{dz}{(z^2+k^2)^n}\right]$, in terms of the same quantity with smaller absolute value of n. Reduction formulas are also called *recursion formulas*.

Example 2. $\int \frac{(3x-2)\,dx}{(x^2-2x+3)^3}$.

The substitution $x-1=z$ leads to an integral of the type

$$\int \frac{3z+1}{(z^2+2)^3}\,dz=3\int \frac{z\,dz}{(z^2+2)^3}+\int \frac{dz}{(z^2+2)^3} \qquad (6)$$

The first term is equal to

$$\frac{3}{2}\int \frac{d\,(z^2+2)}{(z^2+2)^3}=-\frac{3}{4\,(z^2+2)^2} \qquad (7)$$

The constant C is dropped and attached to the second term, which is computed from the formula (5) (putting $k^2=2$, $n=3$):

$$\int \frac{dz}{(z^2+2)^3}=\frac{1}{8}\,\frac{z}{(z^2+2)^2}+\frac{3}{8}\int \frac{dz}{(z^2+2)^2} \qquad (8)$$

Use formula (5) again, putting $k^2=2$, $n=2$:

$$\int \frac{dz}{(z^2+2)^2}=\frac{1}{4}\,\frac{z}{z^2+2}+\frac{1}{4}\int \frac{dz}{z^2+2}=\frac{1}{4}\,\frac{z}{z^2+2}+\frac{1}{4\sqrt{2}}\arctan\frac{z}{\sqrt{2}}+C \qquad (9)$$

From formulas (6) to (9) we find

$$\int \frac{3z+1}{(z^2+2)^3}\,dz=-\frac{3}{4\,(z^2+2)^2}+\frac{1}{8}\,\frac{z}{(z^2+2)^2}+\frac{3}{32}\,\frac{z}{z^2+2}+$$
$$+\frac{3}{32\sqrt{2}}\arctan\frac{z}{\sqrt{2}}+C=\frac{3z^3+10z-24}{32\,(z^2+2)^2}+$$
$$+\frac{3}{32\sqrt{2}}\arctan\frac{z}{\sqrt{2}}+C$$

Returning to the variable x, we obtain

$$\int \frac{(3x-2)\,dx}{(x^2-2x+3)^3}=\frac{3x^3-9x^2+19x-37}{32\,(x^2-2x+3)^2}+\frac{3}{32\sqrt{2}}\arctan\frac{x-1}{\sqrt{2}}+C$$

307. Integration of Rational Functions (General Method)

Rational functions are integrated by the general method as follows:

1. From the given function take out the integral part; it can be integrated directly (Sec. 305, Example 1).

2. Factor the denominator of the remaining proper fraction into real factors of the type $x-a$ and x^2+px+q (*linear terms and irreducible quadratics* [1]).

The factorization is of the form

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=a_0(x-a)(x-b)\ldots$$
$$\ldots(x^2+px+q)(x^2+rx+s)\ldots \qquad (1)$$

Such a factorization is always possible,[2] and it is unique.

3. We attempt to divide the numerator of the proper fraction by each factor of expression (1). If the division is exact, we reduce the fraction by the appropriate factor (Sec. 305, Example 4).

4. Decompose the fraction obtained into a sum of partial fractions and integrate the terms separately (Sec. 306).

Note 1. Every proper fraction is decomposed into a sum of partial fractions in just one way. The method of decomposition is explained below. For a proper understanding, we consider four cases which exhaust all possibilities.

Case 1. Only linear factors enter into the factorization of the denominator and not one of them is repeated.

Then the proper fraction is decomposed into partial fractions by the formula

$$\frac{F(x)}{a_0(x-a)(x-b)\ldots(x-l)}=\frac{A}{x-a}+\frac{B}{x-b}+\ldots+\frac{L}{x-l} \qquad (2)$$

where the constants $A, B, \ldots, L$ are found (by the method of undetermined coefficients) as follows.

(a) Clear of fractions in equality (2).

(b) Equate coefficients of like powers of x on both sides (it may happen that the left-hand side lacks a needed term; in that case we assume the coefficient 0). We thus obtain a system of linear equations for the unknowns $A, B, \ldots, L$.

(c) Solve the system (it always has a unique solution).

Example 1. Evaluate $\int \frac{7x-5}{x^3+x^2-6x}\,dx$.

Solution. The given fraction is a proper fraction. Factor the denominator:

$$x^3+x^2-6x=x(x-2)(x+3) \qquad (3)$$

---

[1] If we have a term x^2+px+q that can be factored into the real factors $x-m$ and $x-n$, then we replace it by these two factors.

[2] In the most elementary cases it is carried out by rearrangement of terms and other algebraic procedures. For the general case see Sec. 308.

The numerator is not divisible by any of the factors, so we cannot cancel. All factors are linear and not one is repeated.

By formula (2)

$$\frac{7x-5}{x(x-2)(x+3)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+3} \qquad (4)$$

To find the constants A, B, C, clear of fractions. This yields

$$7x-5=A(x-2)(x+3)+B(x+3)x+C(x-2)x \qquad (5)$$

or

$$7x-5=(A+B+C)x^2+(A+3B-2C)x-6A \qquad (6)$$

Equate coefficients of like powers of x (on the left-hand side we presume $0\cdot x^2$). The result is the system

$$\left.\begin{aligned} A+B+C&=0,\\ A+3B-2C&=7,\\ -6A&=-5 \end{aligned}\right\} \qquad (7)$$

Solving it we get

$$A=\frac{5}{6},\quad B=\frac{9}{10},\quad C=-\frac{26}{15} \qquad (8)$$

and from (4) we obtain the following decomposition of the given fraction into partial fractions:

$$\frac{7x-5}{x(x-2)(x+3)}=\frac{5}{6}\,\frac{1}{x}+\frac{9}{10}\,\frac{1}{x-2}-\frac{26}{15}\,\frac{1}{x+3}$$

Integrating term by term, we get the desired integral

$$\int\frac{7x-5}{x^3+x^2-6x}\,dx=\frac{5}{6}\ln|x|+\frac{9}{10}\ln|x-2|-\frac{26}{15}\ln|x+3|+C$$

Note 2. The constants A, B, C can also be found in the following manner: take any three values of x and substitute them into (5) We again get a system of three equations, which yields the same values as in (8).

This remark refers to Cases 2, 3 and 4 as well. But in Case 1 this technique can be simplified still more by taking such values of x as make the denominators of the partial fractions vanish; in the given example, the values $x=0$, $x=2$, $x=-3$. Then we get a system of equations $-5=-6A$, $9=10B$, $-26=15C$, which yields the values of A, B, C directly.

Case 2. Only linear factors enter into the factorization of the denominator, and some of them are repeated.

Let a factor $x-a$ be repeated k times. Then in the decomposition (2) it is necessary to replace the corresponding k identical terms by a sum of partial fractions of the form

$$\frac{A_k}{(x-a)^k}+\frac{A_{k-1}}{(x-a)^{k-1}}+\ldots+\frac{A_1}{x-a} \tag{9}$$

The same applies to other repeated factors. Partial fractions corresponding to nonrepeated factors remain the same. The constants of the factorization are determined as in Case 1.

Example 2. Evaluate $\int \frac{(x^3+1)\,dx}{x^4-3x^3+3x^2-x}$.

Solution. The factorization of the denominator is of the form

$$x^4-3x^3+3x^2-x=x(x-1)^3$$

All factors are linear. The factor x is not repeated, the factor $x-1$ is repeated three times. As in Example 1, to the nonrepeated factor corresponds a partial fraction of the form $\frac{A}{x}$, to the repeated factor $(x-1)$, a sum of three partial fractions of the form

$$\frac{B}{(x-1)^3}+\frac{C}{(x-1)^2}+\frac{D}{x-1}$$

The decomposition of the fraction is

$$\frac{x^3+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{(x-1)^3}+\frac{C}{(x-1)^2}+\frac{D}{x-1}$$

Clearing of fractions, we get

$$x^3+1=A(x-1)^3+Bx+Cx(x-1)+Dx(x-1)^2 \tag{10}$$

or

$$\begin{aligned}x^3+1=(A+D)x^3+(-3A+C-2D)x^2+\\+(3A+B-C+D)x-A\end{aligned} \tag{11}$$

Equate coefficients of like powers of x; this yields

$$\left.\begin{aligned}A+D=1,\\-3A+C-2D=0,\\3A+B-C+D=0,\\-A=1\end{aligned}\right\} \tag{12}$$

Solving this system, we get

$$A=-1,\ B=2,\ C=1,\ D=2$$

The decomposition of the fraction is

$$\frac{x^3+1}{x(x-1)^3}=-\frac{1}{x}+\frac{2}{(x-1)^3}+\frac{1}{(x-1)^2}+\frac{2}{x-1}$$

Integrating term by term we find

$$\int \frac{(x^3+1)\,dx}{x^4-3x^3+3x^2-x} = -\ln|x| - \frac{1}{(x-1)^2} - \frac{1}{x-1} +$$
$$+2\ln|x-1| + C = -\frac{x}{(x-1)^2} + \ln\frac{(x-1)^2}{|x|} + C$$

Alternative version. If in (10) we first put $x=0$ and then $x=1$ (see Note 2), we straightway get $A=-1$, $B=2$. Putting in (10) two more values, say $x=2$ and $x=-1$, and taking into account the values of A and B that have been found, we obtain the following system of equations: $2C+2D=6$, $2C-4D=-6$, and from it we find $C=1$, $D=2$.

This method is particularly convenient when there are many nonrepeated factors in the factorization of the denominator, and the multiplicity of the repeated terms is not great.

Case 3. The factorization of the denominator includes irreducible quadratic terms none of which are repeated.

Then, in the partial fraction decomposition, to each factor x^2+px+q there corresponds a partial fraction $\frac{Mx+N}{x^2+px+q} =$ = (Type II). As before, to linear factors (if there are any) there correspond partial fractions of Type I.

Example 3. Find $\int \frac{(7x^2+26x-9)\,dx}{x^4+4x^3+4x^2-9}$.

Solution. Factor the denominator:

$$x^4+4x^3+4x^2-9=(x^2+2x)^2-3^2=(x^2+2x+3)(x^2+2x-3)$$

We obtained two factors of the type x^2+px+q, but only the first is irreducible to real linear factors:

$$\left[q-\left(\frac{p}{2}\right)^2=3-1^2=2>0\right]$$

The second, however,

$$\left[q-\left(\frac{p}{2}\right)^2=-3-1^2=-4<0\right]$$

can be factored:

$$x^2+2x-3=(x-1)(x+3)$$

Therefore, the decomposition into partial fractions is of the form[1]

---

[1] It is not a mistake to seek a decomposition of the form

$$\frac{A'x+B'}{x^2+2x-3}+\frac{Cx+D}{x^2+2x+3}$$ (cont'd on p. 436)

$$\frac{7x^2+26x-9}{(x^2+2x+3)(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}+\frac{Cx+D}{x^2+2x+3} \tag{13}$$

Clearing of fractions, we get

$$7x^2+26x-9=(x^2+2x+3)[A(x+3)+B(x-1)]+ \\ +(Cx+D)(x-1)(x+3) \tag{14}$$

We equate the coefficients of like powers of x:

$$\left.\begin{aligned} A+B+C&=0, \\ 5A+B+2C+D&=7, \\ 9A+B-3C+2D&=26, \\ 9A-3B-3D&=-9 \end{aligned}\right\} \tag{15}$$

Solving the system (15), we obtain

$$A=1, \quad B=1, \quad C=-2, \quad D=5$$

so that

$$\frac{7x^2+26x-9}{(x^2+2x+3)(x-1)(x+3)}=\frac{1}{x-1}+\frac{1}{x+3}+\frac{-2x+5}{x^2+2x+3}$$

Integrating (see Sec. 306, Case B) we find

$$\int\frac{(7x^2+26x-9)\,dx}{x^4+4x^3+4x^2-9}=\ln|x-1|+\ln|x+3|- \\ -\ln(x^2+2x+3)+\frac{7}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C= \\ =\ln\left|\frac{x^2+2x-3}{x^2+2x+3}\right|+\frac{7}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C$$

Alternative version. In order to determine A and B, in (14) we first put $x=1$ and then $x=-3$ (see Note 2). This yields the simple equations $24=24A$, $-24=-24B$ and we find

$$A=1, \quad B=1$$

Setting $x=0$ in (14), we get $-9=9A-3B-3D$, whence $D=5$. Putting $x=-1$, we get $C=-2$.

Case 4. The denominator factors into irreducible quadratics and some of them are repeated.

In the given example the computation is even simplified: we get $A'=2$, $B'=2$ and find

$$\int\frac{(2x+2)\,dx}{x^2+2x-3}=\ln|x^2+2x-3|+C$$

In the general case, we would still have to decompose the fraction $\frac{A'x+B'}{x^2+2x-3}$ into a sum of the partial fractions $\frac{A}{x-1}+\frac{B}{x+3}$.

Then in the decomposition of the fraction, to every factor x^2+px+q repeated k times there corresponds a sum of partial fractions of the form

$$\frac{M_kx+N_k}{(x^2+px+q)^k}+\frac{M_{k-1}x+N_{k-1}}{(x^2+px+q)^{k-1}}+\ldots+\frac{M_1x+N_1}{x^2+px+q} \qquad (16)$$

Example 4. Find $\int\frac{(3x+5)\,dx}{x^5+2x^3+x}$.

Solution. Factor the denominater:

$$x^5+2x^3+x=x\,(x^4+2x^2+1)=x\,(x^2+1)^2$$

The term x^2+1 cannot be factored into real linear factors; it is repeated twice. Therefore the partial fraction decomposition is of the form

$$\frac{3x+5}{x\,(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)^2}+\frac{Dx+E}{x^2+1}$$

Clearing of fractions, we get

$$3x+5=A\,(x^2+1)^2+(Bx+C)\,x+(Dx+E)\,x\,(x^2+1)$$

Equate coefficients of like powers of x:

$$A+D=0,\quad E=0,\quad 2A+B+D=0,$$
$$C+E=3,\quad A+E=5$$

Solving the system, we obtain

$$A=5,\quad B=-5,\quad C=3,\quad D=-5,\quad E=0$$

so that

$$\int\frac{(3x+5)\,dx}{x^5+2x^3+x}=5\int\frac{dx}{x}+\int\frac{(-5x+3)\,dx}{(x^2+1)^2}-5\int\frac{x\,dx}{x^2+1}$$

Computing the middle integral, as explained in Sec. 306 (Case C), we get

$$\int\frac{(3x+5)\,dx}{x^5+2x^3+x}=5\ln|x|+\left[\frac{5}{2\,(x^2+1)}+\frac{3x}{2\,(x^2+1)}+\right.$$
$$\left.+\frac{3}{2}\arctan x\right]-\frac{5}{2}\ln(x^2+1)+C=5\ln\frac{|x|}{\sqrt{x^2+1}}+$$
$$+\frac{3x+5}{2\,(x^2+1)}+\frac{3}{2}\arctan x+C$$

Note 3. The integral of any rational function can theoretically be expressed (cf. Examples 1 to 4) in terms of the logarithms of rational functions, in terms of inverse trigonometric functions and the "algebraic part" (i.e. a rational function). But, as a rule, factors of the type $x-a$, x^2+px+q (i.e. the linear and irreducible quadratic terms into which the denominator of any rational function can be factored) can only be found in approximate fashion (see Sec. 308).

Incidentally, using a method discovered by M. V. Ostrogradsky, the *algebraic* part can always be expressed exactly because it can be found without factoring the denominator.

308. Factoring a Polynomial

The factoring of the polynomial

$$a_0x^n+a_1x^{n-1}+\dots+a_n \tag{1}$$

reduces to solving the equation

$$a_0x^n+a_1x^{n-1}+\dots+a_n=0 \tag{2}$$

Indeed, if we know some root x_1 of Eq. (2), then the polynomial (1) is exactly divisible by $x-x_1$ and we obtain a factorization of the form

$$a_0x^n+a_1x^{n-1}+\dots+a_n=a_0\,(x-x_1)\,(x^{n-1}+b_1x^{n-2}+\dots+b_{n-1}) \tag{3}$$

By methods explained in higher algebra it is always possible to find (approximately, but to any degree of accuracy) one of the roots of any numerical algebraic equation.[1] However, the root x_1 may be imaginary.

After we have obtained the expansion (3) we can find root x_2 of the equation $x^{n-1}+b_1x^{n-2}+\dots+b_{n-1}=0$. The number x_2 will at the same time be a root of Eq. (2). We then have the following factorization for the polynomial (1):

$$a_0x^n+a_1x^{n-1}+\dots+a_n=a_0(x-x_1)(x-x_2)(x^{n-2}+c_1x^{n-3}+\dots+c_{n-2}) \tag{4}$$

etc. Finally, we obtain an expansion into n (real or imaginary) linear factors:

$$a_0x^n+a_1x^{n-1}+\dots+a_n=a_0\,(x-x_1)\,(x-x_2)\dots(x-x_n) \tag{5}$$

This expansion is unique. The numbers $x_1, x_2, \dots, x_n$ are roots of Eq. (2). These numbers are also called the roots of polynomial (1). Some of the roots may prove to be equal. But in this case too it is considered that Eq. (2) has n roots: each of the roots is counted once, twice, etc., depending on how many times the corresponding factor is repeated in the expansion (5).

If all the coefficients of the polynomial (1) are real, then to any complex root $\alpha+\beta i$ there corresponds another complex root $\alpha-\beta i$ (*conjugate roots*). If one of the conjugate roots is repeated, then the other is also repeated the same number of times.

The product of two complex conjugate factors $x-(\alpha+\beta i)$ and $x-(\alpha-\beta i)$ yields a *real* polynomial of the form

$$x^2+px+q$$

Here

$$p=-2\alpha,\quad q=\alpha^2+\beta^2,\quad q-\left(\frac{p}{2}\right)^2=\beta^2>0$$

---

[1] Division of polynomial (1) by a binomial $x-x_1'$, where x_1' is an approximate value of the root, yields a remainder q (equal to the value of the polynomial for $x=x_1'$). q approaches zero as x_1' approaches x_1.

Hence, *every polynomial* (with real coefficients) *can be factored into real factors of the type* $x-x_k$ and x^2+px+q (the latter is an irreducible quadratic).

Note. Although the numbers x_k, p, q which enter into the factors $x-x_k$, x^2+px+q are real, they are, as a rule, irrational. What is more, when the polynomial (1) is of degree five or higher, these numbers cannot, as a rule, be expressed exactly even in terms of radicals. It is therefore not always possible by far to give an exact partial fraction decomposition of a rational function.

309. On the Integrability of Elementary Functions

As a rule, the integral of a rational function *is not a rational function* $\left(\text{for example, } \int \frac{dx}{x} = \ln x + C\right)$. Similarly, as a rule, the integral of an elementary (nonrational) function *is not an elementary function.*

Thus, the integrals

$$\int \frac{x\,dx}{\sqrt{1-x^3}}, \quad \int \frac{dx}{\sqrt{1-x^3}}, \quad \int \frac{dx}{\ln x}, \quad \int \frac{x\,dx}{\ln x}$$

cannot be expressed in terms of elementary functions,[1] although the following integrals, which look very much like them,

$$\int \frac{x\,dx}{\sqrt{1-x^2}}, \quad \int \frac{dx}{\sqrt{1-x^2}}, \quad \int \ln x\,dx, \quad \int x \ln x\,dx$$

are elementary functions.

By the rules of differential calculus, we can find for any elementary function the derivative (which is also elementary). In integral calculus, such rules for finding the antiderivative are *fundamentally impossible*.

But for certain classes of elementary functions the integral is always an elementary function (though it is often an involved expression). In Sec. 307 we studied one such class (rational functions). In Secs. 310-313 we will consider other important classes and indicate general rules for evaluating their integrals. Incidentally, in many cases specific techniques are preferable. They are usually suggested by experience.

310. Some Integrals Dependent on Radicals

The symbol $R(x, y)$ will be used from now on to denote a fraction whose numerator and denominator are polynomials in x and y. Such a fraction is termed a *rational function* of the two variables x, y (cf. Sec. 304). If the denominator is a constant quantity (polynomial of degree zero), the rational function is termed *entire (integral)*.

---

[1] Nevertheless, for every continuous function there is an indefinite integral (which is a continuous function)

We define in similar fashion a rational function of three variables $R(x, y, z)$, of four, etc.

An Integral of the form[1]

$$I=\int R\left[x, \left(\frac{px+q}{rx+s}\right)^{\alpha}, \left(\frac{px+q}{rx+s}\right)^{\beta}, \ \ldots\right] dx \qquad (1)$$

where α, β are *rational* numbers and p, q, r, s are constants (numerical or literal), is reducible to an integral of a rational function and, hence, is expressible in terms of elementary functions. This purpose is served by the substitution[2] $\frac{px+q}{rx+s}=t^n$, where n is the common denominator of the fractions α, β, ...

To be specific, the integral

$$I=\int R\left[x, x^{\alpha}, x^{\beta}, \ \ldots\right] dx \qquad (2)$$

is computed by the substitution $x=t^n$.

Note. Reduction of a given integral to the integral of a rational function is called *rationalization*.

Example 1. $I=\int \frac{x\,dx}{\sqrt[3]{1+x}-\sqrt{1+x}}$.

Here $p=q=s=1$, $r=0$, $\alpha=\frac{1}{3}$, $\beta=\frac{1}{2}$. The common denominator is $n=6$. The integral is rationalized by the substitution

$$1+x=t^6, \quad dx=6t^5\,dt$$

We get

$$I=\int \frac{(t^6-1)\,6t^5\,dt}{t^2-t^3}=-6\int (t^8+t^7+t^6+t^5+t^4+t^3)\,dt=$$

$$=-6t^4\left(\frac{t^5}{9}+\frac{t^4}{8}+\frac{t^3}{7}+\frac{t^2}{6}+\frac{t}{5}+\frac{1}{4}\right)+C$$

where $t=\sqrt[6]{1+x}$.

---

[1] From here on the letter I denotes an integral.

[2] It is assumed that $\frac{p}{r}\neq\frac{q}{s}$; when $\frac{p}{r}=\frac{q}{s}$ the fraction $\frac{px+q}{rx+s}$ reduces to a constant and no substitution is required.

Example 2. $I=\int x^{-\frac{1}{2}}\left(x^{\frac{1}{3}}+1\right)^{-2}dx.$

This is an integral of type (2). Put $x=t^6$. We then have

$$I=6\int\frac{t^2\,dt}{(1+t^2)^2}=-\frac{3t}{1+t^2}+3\arctan t+C$$

where $t=\sqrt[6]{x}$.

311. The Integral of a Binomial Differential

A *binomial differential* is a differential of the form

$$x^m(a+bx^n)^p\,dx$$

where m, n, p are rational numbers and a and b are constants not equal to zero. The integral

$$I=\int x^m(a+bx^n)^p\,dx \tag{1}$$

is expressible in terms of elementary functions in the following three cases.

Case 1. p is an integer. Then the integral fits the type of Sec. 310.

See Example 2, Sec. 310, where $m=-\frac{1}{2}$, $n=\frac{1}{3}$, $p=-2$.

Case 2. p is a fraction $\left(p=\frac{r}{s}\right)$, but $\frac{m+1}{n}$ is an integer. Then the integral is rationalized by the substitution

$$a+bx^n=z^s$$

(where s is the denominator of the fraction p).

Example 1.

$$I=\int x^{\frac{1}{5}}\left(3-2x^{\frac{3}{5}}\right)^{-\frac{1}{2}}dx \tag{2}$$

Here $m=\frac{1}{5}$, $n=\frac{3}{5}$, $\frac{m+1}{n}=2$ is an integer. We put

$$3-2x^{\frac{3}{5}}=z^2 \tag{3}$$

We can express x in terms of z and substitute into (2). But it is simpler to differentiate (3):

$$x^{-\frac{2}{5}}\,dx=-\frac{5}{3}z\,dz \tag{4}$$

and transform I with the aid of (3) and (4) as follows:

$$I=\int\left(3-2x^{\frac{3}{5}}\right)^{-\frac{1}{2}}x^{\frac{3}{5}}\left(x^{-\frac{2}{5}}\,dx\right)=$$

$$=\int(z^2)^{-\frac{1}{2}}\,\frac{3-z^2}{2}\left(-\frac{5}{3}z\,dz\right)=-\frac{5}{6}\int(3-z^2)\,dz=$$

$$=-\frac{5}{2}z+\frac{5}{18}z^3+C$$

where $z=\left(3-2x^{\frac{3}{5}}\right)^{\frac{1}{2}}$.

Case 3. Both numbers $p=\frac{r}{s}$ and $\frac{m+1}{n}$ are fractions but their sum $\frac{m+1}{n}+p$ is an integer.

Then the integral is rationalized by the substitution

$$ax^{-n}+b=z^s$$

where s is the denominator of the fraction p.

Example 2.

$$I=\int x^{-6}(1+2x^3)^{\frac{2}{3}}\,dx$$

Here $m=-6$, $n=3$, $p=\frac{2}{3}$ (fraction), $\frac{m+1}{n}=-\frac{5}{3}$ (fraction), $\frac{m+1}{n}+p=-1$ (integer).

Put

$$x^{-3}+2=z^3,\quad x^{-4}\,dx=-z^2\,dz$$

Representing $1+2x^3$ as $x^3(x^{-3}+2)$, we obtain

$$I=\int x^{-4}(x^{-3}+2)^{\frac{2}{3}}\,dx=\int z^2(-z^2\,dz)=-\frac{1}{5}z^5+C=$$

$$=-\frac{1}{5}x^{-5}(1+2x^3)^{\frac{5}{3}}+C$$

Newton had already pointed out these three cases. Euler, who has never been surpassed by any mathematician in the art of transformation, sought in vain for new cases of the integrability of the binomial differential. He was convinced that these three cases were the only ones. In 1853, P. L. Chebyshev succeeded in proving Euler's assertion. D. D. Mordukhai-Boltovskoi, in 1926, proved the appropriate theorem for an integral of type (1) for irrational exponents m, n, p.

312. Integrals of the form $\int R(x, \sqrt{ax^2+bx+c})\,dx$

Integrals of this form [1] are rationalized by one of Euler's substitutions.

First Euler substitution. It is applicable for $a>0$. Put [2]

$$\sqrt{ax^2+bx+c}+x\sqrt{a}=t \tag{1}$$

Then

$$ax^2+bx+c=(t-x\sqrt{a})^2$$

The terms containing x^2 cancel and x (hence also dx) is expressed in terms of t in rational fashion. Putting this expression into (1), we find a rational expression for the radical $\sqrt{ax^2+bx+c}$ as well.

Example 1.

$$I=\int\frac{dx}{\sqrt{k^2+x^2}}$$

Put

$$\sqrt{k^2+x^2}=t-x$$

Whence

$$x=\frac{t^2-k^2}{2t}, \qquad dx=\frac{(t^2+k^2)\,dt}{2t^2},$$

$$\sqrt{k^2+x^2}=t-x=\frac{t^2+k^2}{2t}$$

Consequently

$$I=\int\frac{(t^2+k^2)\,dt}{2t^2}:\frac{t^2+k^2}{2t}=\int\frac{dt}{t}=\ln|t|+C,$$

$$I=\ln(x+\sqrt{k^2+x^2})+C$$

Third Euler substitution (note below discusses second). This substitution is applicable every time the trinomial

---

[1] It may be taken that $a\neq0$. for when $a=0$ we get the case of Sec. 310.

[2] We can just as readily put

$$\sqrt{ax^2+bx+c}-x\sqrt{a}=t$$

ax^2+bx+c has real roots and, in particular, for $a<0$.[1]

Let the roots be x_1, x_2. Then put

$$\sqrt{\frac{a(x-x_1)}{x-x_2}}=t \tag{2}$$

whence we find a rational expression of x in terms of t:

$$x=\frac{x_2t^2-ax_1}{t^2-a} \tag{3}$$

Rationalizing, we find

$$\sqrt{ax^2+bx+c}=\sqrt{a(x-x_1)(x-x_2)}=$$
$$=\sqrt{\frac{a(x-x_1)}{x-x_2}(x-x_2)^2}=t\,|x-x_2| \tag{4}$$

Example 2. $I=\int\frac{dx}{(x-1)\sqrt{-x^2+3x-2}}$.

The trinomial $-x^2+3x-2$ has the roots $x_1=1$, $x_2=2$:

$$-x^2+3x-2=-(x-2)(x-1)$$

The radicand is positive for $1<x<2$ (for $x=1$ and $x=2$ the integrand becomes infinite).

Put[2]

$$\sqrt{\frac{-(x-1)}{x-2}}=t \tag{5}$$

From this we get

$$x=\frac{2t^2+1}{t^2+1}, \quad dx=\frac{2t\,dt}{(t^2+1)^2}, \tag{6}$$

$$\sqrt{-(x-2)(x-1)}=\sqrt{\frac{-(x-1)}{x-2}}\,|x-2|=t\,|x-2|=-t(x-2)$$

(by virtue of the inequality $1<x<2$ the quantity $x-2$ is negative). Substituting into the right side the expression of

---

[1] For $a<0$ the trinomial ax^2+bx+c could have complex roots too (if $4ac-b^2>0$), but then by virtue of the identity $ax^2+bx+c=\frac{1}{4a}[(2ax+b)^2+(4ac-b^2)]$ the trinomial would always have negative values so that the root $\sqrt{ax^2+bx+c}$ would be imaginary for any value of x.

[2] We can put $x_1=2$, $x_2=1$. Then Euler's third substitution is modified $\left(\text{we have to put } \sqrt{\frac{-(x-2)}{x-1}}=t\right)$.

x in terms of t, we find

$$\sqrt{-(x-2)(x-1)}=\frac{t}{t^2+1} \tag{7}$$

By (6) and (7) we have

$$I=\int\frac{dx}{(x-1)\sqrt{-(x-2)(x-1)}}=\int\frac{2dt}{t^2}=-\frac{2}{t}=-2\sqrt{\frac{x-2}{-(x-1)}}+C$$

Note. The first and third Euler substitutions are sufficient to compute any integral of the type under consideration. To complete the picture, we give Euler's *second substitution*:

$$\sqrt{ax^2+bx+c}=tx+\sqrt{c} \tag{8}$$

It is applicable for $c>0$. Squaring and dividing by x, we obtain a rational expression of x in terms of t; then (8) rationalizes the radical.

313. Integrals of the Form $\int R(\sin x, \cos x)\,dx$

Integrals of this type are rationalized by the substitution

$$\tan\frac{x}{2}=z \tag{1}$$

whence

$$\sin x=\frac{2z}{1+z^2}, \quad \cos x=\frac{1-z^2}{1+z^2}, \tag{2}$$

$$dx=\frac{2dz}{1+z^2} \tag{3}$$

Example. $I=\int\frac{dx}{3+5\cos x}$

By means of (2) and (3) we get

$$I=\int\frac{2dz}{(1+z^2)\left(3+5\frac{1-z^2}{1+z^2}\right)}=\int\frac{dz}{4-z^2}=\frac{1}{4}\ln\left|\frac{2+z}{2-z}\right|+C$$

Substituting $z=\tan\frac{x}{2}$, we find

$$I=\frac{1}{4}\ln\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right|+C$$

314. The Definite Integral[1]

Let a function $f(x)$ be continuous over an interval (a, b) and at its end-points. In the interval take n consecutive points $x_1, x_2, x_3, x_4, \ldots, x_n$ (Fig. 327, where $n=5$); for the sake of uniformity, denote $a=x_0$ and $b=x_{n+1}$. The interval (a, b) is partitioned into $n+1$ subintervals (x_0, x_1), (x_1, x_2), (x_2, x_3), $\ldots$, (x_{n-1}, x_n), (x_n, x_{n+1}).

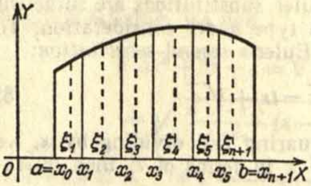

Fig. 327

In each of the subintervals (in the interior or at one of the end-points) take a point [point ξ_1 in (x_0, x_1), ξ_2 in (x_1, x_2), etc.].

Form the sum

$$S_n = f(\xi_1)(x_1-x_0) + f(\xi_2)(x_2-x_1) + \ldots + f(\xi_{n+1})(x_{n+1}-x_n) \quad (1)$$

The following theorem holds true.

Theorem. If as the number of subintervals (x_0, x_1), (x_1, x_2), $\ldots$ increases without bound the largest one tends to zero, then the sum S_n tends to some limit S. The number S is the same no matter how the partitioning into subintervals was performed or what the choice of points $\xi_1, \xi_2, \ldots$ was.

Fig. 328 gives a pictorial explanation of the theorem. The sum S_n is numerically equal to the hatched area of the step-like figure [the base of the left step is equal to x_1-x_0, the altitude is $KL=f(\xi_1)$; hence, the area is equal to $f(\xi_1)(x_1-x_0)$, etc.]. The narrower the steps, the closer is the area of the step-like figure to the area of the "curvilinear trapezoid" x_0ABx_{n+1} so that the limit S of the sum S_n is numerically equal to the area of the figure x_0ABx_{n+1}.

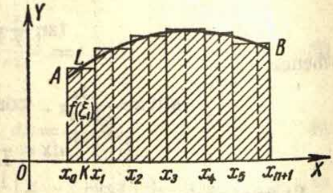

Fig. 328

The sum (1) is often abbreviated to

$$\sum f(\xi_i)(x_i-x_{i-1}) \quad (2)$$

---

[1] It is advisable first to read Sec. 292, Item 2.

The symbol $\sum$ (sigma) indicates that the expression (2) is the sum of terms of a single type. The expression $f(\xi_i)(x_i-x_{i-1})$ indicates the law of formation of the terms: for $i=1$ we have the first term, for $i=2$, the second, etc. An expanded notation is

$$\sum_{i=1}^{i=n+1} f(\xi_i)(x_i-x_{i-1}) \tag{2a}$$

Here, the first term is $i=1$, and the last, $i=n+1$.

Definition. The limit to which the sum (1) tends as the largest subinterval approaches zero is called the *definite integral* of the function $f(x)$. The end-points a, b of the given interval (the *interval of integration*) are called the *limits of integration*: lower limit (a) and upper limit (b).

The definite integral is denoted by

$$\int_a^b f(x)\,dx \tag{3}$$

and is read "*the integral of* $f(x)$ *with respect to* x *between the limits* a *and* b."

The value of the definite integral depends on the form of the function $f(x)$ and on the values of the upper and lower limits. The argument of the function may be denoted by any letter, say y, so that the expression

$$\int_a^b f(y)\,dy \tag{4}$$

represents the same number as (3).

Note. The upper limit b may be greater or less than the lower limit a. In the first case,

$$a < x_1 < x_2 < \dots < x_{n-1} < x_n < b \tag{5}$$

In the second,

$$a > x_1 > x_2 > \dots > x_{n-1} > x_n > b \tag{6}$$

Supplement to definition. The definition assumes that $a \neq b$. But the concept of the definite integral is extended to the case of $a=b$; an integral with identical limits is considered equal to zero:

$$\int_a^a f(x)\,dx = 0 \tag{7}$$

[this agreement is justified on the grounds that the integral (3) tends to zero as a and b approach one another, cf. Fig. 327].

Example. Find $\int_a^b 2x\,dx$. Here

$$f(x)=2x \tag{8}$$

Solution. First method. Divide the interval (a, b) into equal parts (Fig. 329); then the abscissas

$$x_0=a,\ x_1,\ x_2,\ \ldots,\ x_n,\ x_{n+1}=b$$

form an arithmetic progression with difference

$$x_1-x_0=x_2-x_1=\ldots=\frac{b-a}{n+1} \tag{9}$$

For the points $\xi_1, \xi_2, \ldots$ we take the right end-points[1] of the successive intervals $(a, x_1), (x_1, x_2), \ldots$, so that

$$\xi_1=x_1,\ \xi_2=x_2,\ \ldots,\ \xi_n=x_n,\ \xi_{n+1}=b;$$
$$f(\xi_1)=2x_1,\ f(\xi_2)=2x_2,\ \ldots,\ f(\xi_n)=2x_n,\ f(\xi_{n+1})=2b \tag{10}$$

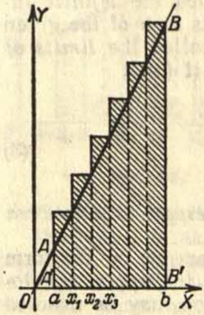

Fig. 329

By virtue of (8) and (10), the sum (1) takes the form

$$S_n=2x_1(x_1-x_0)+2x_2(x_2-x_1)+\ldots+2x_n(x_n-x_{n-1})+$$
$$+2x_{n+1}(x_{n+1}-x_n)=2\frac{b-a}{n+1}(x_1+x_2+\ldots+x_{n+1})$$

Summing the arithmetic progression, we find

$$S_n=2\frac{b-a}{n+1}\frac{(x_1+x_{n+1})(n+1)}{2}=(b-a)(x_1+b) \tag{11}$$

As the number of equal intervals increases without bound, their lengths approach zero; in the process, x_1 tends to a. Therefore

$$\lim S_n=(b-a)(a+b)=b^2-a^2$$

Consequently,

$$\int_a^b 2x\,dx=b^2-a^2 \tag{12}$$

---

[1] In other words, the rectangles lie to the right of the ordinates of the straight line $y=2x$.

In exactly the same way

$$\int_a^b 2y\,dy = b^2 - a^2,$$

$$\int_a^b 2t\,dt = b^2 - a^2$$

and so forth.

The quantity $b^2 - a^2$ is the area S of the trapezoid $A'ABB'$ (Fig. 329); indeed

$$S = \frac{1}{2}(A'A + B'B)\,A'B' = \frac{1}{2}(2a + 2b)(b - a) = b^2 - a^2$$

Second method. Partition the interval (a, b) into unequal parts so that the points $x_0, x_1, x_2, \ldots, x_n, x_{n+1}$ form a geometric progression[1] (Fig. 330):

$$x_0 = a,\ x_1 = aq,\ \ldots,\ x_n = aq^n,$$
$$x_{n+1} = b = aq^{n+1} \qquad (13)$$

From this equality we have

$$q^{n+1} = \frac{b}{a} \qquad (14)$$

For the points $\xi_1, \xi_2, \ldots$ we take the left end-points[2] of successive intervals $(a, x_1)(x_1, x_2), \ldots$, so that

$$\xi_1 = a,\ \xi_2 = x_1,\ \ldots,\ \xi_n = x_{n-1},$$
$$\xi_{n+1} = x_n$$

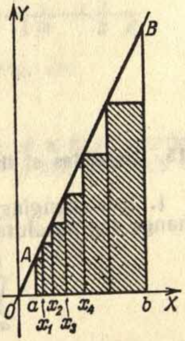

Fig. 330

Sum (1) takes the form

$$S_n = 2x_0(x_1 - x_0) + 2x_1(x_2 - x_1) + \ldots + 2x_n(x_{n+1} - x_n) =$$
$$= 2a^2(q - 1)[1 + q^2 + q^4 + \ldots + q^{2n}]$$

---

[1] This is possible if both limits a and b have the same sign (neither can be equal to zero). In the preceding method, the numbers a and b may be arbitrary.

[2] The rectangles border on the ordinates of the straight line $y = 2x$ on the left.

In square brackets we have a geometric progression with ratio q^2. Summing, we find

$$S_n = 2a^2(q-1)\frac{q^{2(n+1)}-1}{q^2-1} = \frac{2a^2[(q^{n+1})^2-1]}{q+1}$$

or, by virtue of (14),

$$S_n = \frac{2a^2\left[\left(\frac{b}{a}\right)^2-1\right]}{q+1} = \frac{2(b^2-a^2)}{q+1} \tag{15}$$

As the number n increases without bound, the ratio q, as is evident from (14), tends to unity:

$$\lim q = 1 \tag{16}$$

The lengths of all subintervals approach zero. By virtue of (15) and (16) we have

$$\lim S_n = b^2 - a^2$$

or

$$\int_a^b 2x\,dx = b^2 - a^2$$

315. Properties of the Definite Integral

1. Interchanging the limits of a definite integral does not change the absolute value but only the sign:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{1}$$

This follows from comparing the sums S_n which correspond to the two integrals.

2.

$$\int_a^b (x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{2}$$

This property is explained in Fig. 331 (area $aABb$ = area $aACc$ + area $cCBb$), but it also holds true when the point c is exterior to the interval (a, b).

2a. In place of one additional point c, we can take several. For three points k, l, m we have

$$\int_a^b f(x)\,dx = \int_a^k f(x)\,dx + \int_k^l f(x)\,dx + \int_l^m f(x)\,dx + \int_m^b f(x)\,dx$$

The order of the points is immaterial; of practical importance is the case when a, k, l, m, b are taken in increasing (Fig. 332) or decreasing order.

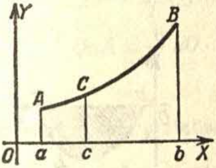

Fig. 331

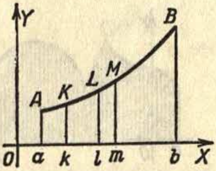

Fig. 332

3. The integral of an algebraic sum of a fixed number of terms is equal to the algebraic sum of the integrals of the separate terms. For three terms,

$$\int_a^b [f_1(x) + f_2(x) - f_3(x)]dx =$$

$$= \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx - \int_a^b f_3(x)\,dx \qquad (3)$$

4. A constant factor can be taken outside the integral sign:

$$\int_a^b mf(x)\,dx = m\int_a^b f(x)\,dx \qquad (4)$$

316. Geometrical Interpretation of the Definite Integral

Consider the integral

$$\int_a^b f(x)\,dx \tag{1}$$

where the lower limit is less than the upper ($a < b$).[1]

If the function $f(x)$ is positive within the interval (a, b) (Fig. 333), then the integral is numerically equal to (Sec. 314) the area covered by the ordinates of the graph $y=f(x)$ ($aADEBb$ in Fig. 333)

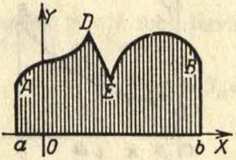

Fig. 333

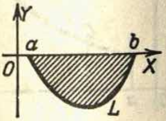

Fig. 334

If the function $f(x)$ is negative within (a, b) (Fig. 334), then the absolute value of the integral is equal to the area covered by the ordinates, but is negative.

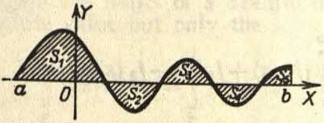

Fig 335

Now let $f(x)$ change sign once or several times in (a, b) (Fig. 335) The integral is then equal to the difference of two numbers, one of which (diminuend) expresses the area covered by positive ordinates, and the other (subtrahend), the area covered by negative ordinates (cf. Sec. 315, Item 2a).

---

[1] The case $a > b$ reduces to that under consideration by virtue of Sec. 315, Item 1.

Thus, for the case depicted in Fig. 335,

$$\int_a^b f(x)\,dx=(S_1+S_3+S_5)-(S_2+S_4)$$

Example. The integral $\int_{-2}^{1} 2\,x\,dx$ is equal (Sec. 314, Example) to $1^2-(-2)^2=-3$. This number is equal to the difference between the areas (Fig. 336)

$$ObB=\frac{1}{2}\,Ob\cdot bB=1$$

and

$$OaA=\frac{1}{2}\,aO\cdot Aa=4$$

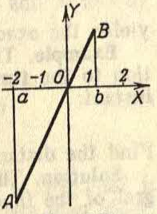

Fig. 336

317. Mechanical Interpretation of the Definite Integral

1. **The path of a material particle.** Let a particle be in motion in some direction with a velocity

$$v=f(t)$$

where t is the duration of motion. It is required to find the distance s covered by the particle from time $t=T_1$ to time $t=T_2$.

If the velocity is constant, then

$$s=v(T_2-T_1)$$

But if the velocity varies, then in order to find s we have to partition the interval of time into subintervals

$$(T_1,\ t_1),\ (t.\ t_2),\ \ldots,\ (t_{n-1},\ t_n),\ (t_n,\ T_2)$$

Let τ_1 be some instant of the interval $(T_1,\ t_1)$, τ_2 some instant of the interval $(t_1,\ t_2)$, etc.

The quantity $f(\tau_1)$ is the velocity at time τ_1; the product $f(\tau_1)(t_1-T_1)$ expresses approximately the distance covered during the first interval of time. In exactly the same way, $f(\tau_2)(t_2-t_1)$ approximately expresses the distance covered in the second interval, etc. The sum

$$s_n=f(\tau_1)(t_1-T_1)+f(\tau_2)(t_2-t_1)+\ldots+f(\tau_{n+1})(T_2-t_n)$$

describes the distance s the more precisely, the smaller the subintervals. The limit of the sum s_n, i.e. the integral

$$\int_{T_1}^{T_2} f(t)\,dt$$

yields the exact value of the distance s.

Example. The velocity of a particle increases in proportion to the time that has elapsed starting from some initial instant:

$$v = mt$$

Find the distance covered from the initial time to time T.

Solution. The required distance is expressed by the integral of the function mt; the lower limit is equal to zero; the upper limit is T:

$$s = \int_0^T mt\,dt = m\int_0^T t\,dt$$

(Sec. 315, Item 4) We know (Sec. 314, Example) that $\int_a^b 2t\,dt = b^2 - a^2$. Hence (Sec. 315, Item 4), $\int_a^b t\,dt = \frac{b^2-a^2}{2}$.

For $a=0$, $b=T$ we have

$$s = m\int_0^T t\,dt = \frac{1}{2}mT^2$$

2. The work of a force. If a constant force P acts on a material particle moving in the direction of the force, then the work A of the force over the line segment (s_1, s_2) is found from the formula

$$A = P(s_2 - s_1)$$

But if the force P maintains its direction of motion and its magnitude changes depending on the distance s, i.e. $P = f(s)$, then the work is found from the formula

$$A = \int_{s_1}^{s_2} f(s)\,ds$$

318. Evaluating a Definite Integral

Theorem 1. If m is the smallest and M the greatest value of a function $f(x)$ in an interval (a, b), then the value of the integral $\int_a^b f(x)\,dx$ lies between $m(b-a)$ and $M(b-a)$.

For $a < b$ we have

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a) \tag{1}$$

For $a > b$ the inequalities are reversed.

Geometrically, the figure shown hatched in Fig. 337 is (in area) greater than the rectangle $ablk$ and less than $abLK$.

Example. Evaluate the integral $\int_4^6 2x\,dx$.

Solution. The smallest value of the function $2x$ in the interval $(4, 6)$ is $m = 2\cdot 4 = 8$, the greatest value $M = 2\cdot 6 = 12$.

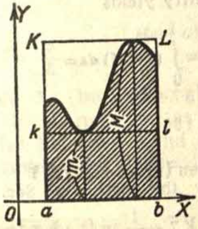

Fig. 337

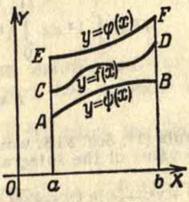

Fig. 338

Finally, $b-a = 6-4 = 2$. Hence, the integral lies between $8\cdot 2 = 16$ and $12\cdot 2 = 24$;

$$16 < \int_a^b 2x\,dx < 24$$

Its exact value (Sec. 314, Example) is 20.

Theorem 2. If at every point of the interval (a, b) the inequalities

$$\psi(x) \leqslant f(x) \leqslant \varphi(x) \tag{2}$$

hold, then

$$\int_a^b \psi(x)\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b \varphi(x)\,dx \qquad (3)$$

Geometrically (Fig. 338), the area $aABb <$ area $aCDb <$ area $aEFb$. Theorem 1 is a particular case of Theorem 2 $[\psi(x)=m,\ \varphi(x)=M]$.

Note. Theorem 2 asserts that the inequalities can be integrated. But the inequalities cannot be differentiated.

318a. The Bunyakovsky Inequality [1]

The evaluation of an integral by formula (1), Sec. 318, is ordinarily very rough. There are a number of formulas that enable one to obtain a more exact evaluation. An important one is the Bunyakovsky inequality:

$$\left[\int_a^b f(x)\,\varphi(x)\,dx\right]^2 \leqslant \int_a^b [f(x)]^2\,dx \cdot \int_a^b [\varphi(x)]^2\,dx$$

Example. Evaluate the integral $I=\int_0^1 \sqrt{1+x^2}\,dx$.

Represent the integrand in the form $1\cdot\sqrt{1+x^2}$ so that $f(x)=1$, $\varphi(x)=\sqrt{1+x^2}$ The Bunyakovsky inequality yields

$$I^2 \leqslant \int_0^1 1^2\,dx \cdot \int_0^1 (\sqrt{1+x^2})^2\,dx = \int_0^1 (1+x^2)\,dx = \frac{4}{3}$$

whence

$$I \leqslant \frac{2}{\sqrt{3}} < 1.155$$

Formula (1), Sec. 318, would have given $(M=\sqrt{2})$: $I \leqslant \sqrt{2} < 1.415$. The true value of the integral (found by the method of Sec. 323) is

$$I=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)\Big|_0^1=\frac{1}{2}\sqrt{2}+\frac{1}{2}\ln\left(1+\sqrt{2}\right)=1.147\ldots$$

319. The Mean-Value Theorem of Integral Calculus

The definite integral [2] is equal to the product of the length of the interval of integration (a, b) by the value of the integrand at some point ξ of the interval (a, b):

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[1] V. Ya. Bunyakovsky (1804-1889), noted Russian mathematician, made contributions to number theory and probability. *Translator's note*: the names Schwarz, Cauchy and Cauchy-Schwarz inequality are also encountered in the literature.

[2] The mean-value theorem breaks down when the concept of the integral is extended to the case of a discontinuous function (Sec. 328).

$$\int_a^b f(x)\,dx=(b-a)f(\xi)\quad (a\leqslant\xi\leqslant b) \tag{1}$$

Explanation. In Fig. 339, let us displace KL from the position CD to the position EF. At the beginning, the area $AKLB$ is less than $\int_a^b f(x)\,dx$ (cf. Sec. 318, Theorem 1), at the end, it is greater. At some intermediate time the following equality must hold: area $AKLB=\int_a^b f(x)\,dx$. The base of the rectangle $AKLB$ is $AB=b-a$, and the altitude, the ordinate NM, corresponding to the point $N\,(\xi)$ of the interval AB. Hence

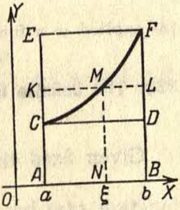

Fig. 339

$$(b-a)f(\xi)=\int_a^b f(x)\,dx$$

Note 1. The mean-value theorem establishes that Eq. (1), where ξ is regarded as the unknown, has at least one root that lies between a and b.

Example. For $f(x)=2x$, formula (1) takes the form

$$\int_a^b 2x\,dx=(b-a)\,2\xi \tag{2}$$

The theorem states that ξ lies between a and b. Indeed, the integral is equal to b^2-a^2 and formula (2) yields

$$\xi=\frac{b^2-a^2}{2(b-a)}=\frac{a+b}{2}$$

i.e. ξ is the arithmetic mean [1] of a and b.

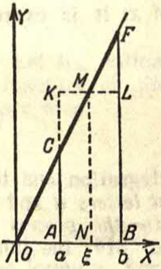

Fig. 340

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[1] The geometrical formula (2) expresses the familiar theorem on the area of a trapezoid ($ACFB$ in Fig. 340; $AB=b-a$ is the altitude, $NM=2\xi$ is the median).

Note 2. Actually, the mean-value theorem of differential calculus (Sec. 264) differs from the theorem of this section in notations alone. Denote the integral of formula (1) by $f'(x)$. Formula (1) becomes

$$\int_a^b f'(x)\,dx=(b-a)\,f'(\xi)$$

Here the left-hand side is equal to $f(b)-f(a)$ (see Sec. 322 below) and we get the Lagrange formula

$$f(b)-f(a)=(b-a)\,f'(\xi)$$

[as applied to a function $f(x)$ with a continuous derivative].

320. The Definite Integral as a Function of the Upper Limit

Given fixed limits a and b, the integral $\int_a^b f(x)\,dx$ of any function $f(x)$ has a *definite numerical value*. But if the upper (or lower) limit is capable of taking on a variety of values, then the integral becomes a *function* of the upper (or lower) limit. Its aspect depends on the form of the integrand $f(x)$ (and also on the value of the constant lower limit). The character of the dependence is discussed in Sec. 321, Theorem 2.

Example 1. The integral $\int_0^1 2t\,dt$ has the numerical value 1, the integral $\int_0^2 2t\,dt$ has the value 4, the integral $\int_0^3 2t\,dt$, the value 9, etc. Hence, $\int_0^x 2t\,dt$ is a function of x; it is expressed by the formula

$$\int_0^x 2t\,dt=x^2 \tag{1}$$

Note. In formula (1), the variable of integration and the variable upper limit are denoted by *different letters* (t and x) because these variables play different roles *in the process of integration*. Namely, we *first* evaluate (Sec. 314) the limit of the sum

$$S_n=2\tau_1(t_1-0)+2\tau_2(t_2-t_1)+\ldots+2\tau_{n+1}(x-t_n)$$

where $t_1, t_2, \ldots, t_n$ are taken from 0 to x, and the numbers $\tau_1, \tau_2, \ldots$ belong to the intervals $(0, t_1)$ $(t_1, t_2), \ldots$. In this process, x remains constant.

Then x is subject to change; this time we do not deal with the variable t.

If in place of (1) we write

$$\int_0^x 2x\,dx = x^2 \tag{2}$$

then the above-indicated difference is blurred.

Nevertheless, the notation (2) is frequently employed and, generally, we have

$$\int_a^x f(x)\,dx \tag{3}$$

$\left[\text{or, also, } \int_a^t f(t)\,dt, \int_a^s f(s)\,ds, \text{ etc.}\right]$. The fact of the matter is that *after performing the integration* the variable limit has *the same meaning* (geometrical, mechanical, etc.) as the variable of integration (see Examples 2 and 3).

Example 2. The area S of a triangle OPM (Fig. 341) is expressed by the integral $\int_0^a x\,dx$:

$$S = \int_0^a x\,dx = \frac{a^2}{2} \tag{4}$$

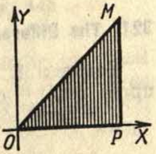

Fig. 341

Let the ordinate PM be moving; then the integral (4) is a function of the upper limit; accordingly, we write t in place of a:

$$S = \int_0^t x\,dx = \frac{t^2}{2} \tag{5}$$

The notation (5) is flawless but inconvenient because in the formula $S = \frac{t^2}{2}$ the letter t is the abscissa, whereas we used x to denote the abscissa. That is why preference is frequently given to the not quite exact notation

$$S=\int_0^x x\,dx=\frac{x^2}{2} \tag{6}$$

Example 3. The velocity v of a freely falling body is given by the formula

$$v=gt \tag{7}$$

The distance s covered by a falling body in the time interval $(0, T)$ is (Sec. 317, Item 1) equal to the integral $\int_0^T gt\,dt$:

$$s=\int_0^T gt\,dt=\frac{1}{2}gT^2 \tag{8}$$

The notation (8) is flawless, but in formulas (7) and (8) the arguments are denoted differently, whereas their physical meaning is the same. Therefore in place of (8) one writes

$$s=\int_0^t gt\,dt=\frac{1}{2}gt^2$$

321. The Differential of an Integral

Theorem 1. The differential of an integral with variable upper limit[1] coincides with the integrand expression:

$$d\int_a^x f(x)\,dx=f(x)\,dx \tag{1}$$

Formula (1) may be written more precisely as $d\int_a^x f(t)\,dt=f(x)dx$ (see Sec. 320).

Example.

$$d\int_0^x 2x\,dx=2x\,dx \tag{1a}$$

---

[1] An integral with variable upper limit x is always a *differentiable* function of x.

Let us verify this equality. We have (Sec. 320)

$$\int_0^x 2x\,dx = x^2$$

Differentiating, we obtain (1a).

Note. From (1) we get

$$\frac{d}{dx}\int_a^x f(x)\,dx = f(x) \tag{2}$$

In words, the derivative of an integral with respect to the upper limit coincides with the integrand function. This proposition may be stated otherwise.

Theorem 2. An integral with variable upper limit is one of the antiderivatives (Sec. 293) of the integrand function.

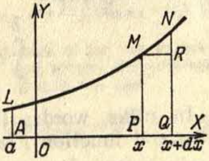

Fig. 342

Explanation of formula (1). The area of $ALMP$ (Fig. 342) is expressed by the integral $\int_a^x f(x)\,dx$. When x is increased by $dx = PQ$, the area of $ALMP$ receives an increment $PMNQ$, which is divided into the rectangle $PMRQ$ and the curvilinear triangle MNR. The area of the rectangle is equal to $PM \cdot PQ = f(x)\,dx$; it is proportional to dx, while the area of the triangle MNR is of higher order than dx (in Fig. 342, it is less than $MR \cdot RN = dx\,\Delta y$). Hence (Sec. 230) the integrand $f(x)\,dx$ is the differential of the integral $\int_a^x f(x)\,dx$.

Explanation of formula (2). If $f(t)$ is the velocity of a particle at time t, then $\int_a^t f(t)\,dt$ (Sec. 317, Item 1) is the distance s covered by the particle during the time that has elapsed between the initial time a and the time t:

$$s = \int_a^t f(t)\,dt \tag{3}$$

The derivative $\frac{ds}{dt}=\frac{d}{dt}\int_a^t f(t)\,dt$ is the velocity of the particle (Sec. 223). Hence, $\frac{d}{dt}\int_a^t f(t)\,dt=f(t)$.

322. The Integral of a Differential. The Newton-Leibniz Formula

The theorem given below reduces the computation of a definite integral to finding an indefinite integral (cf. Sec. 323).

Theorem. The integral of the differential of a function $F(x)$ is equal to the increment to the function $F(x)$ over the interval of integration:

$$\int_a^b dF(x)=F(b)-F(a) \tag{1}$$

In other words: if $F(x)$ is some antiderivative of the integrand function $f(x)$, then

$$\int_a^b f(x)\,dx=F(b)-F(a) \tag{2}$$

Formula (2) is sometimes called the *Newton-Leibniz formula.* [1]

Example 1. We have (Sec. 314)

$$\int_a^b 2x\,dx=b^2-a^2 \tag{3}$$

The integrand expression is the differential of the function x^2 ($dx^2=2x\,dx$). When passing from $x=a$ to $x=b$, the function x^2 receives the increment b^2-a^2. Formula (3) expresses the fact that the integral is equal to this increment.

---

[1] The name is justified only insofar as Newton and Leibniz were the first to use the relationship between differentiation and integration to find integrals. But they did not give formula (2) either in words or as a literal expression. In geometrical form the theorem of this section (as also Theorem 1, Sec 321) was established by Barrow, Newton's teacher.

Example 2. Find the integral $\int_a^b 3x^2\,dx$.

Solution. Noting that the integrand expression is the differential of the function x^3, we obtain, by formula (2)

$$\int_a^b 3x^2\,dx=\int_a^b d(x^3)=b^3-a^3 \tag{4}$$

Explanation. The integral $\int_a^b 3x^2\,dx$ is (Sec. 314) equal to the limit of the sum

$$3x_0^2(x_1-x_0)+3x_1^2(x_2-x_1)+\ldots+3x_{n-1}^2(x_n-x_{n-1})+3x_n^2(x_{n+1}-x_n) \tag{5}$$

The first term is the differential of the function x^3 for the interval (x_0, x_1), the second, for the interval (x_1, x_2), and so on. Replace each differential by an appropriate increment. We obtain the sum

$$(x_1^3-x_0^3)+(x_2^3-x_1^3)+\ldots+(x_n^3-x_{n-1}^3)+(x_{n+1}^3-x_n^3) \tag{6}$$

Remove parentheses. All terms, except $x_0^3=a^3$ and $x_{n+1}^3=b^3$ cancel, so that the sum (6) is exactly equal to b^3-a^3.

A number of errors are committed in passing from (5) to (6) but each one of them is of higher order than the corresponding increment of the argument. Therefore, despite the accumulation of errors, their sum is infinitely small, which means that for an unbounded increase in the number of terms of (5), the expression (5) differs from b^3-a^3 by an infinitesimal. In other words, b^3-a^3 is the limit of the sum (5), i.e.

$$\int_a^b 3x^2\,dx=b^3-a^3$$

The general formula (1) is derived in the same manner.

Mechanical interpretation. Let a particle be in motion in some direction and let $F(t)$ be the distance of the particle from the initial position at time t. The derivative $\frac{dF(t)}{dt}=f(t)$ is (Sec. 223) the velocity. Hence the integral $\int_a^b f(t)\,dt$ expresses (Sec. 317) the distance s covered between time $t=a$

and $t=b$:

$$s=\int_a^b f(t)\,dt \tag{7}$$

But at time $t=a$ the distance from the initial point is $F(a)$, at time $t=b$ the distance is $F(b)$. Hence

$$s=F(b)-F(a) \tag{8}$$

Comparing (7) and (8), we get

$$\int_a f(t)\,dt=F(b)-F(a)$$

323. Computing a Definite Integral by Means of the Indefinite Integral

Rule. In order to evaluate the definite integral $\int_a^b f(x)\,dx$, it is sufficient to find the indefinite integral $\int f(x)\,dx$, substitute into this expression, in place of x, first the upper limit, then the lower limit, and then subtract the second quantity from the first.

This rule is based on the theorem in Sec. 322.

Note. The additive constant of the indefinite integral need not be written out, since it disappears in subtraction.

Example 1. Find $\int_{-2}^{3} 3x^2\,dx$.

Solution. We find the indefinite integral

$$\int 3x^2\,dx=x^3+C$$

Substituting $x=3$, we find $27+C$; for $x=-2$ we get $-8+C$ Subtracting the second from the first, we obtain

$$\int_{-2}^{3} 3x^2\,dx=(27+C)-(-8+C)=27-(-8)=35 \tag{1}$$

The additive constant C disappeared in the subtraction.

Example 2. Find $\int_0^{\pi} \sin x\,dx$.

Solution. We have $\int \sin x\,dx = -\cos x$ (we drop the additive constant) Hence

$$\int_0^{\pi} \sin x\,dx = -[\cos\pi - \cos 0] = 2 \tag{2}$$

Notation of substitution The notation

$$F(x)\Big|_a^b \text{ or } [F(x)]_a^b \tag{3}$$

denotes the same as $F(b) - F(a)$. For example, in place of

$-[\cos\pi - \cos 0]$ one writes $-\cos x\Big|_0^{\pi}$ or $[-\cos x]_0^{\pi}$.

Example 3. Find $\int_0^a \frac{dx}{a^2+x^2}$.

Solution.

$$\int_0^a \frac{dx}{a^2+x^2} = \left[\frac{1}{a}\arctan\frac{x}{a}\right]_0^a = \frac{1}{a}\arctan 1 - \frac{1}{a}\arctan 0 = \frac{\pi}{4a}$$

Example 4.

$$\int_3^5 \frac{dx}{(x-2)^2} = -\frac{1}{x-2}\Big|_3^5 = -\frac{1}{3} + \frac{1}{1} = \frac{2}{3}$$

324. Definite Integration by Parts

Integration by parts (Sec 301) may be applied directly to a definite integral through the use of the formula

$$\int_{x_1}^{x_2} u\,dv = uv\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} v\,du \tag{1}$$

It is better to use formula (1) than to first compute the indefinite integral, particularly when integration by parts is repeated.

Example 1. $I=\int\limits_0^{\sqrt{3}} \frac{x^2\,dx}{(1+x^2)^2}$.

Setting

$$u=x,\ dv=\frac{x\,dx}{(1+x^2)^2}=d\left[-\frac{1}{2(1+x^2)}\right]$$

we find

$$I=\int\limits_0^{\sqrt{3}} x\,d\left[-\frac{1}{2(1+x^2)}\right]=-\frac{x}{2(1+x^2)}\Bigg|_0^{\sqrt{3}}+$$

$$+\int\limits_0^{\sqrt{3}}\frac{dx}{2(1+x^2)}=-\frac{\sqrt{3}}{8}+\frac{1}{2}\arctan\sqrt{3}=-\frac{\sqrt{3}}{8}+\frac{\pi}{6}\approx 0.307$$

Example 2. $I=\int\limits_0^{\frac{\pi}{2}} x\sin x\,dx.$

We have

$$I=\int\limits_0^{\frac{\pi}{2}} x\,d(-\cos x)=-x\cos x\Bigg|_0^{\frac{\pi}{2}}+\int\limits_0^{\frac{\pi}{2}}\cos x\,dx$$

The first term vanishes and we obtain

$$I=\sin x\Bigg|_0^{\frac{\pi}{2}}=1$$

325. The Method of Substitution in a Definite Integral

Rule. To evaluate the integral $\int\limits_{x_1}^{x_2} f(x)\,dx$ we can introduce an auxiliary variable z connected with x by some relation. The integrand is transformed, as in the case of indefinite integration (Sec. 300), to the form $f_1(z)\,dz$. It is also necessary to replace the limits x_1 and x_2 by new limits z_1 and z_2

in such fashion that these values of the variable z are associated with the given values x_1, x_2 of the variable x. If this is possible, then we have[1]

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{z_1}^{z_2} f_1(z)\,dz \tag{1}$$

Example 1. Find

$$\int_{5}^{13} \sqrt{2x-1}\,dx$$

Solution. Introduce the auxiliary variable z connected with x by the relation

$$z = 2x - 1 \tag{2}$$

Expressing x in terms of z, we obtain

$$x = \frac{z+1}{2} \tag{3}$$

The integrand $\sqrt{2x-1}\,dx$ is transformed to

$$\frac{1}{2} z^{\frac{1}{2}}\,dz$$

The limits $x_1 = 5$, $x_2 = 13$ must be replaced by new limits z_1, z_2 by formula (2):

$$z_1 = 2x_1 - 1 = 9, \quad z_2 = 2x_2 - 1 = 25$$

According to (1) we have

$$\int_{5}^{13} \sqrt{2x-1}\,dx = \int_{9}^{25} \frac{1}{2} z^{\frac{1}{2}}\,dz = \frac{1}{3} z^{\frac{3}{2}} \Big|_{9}^{25} = 32\frac{2}{3}$$

Example 2. Find $\int_{-a}^{+a} \sqrt{a^2 - x^2}\,dx$.

Solution. The substitution

$$x = a \sin t \tag{4}$$

---

[1] It is assumed that: (1) the relationship between x and z may be expressed by the formula $x = \varphi(z)$, where the function $\varphi(z)$ has a continuous derivative in the interval (z_1, z_2); (2) the function $f(x)$ is continuous for all values assumed by x when z varies in the interval (z_1, z_2).

reduces (Sec. 303, Example 1) the integrand to the form

$$a^2\sqrt{1-\sin^2 t}\cos t\,dt=\pm a^2\cos^2 t\,dt \tag{5}$$

The upper sign is taken if t belongs to the first or fourth quadrant, the lower sign if it lies in the second or third.

The new limits t_1, t_2 must be chosen so that

$$-a=a\sin t_1,\quad a=a\sin t_2$$

This is possible (and in two ways). We can take

$$t_1=-\frac{\pi}{2},\quad t_2=\frac{\pi}{2}$$

Then t varies within the fourth and first quadrants, and so we take the upper sign in (5) and obtain

$$\int_{-a}^{+a}\sqrt{a^2-x^2}\,dx=a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt=\frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=$$

$$=\frac{\pi a^2}{2}$$

We can also take

$$t_1=\frac{3\pi}{2},\quad t_2=\frac{\pi}{2}$$

But then we take the lower sign in (5) and obtain

$$\int_{-a}^{+a}\sqrt{a^2-x^2}\,dx=-a^2\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt=\frac{\pi a^2}{2}$$

If we take the upper sign in (5), the result will be incorrect: $-\frac{\pi a^2}{2}$.

Note. The method of substitution can lead to mistakes if the condition specified in the rule is not fulfilled: namely, if there are no such values z_1, z_2 to which the given values x_1, x_2 of the limits of intergration can correspond. Mistakes of this nature are encountered in Example 3.

Example 3. Evaluate the integrals

$$I_1=\int\limits_{-1}^{-\frac{1}{2}} \sqrt{1-x^2}\,dx,\quad I_2=\int\limits_{-1}^{+1} \sqrt{1-x^2}\,dx$$

which differ solely in the values of the upper limit.

Let us try to introduce an auxiliary function:

$$x^2=z \tag{6}$$

When transforming the integrand $\sqrt{1-x^2}\,dx$, we replace the factor $\sqrt{1-x^2}$ by the expression $\sqrt{1-z}$. To transform the factor dx, express x in terms of z: this yields

$$x=\pm\sqrt{z} \tag{7}$$

The simplest thing would seem to be to take the substitution

$$x=+\sqrt{z} \tag{8}$$

But then it is impossible to replace, in the integrals I_1 and I_2, the lower limit $x_1=-1$ with the new limit z_1, to which, by formula (8), would correspond the value $x=-1$. We therefore take the substitution

$$x=-\sqrt{z} \tag{9}$$

and attempt to apply it first to the integral I_1. Now, the values of the limits $x_1=-1$, $x_2=-\frac{1}{2}$ correspond to the values $z_1=1$, $z_2=\frac{1}{4}$. From (9) we obtain

$$dx=-\frac{dz}{2\sqrt{z}} \tag{10}$$

and by formula (1) we get

$$I_1=\int\limits_{-1}^{-\frac{1}{2}} \sqrt{1-x^2}\,dx=-\frac{1}{2}\int\limits_{1}^{\frac{1}{4}} \sqrt{\frac{1-z}{z}}\,dz \tag{11}$$

or, interchanging the limits,

$$I_1=\frac{1}{2}\int\limits_{\frac{1}{4}}^{1} \sqrt{\frac{1-z}{z}}\,dz \tag{12}$$

Introduce a new auxiliary function:

$$\sqrt{\frac{1-z}{z}}=t \tag{13}$$

which yields the new limits

$$t_1=\sqrt{\frac{1-\frac{1}{4}}{\frac{1}{4}}}=\sqrt{3}, \quad t_2=0$$

The corresponding substitution will be

$$z=\frac{1}{1+t^2} \qquad dz=-\frac{2t\,dt}{(1+t^2)^2} \tag{14}$$

and we obtain

$$I_1=\frac{1}{2}\int\limits_{\sqrt{3}}^{0} t\left[-\frac{2t\,dt}{(1+t^2)^2}\right]=\int\limits_{0}^{\sqrt{3}}\frac{t^2\,dt}{(1+t^2)^2}$$

As in Example 1, Sec. 324, we find

$$I_1=-\frac{\sqrt{3}}{8}+\frac{\pi}{6}\approx 0.307$$

If in place of (9) we employ the substitution (8), without noticing that the condition given in the rule is violated, then in place of (12) we get $-\frac{1}{2}\int\limits_{\frac{1}{4}}^{1}\sqrt{\frac{1-z}{z}}\,dz$ and for I_1 we obtain the wrong value (-0.307).

As for the integral I_2, neither substitution [(8) or (9)] is suitable because the former fails to yield the lower limit, the latter, the upper limit. If by mistake we take, say, the substitution (8), then for I_2 we get $\frac{1}{2}\int\limits_{1}^{1}\sqrt{\frac{1-z}{z}}\,dz$, which is zero. This is absurd since the integrand $\sqrt{1-x^2}$ is positive throughout the interval of integration.

To evaluate I_2 we can split it up into two terms as follows:

$$I_2=\int\limits_{-1}^{0}\sqrt{1-x^2}\,dx+\int\limits_{0}^{1}\sqrt{1-x^2}\,dx$$

We apply substitution (9) to the first summand, and substitution (8) to the second. Each of the two terms becomes

$$\frac{1}{2}\int\limits_{0}^{1}\sqrt{\frac{1-z}{z}}\,dz$$

so that

$$I_2=\int_0^1 \sqrt{\frac{1-z}{z}}\,dz \tag{15}$$

This is an improper integral (Sec. 328) because the integrand has a discontinuity at the point $z=0$. However, we can still apply substitution (14) (because the function $z=\frac{1}{1+t^2}$ is monotonic; see Sec. 328, Note 3).

We find that

$$I_2=\int_0^{+\infty} \frac{t^2\,dt}{(1+t^2)^2}$$

This integral is also improper (Sec. 327). Nevertheless, we can apply (Sec. 327, Note 1) integration by parts, so that proceeding as in Example 1, Sec. 324, we find

$$I_2=-\frac{t}{2\,(1+t^2)}\Bigg|_0^{+\infty}+\int_0^{+\infty}\frac{dt}{2\,(1+t^2)}$$

The first term is equal to zero, and we finally obtain

$$I_2=\frac{1}{2}\int_0^{+\infty}\frac{dt}{1+t^2}=\frac{1}{2}\arctan t\,\Bigg|_0^{+\infty}=\frac{\pi}{4}\approx 0.785$$

326. On Improper Integrals

The concept of a definite integral was introduced (Sec. 314) for a finite interval (a, b) and for a continuous function $f(x)$. A number of specific problems (see examples in Secs. 327 and 328) lead to an extension of the concept of an integral to cases of infinite intervals and discontinuous functions. This is attained by performing another passage to the limit in addition to the limit process of Sec. 314. Integrals obtained by such a double passage to the limit are called *improper* integrals in contrast to the integrals introduced in Sec. 314, which are called *proper* integrals. In Sec. 327 we consider improper integrals of the *first type* (with one or two infinite limits), in Sec. 328 we consider elementary improper integrals of the *second type* (of a discontinuous function).

327. Integrals with Infinite Limits

Definition 1. If an integral

$$\int_a^{x'} f(x)\,dx \tag{1}$$

has a finite limit as $x' \to +\infty$, then this limit is called the *integral of the function* $f(x)$ *from a to infinity* and is denoted

$$\int_a^{+\infty} f(x)\,dx \tag{2}$$

Thus, by definition

$$\int_a^{+\infty} f(x)\,dx = \lim_{x' \to +\infty} \int_a^{x'} f(x)\,dx \tag{3}$$

If, as $x' \to +\infty$, integral (1) has an infinite limit[1] or has no limit at all, then we say that the *improper integral (2) diverges*. If integral (2) has a finite limit, we say that the improper integral (2) *converges*.

Example 1. Find the integral $\int_0^{+\infty} 2^{-x}\,dx$.

Solution. We have

$$\int_0^{x'} 2^{-x}\,dx = \frac{1}{\ln 2}\left(-2^{-x}\right)\Big|_0^{x'} = \frac{1}{\ln 2}\left(1 - \frac{1}{2^{x'}}\right)$$

---

[1] When the integral $\int_a^{x'} f(x)\,dx$ has an infinite limit as $x' \to +\infty$, we say (conditionally) that the improper integral $\int_a^{+\infty} f(x)\,dx$ has an infinite value, and we write $\int_a^{+\infty} f(x)\,dx = \infty$.

As $x' \to +\infty$ this expression has the limit $\frac{1}{\ln 2}$. Hence

$$\int\limits_0^{+\infty} 2^{-x}\,dx = \lim_{x' \to +\infty} \int\limits_0^{x'} 2^{-x}\,dx = \frac{1}{\ln 2} \approx 1.4$$

Geometrical interpretation. The integral $\int\limits_0^{x'} 2^{-x}\,dx$ is depicted by the area $OBB'D$ (Fig. 343) under the line $y = 2^{-x}$. As the ordinate BB' recedes, the area $OBB'D$ increases but not without bound. It tends to $\frac{1}{\ln 2}$. We say that the area of the infinite strip under the line $y = 2^{-x}$ is equal to $\frac{1}{\ln 2}$.

Explanation. Let us consider the step-like figure in Fig. 343. The first rectangle $OACD$ has an area $OD \cdot OA = 1 \cdot 1 = 1$, the second, an area $AK\ AN = \frac{1}{2} \cdot 1 = \frac{1}{2}$, the third, an area $\frac{1}{4}$, etc. As the number of rectangles increases, their overall area tends to 2 (the sum of an infinitely decreasing progression). The number 2 is naturally considered a measure of the area of the infinite step-like strip. The area of the infinite curvilinear strip is still less.

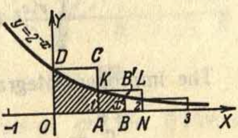

Fig. 343

Example 2. Find $\int\limits_1^{+\infty} \frac{dx}{x}$.

Solution. The integral $\int\limits_1^{x'} \frac{dx}{x} = \ln x'$ has an infinite limit as $x' \to +\infty$. The required improper integral diverges.[1]

---

[1] It then has an infinite value: $\left(\int\limits_1^{+\infty} \frac{dx}{x} = +\infty \right)$ See footnote on p. 472.

Geometrically, the area of the strip $AA'B'B$ (Fig. 344) under the hyperbola $y=\frac{1}{x}$ increases without bound (the infinite curvilinear strip has an infinite area).

Example 3. Two electrified balls with positive charges e_1 and e_2 electrostatic units are attached to a plane surface R cm apart. The ball carrying charge e_2 is released and it recedes from e_1 due to the force of repulsion of charges $F=\frac{e_1e_2}{r^2}$ (r is the variable distance between the centres in centimetres, F is the magnitude of the force in dynes).

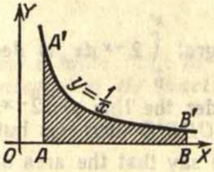

Fig. 344

The work of the force F over the interval (R, r') is expressed (in ergs) by the integral (Sec. 317)

$$\int_R^{r'} \frac{e_1e_2}{r^2}\,dr = e_1e_2\left(\frac{1}{R}-\frac{1}{r'}\right)$$

The improper integral

$$e_1e_2\int_R^{+\infty}\frac{dr}{r^2}=\lim_{r'\to\infty}\left[e_1e_2\left(\frac{1}{R}-\frac{1}{r'}\right)\right]=\frac{e_1e_2}{R}$$

expresses the total *reserve of work* of the system at hand. In physics this quantity is termed the *potential*.

Definition 2. The limit of the integral $\int_{x''}^{a} f(x)\,dx$ as $x''\to-\infty$ is called the *integral of the function* $f(x)$ *from* $-\infty$ *to* a:

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{x''\to-\infty}\int_{x''}^{a} f(x)\,dx \tag{4}$$

Convergence and divergence of the improper integral $\int_{-\infty}^{a} (x)\,dx$ is to be understood as in Definition 1.

Definition 3. The *integral of a function* $f(x)$ *from* $-\infty$ *to* $+\infty$:

$$\int_{-\infty}^{+\infty} f(x)\,dx \tag{5}$$

is the sum

$$\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx \tag{6}$$

It is independent of the choice of a. It is assumed that both improper integrals (6) converge.

Integral (5) expresses the area of a strip under the curve $y=f(x)$ extending to infinity in both directions (curve VAU in Fig. 345).

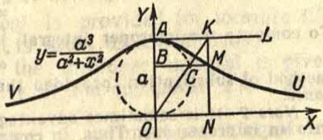

Fig. 345

Example 4. Find the area of the infinite strip under the curve $y=\frac{a^3}{a^2+x^2}$ (the *versiera*, or the *Witch of Agnesi*, Fig. 345; see Sec. 506).

Solution. The desired area is given by the integral

$$\int_{-\infty}^{+\infty} \frac{a^3\,dx}{a^2+x^2} = \int_{-\infty}^{0} \frac{a^3\,dx}{a^2+x^2} + \int_{0}^{+\infty} \frac{a^3\,dx}{a^2+x^2} \tag{7}$$

Since $\int_{0}^{x'} \frac{a^3\,dx}{a^2+x^2} = a^2 \arctan \frac{x'}{a}$, it follows that

$$\int_{0}^{+\infty} \frac{a^3\,dx}{a^2+x^2} = a^2 \lim_{x'\to+\infty} \arctan \frac{x'}{a} = \frac{\pi a^2}{2}$$

The first summand is evaluated in similar fashion and we get

$$\int_{-\infty}^{+\infty} \frac{a^3\,dx}{a^2+x^2} = \pi a^2 \tag{8}$$

Note 1 The basic formula

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

as applied to the convergent integral $\int_a^{+\infty} f(x)\,dx$ is of the form

$$\int_a^{\infty} f(x)\,dx = F(\infty) - F(a)$$

The symbol $F(\infty)$ denotes $\lim\limits_{x' \to \infty} F(x')$

The formula for integration by parts is applied in similar fashion. To compute the improper integral $\int_a^{\infty} f(x)\,dx$ we can also use the method of substitution, provided that the function $x = \varphi(z)$ is monotonic.

Note 2 It is sometimes advantageous to convert a proper integral into an improper one. Thus, in computing the integral

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x \cos^2 x\,dx}{(\sin^3 x + \cos^3 x)^2} \tag{9}$$

It is best to introduce the auxiliary function

$$\tan x = z \tag{10}$$

This yields

$$\int_0^{\infty} \frac{z^2\,dz}{(1+z^3)^2} = -\frac{1}{3(1+z^3)}\Bigg|_0^{\infty} = \frac{1}{3} \tag{11}$$

Transforming integral (9) to the form (11), we regard the given integral as the limit of the integral

$$\int_0^{x'} \frac{\sin^2 x \cos^2 x\,dx}{(\sin^3 x + \cos^3 x)^2} \text{ as } x' \to \frac{\pi}{2}$$

328. The Integral of a Function with a Discontinuity

Definition 1. Let a function $f(x)$ be discontinuous at the point $x = b$ and continuous at all other points of the interval (a, b).

If the integral

$$\int_a^{x'} f(x)\,dx \tag{1}$$

has a finite limit as x' tends to b (remaining less than b), then this limit is called the *improper integral of the function* $f(x)$ *from* a *to* b and is denoted in the same way as the similar proper integral:

$$\int_a^b f(x)\,dx = \lim_{x' \to b-0} \int_a^{x'} f(x)\,dx \tag{2}$$

For proper integrals, *proof* is provided for formula (2), but for improper integrals it is *taken as a definition*.

A similar definition for the improper integral is given when $f(x)$ has a discontinuity at the end-point $x=a$ alone of the interval (a, b).

Convergence and divergence of an improper integral are to be understood as in Sec. 327.

Definition 2. If $f(x)$ has a discontinuity only at some interior point c of the interval (a, b), then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{3}$$

It is assumed that both improper integrals on the right converge.

Formula (3) is *proved* for proper integrals; here, it serves as a *definition* of the improper integral $\int_a^b f(x)\,dx$.

Note 1. Definition 2 may be extended to the case when there are two, three, etc. points of discontinuity in the interval (a, b) For instance, for two points c', c'' we have

$$\int_a^b f(x)\,dx = \int_a^{c'} f(x)\,dx + \int_{c'}^{c''} f(x)\,dx + \int_{c''}^b f(x)\,dx \tag{3a}$$

Example 1. Find

$$\int_0^a \frac{a^2\,dx}{\sqrt{a^2-x^2}}$$

This integral is improper because the integrand function is discontinuous (becomes infinite) at $x=a$. The integral converges because the function

$$\int_0^{x'} \frac{a^2\,dx}{\sqrt{a^2-x^2}} = a^2 \arcsin\frac{x'}{a} \tag{4}$$

tends to the limit $\frac{\pi a^2}{2}$ as $x' \to a$. Hence

$$\int_0^{a} \frac{a^2\,dx}{\sqrt{a^2-x^2}} = \frac{\pi a^2}{2} \tag{5}$$

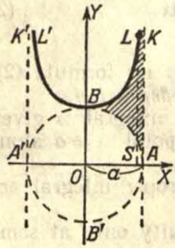

Fig. 346

Geometrically, the area of the infinite strip $KAOBL$[1] (that is, the limit of the area $LSOB$ as the point S tends to A; Fig. 346) is equal to the area of the semicircle $BOB'A$; hence, the hatched figure going off to infinity is equivalent to the sector AOB'.

Example 2. Find $\int_{-a}^{+a} a\sqrt[3]{\frac{a^2}{x^2}}\,dx$.

This integral is improper because the integral becomes infinite at $x=0$ inside the interval $(-a, +a)$. By Definition 2 we have

$$\int_{-a}^{+a} a\sqrt[3]{\frac{a^2}{x^2}}\,dx = a^{5/3}\int_{-a}^{0} x^{-2/3}\,dx + a^{5/3}\int_{0}^{a} x^{-2/3}\,dx \tag{6}$$

By Definition 1,

$$\int_{-a}^{0} x^{-2/3}\,dx = \lim_{x'\to 0}\int_{-a}^{x'} x^{-2/3}\,dx =$$

$$= \lim_{x'\to 0} 3\left(a^{1/3} - x'^{1/3}\right) = 3a^{1/3}$$

---

[1] The radius a of circle O is a mean proportional between the ordinate of the line $L'BL$ and the corresponding ordinate of the semicircle $A'BA$; this makes it easy to construct the line $L'BL$.

The same goes for the second term in formula (6). We finally get

$$\int_{-a}^{+a} a\sqrt[3]{\frac{a^2}{x^2}}\,dx=6a^2$$

Geometrically, the area of the infinite strip $ADLL'D'A'$ (Fig. 347) is three times the area of the rectangle $A'ADD'$ (so that the "infinite spire" $DLL'D'$ is equal to the square constructed on DD').

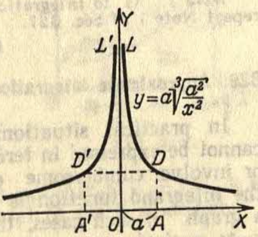

Fig. 347

Example 3. In the expression $\int_{-1}^{+1}\frac{dx}{x^2}$ the integrand is discontinuous at the point $x=0$, and the improper integrals $\int_{-1}^{0}\frac{dx}{x^2}$ and $\int_{0}^{1}\frac{dx}{x^2}$ diverge (because the integrals $\int_{-1}^{x'}\frac{dx}{x^2}=1-\frac{1}{x'}$ and $\int_{x''}^{1}\frac{dx}{x^2}=\frac{1}{x''}-1$ have infinite limits as $x'\to-0$ and $x'\to+0$). Hence, this expression is not in the least an improper integral (in the meaning of Definition 2). The infinite strip $ADLL'D'A'$ (Fig. 348) under the line $y=\frac{1}{x^2}$ has an infinite area.

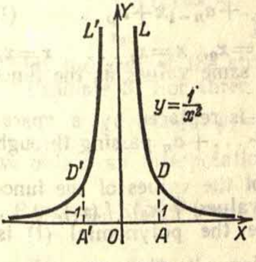

Fig. 348

Note 2. Applying the basic formula of integral calculus

$$\int_{a}^{b} f(x)\,dx=F(b)-F(a) \qquad (7)$$

to expression $\int_{-1}^{1}\frac{dx}{x^2}$, we would get the negative number -2; this absurd result $\left(\text{the integrand } \frac{1}{x^2} \text{ is everywhere positive!}\right)$ is obtained because the expression $\int_{-1}^{+1}\frac{dx}{x^2}$ is meaningless. Now if the improper integrals

$$\int_a^c f(x)\,dx, \quad \int_c^b f(x)\,dx$$

entering into (3) converge, then formula (7) is always *true* for the improper integral $\int_a^b f(x)\,dx$.

Note 3. As to integration by parts and by substitution, we can repeat Note 1 of Sec. 327.

329. Approximate Integration

In practical situations we often encounter integrals that cannot be expressed in terms of elementary functions (Sec. 309) or involve cumbersome expressions. It often happens that the integrand function is specified in the form of a table or a graph. In such cases, the integrals are found by approximate methods.

Historically, the first was worked out by Newton in his *method of infinite series* (see Sec. 270). It is still used (on a more rigorous basis; see Sec. 402).

Another method, often called the *method of quadratures*,[1] consists in substituting for the integrand function $y=f(x)$ a polynomial of degree n

$$P(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n \tag{1}$$

such that for the given values $x=x_0$, $x=x_1$ . . . , $x=x_n$ (their number is $n+1$) it has the same values as the function $f(x)$.

Geometrically, the curve $y=f(x)$ is replaced by a "parabola of degree n" $y=a_0x^n+a_1x^{n-1}+\ldots+a_n$ passing through $n+1$ points of the given curve.

The approximate computation of the values of the function $f(x)$ from several of its given values $f(x_0)$, $f(x_1)$, . . . , $f(x_n)$ is called *interpolation*, while the polynomial (1) is called the *interpolation polynomial*.

Integrating the interpolation polynomial, we obtain an approximate value of the integral of the function $f(x)$.

---

[1] It is also based on the ideas of Newton and was first developed by Taylor, Simpson, and others. Some of the latest work has been done by the Soviet mathematicians V. P. Vetchinkin and F. M. Kogan.

Example 1. For one given value $y_0=f(x_0)$ we obtain an interpolation polynomial of degree zero:

$$P(x)=y_0 \tag{2}$$

The curve $y=f(x)$ is replaced by the horizontal straight line UV (Fig. 349) passing through the given point $M_0(x_0, y_0)$.

The approximate value of the integral

$$\int\limits_{x_0-\frac{h}{2}}^{x_0+\frac{h}{2}} f(x)\,dx \approx \int\limits_{x_0-\frac{h}{2}}^{x_0+\frac{h}{2}} y_0\,dx=y_0h \tag{3}$$

yields the area of the rectangle $AUVB$ (in place of the area of the curvilinear trapezoid $AA'B'B$).

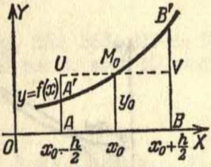

Fig. 349

Example 2. For two given values $y_0=f(x_0)$, $y_1=f(x_0+h)$ we obtain an interpolation polynomial of the first degree:

$$P(x)=y_0+\frac{y_1-y_0}{h}(x-x_0) \tag{4}$$

It represents the straight line M_0M_1 (Fig. 350) passing through the points $M_0(x_0, y_0)$, $M_1(x_0+h, y_1)$. The corresponding approximate value of the integral

$$\int\limits_{x_0}^{x_0+h} f(x)\,dx \approx \int\limits_{x_0}^{x_0+h} P(x)\,dx=\frac{1}{2}(y_0+y_1)\,h \tag{5}$$

yields the area of the straight-line trapezoid $x_0M_0M_1x_1$.

Example 3. For three given values

$$y_0=f(x_0), \quad y_1=f(x_0+h), \quad y_2=f(x_0+2h)$$

we obtain an interpolation polynomial of degree two:

$$P(x)=y_0+\frac{y_1-y_0}{h}(x-x_0)+\frac{y_2-2y_1+y_0}{2h^2}(x-x_0)\,[x-(x_0+h)] \tag{6}$$

We can verify the validity of formula (6) by substituting successively

$$x=x_0, \quad x=x_0+h, \quad x=x_0+2h$$

We obtain

$$P(x_0)=y_0, \quad P(x_0+h)=y_1, \quad P(x_0+2h)=y_2$$

The formula becomes complicated if we arrange $P(x)$ in powers of x. The expressions y_1-y_0 $(=\Delta y_0)$ and $y_2-2y_1+y_0$ $(=\Delta^2 y_0)$ are the first and second differences (Sec. 258) of the function $f(x)$.

The polynomial (6) is a parabola with vertical axis (Fig. 351) passing through three points: $M_0(x_0, y_0)$, $M_1(x_0+h, y_1)$, $M_2(x_0+2h, y_2)$.

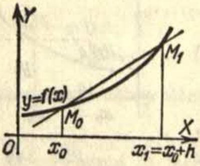

Fig. 350

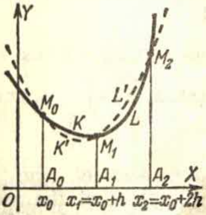

Fig. 351

The approximate value[1]

$$\int_{x_0}^{x_0+2h} f(x)\,dx \approx \int_{x_0}^{x_0+2h} P(x)\,dx = \frac{1}{3}(y_0+4y_1+y_2)h \qquad (7)$$

yields the area of the *parabolic trapezoid* $A_0M_0K'M_1L'M_2A_2$ (in place of the curvilinear trapezoid $A_0M_0KM_1LM_2A_2$).

Formulas (4), (6) generalize to an arbitrary number of equidistant values of x. For four values we have

$$P(x)=y_0+\frac{y_1-y_0}{h}(x-x_0)+\frac{y_2-2y_1+y_0}{2!h^2}(x-x_0)[x-(x_0+h)]+$$
$$+\frac{y_3-3y_2+3y_1-y_0}{3!h^3}(x-x_0)[x-(x_0+h)][x-(x_0+2h)]$$

or, more succinctly,

$$P(x)=y_0+\frac{\Delta y_0}{\Delta x_0}(x-x_0)+\frac{\Delta^2 y_0}{2!\Delta x_0^2}(x-x_0)(x-x_1)+$$
$$+\frac{\Delta^3 y_0}{3!\Delta x_0^3}(x-x_0)(x-x_1)(x-x_2)$$

---

[1] Evaluation of the integral $\int_{x_0}^{x_0+2h} P(x)\,dx$ is simplified if we introduce an auxiliary variable $x-(x_0+h)=z$.

The corresponding general formula is known as *Newton's interpolation formula*. From it, Taylor obtained (in nonrigorous fashion) an expansion of the function $f(x)$ in a power series (he put $x_1 = x_0$, $x_2 = x_0$, etc. and replaced the differences Δx_0, Δy_0, $\Delta^2 y_0$, etc. by the differentials dx_0, dy_0, $d^2 y_0$, etc.) (cf. Sec. 270).

330. Rectangle Formulas

Using points $x_1, x_2, \ldots, x_{n-1}$ (Figs. 352, 353) divide the interval of integration (a, b) into n equal parts, each of length

$$h = \frac{b-a}{n}$$

For the sake of uniformity, put $a = x_0$, $b = x_n$. Use $x_{1/2}$, $x_{3/2}$, $x_{5/2}$, ... (Fig. 354) to denote the midpoints of the subintervals (x_0, x_1), (x_1, x_2), (x_2, x_3), ... We set

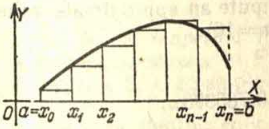

Fig. 352

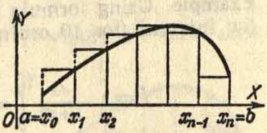

Fig. 353

$$f(x_0) = y_0, \quad f(x_1) = y_1, \quad f(x_2) = y_2, \ \ldots;$$
$$f(x_{1/2}) = y_{1/2}, \quad f(x_{3/2}) = y_{3/2}, \quad f(x_{5/2}) = y_{5/2}, \ \ldots$$

The *rectangle formulas* are the following approximate equalities:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\,[y_0 + y_1 + \ldots + y_{n-1}], \tag{1}$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\,[y_1 + y_2 + \ldots + y_n], \tag{2}$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[y_{1/2} + y_{3/2} + \cdots\ y_{\frac{2n-1}{2}}\right] \tag{3}$$

Expressions (1), (2), (3) yield the areas of the step-like figures in Figs. 352, 353, 354 (cf. Sec. 329, Example 1).

In most cases, for a given n, formula (3) is more accurate than (1) and (2). As n increases, the accuracy of formulas (1), (2), and (3) increases without bound.

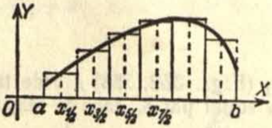

Fig. 354

Note. The limiting error of formula (3) is

$$\frac{(b-a)^3}{24n^2}M_2 \tag{4}$$

where M_2 is the greatest value of $|f''(x)|$ in the interval (a, b). For empirical functions, in place of M_2 we take the greatest value of the quantity $\left|\frac{\Delta^2 y}{\Delta x^2}\right|$

Example. Using formula (3) compute an approximate value of the integral (for 10 ordinates: $n=10$)

$$I=\int\limits_0^1 \frac{dx}{1+x^2}\left(=\frac{\pi}{4}=0.785398\ldots\right)$$

$$\begin{array}{ll}
x_{1/2}=0.05 & y_{1/2}=0.9975 \\
x_{3/2}=0.15 & y_{3/2}=0.9780 \\
x_{5/2}=0.25 & y_{5/2}=0.9412 \\
x_{7/2}=0.35 & y_{7/2}=0.8909 \\
x_{9/2}=0.45 & y_{9/2}=0.8316 \\
x_{11/2}=0.55 & y_{11/2}=0.7678 \\
x_{13/2}=0.65 & y_{13/2}=0.7029 \\
x_{15/2}=0.75 & y_{15/2}=0.6400 \\
x_{17/2}=0.85 & y_{17/2}=0.5806 \\
x_{19/2}=0.95 & y_{19/2}=0.5256
\end{array}
\qquad \frac{b-a}{n}=\frac{1}{10}$$

$$\text{the sum } \sum y=7.8561$$

$$I\approx\frac{b-a}{n}\sum y=\underline{\underline{0.78561}}$$

The error amounts to roughly 0.0002.

We have $f''(x) = 2\frac{3x^2-1}{(1+x^2)^3}$. The greatest value of $|f''(x)|$ in the interval (0, 1) is equal to 2 (it is reached at $x=0$). Substituting into (4) $n=10$, $M_2=2$, we find the limiting error 0.00085. Hence, there is no sense in computing $y_{1/2}$, $y_{3/2}$ and so on to more than four places.

By formulas (1) and (2) (the values y_0, y_1, y_2, ... are given in Sec. 331), we obtain $I \approx 0.8099$ and $I \approx 0.7599$, i.e. the error is approximately greater by a factor of 50.

331. Trapezoid Rule

Using the notations of Sec. 330, we have

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[\frac{y_0+y_n}{2}+y_1+y_2+\ldots+y_{n-1}\right] \tag{1}$$

This is the *trapezoid formula*. It yields the total area of the trapezoids shown in Fig. 355 (cf. Sec. 329, Example 2).

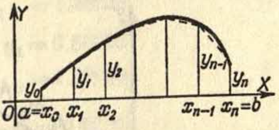

Fig. 355

Note. The limiting error of formula (1) amounts to $\frac{(b-a)^3}{12n^2}M_2$, where M_2 is the largest value of $|f''(x)|$ in the interval (a, b) (cf. Sec. 330, Note.)

Example. Let us compute the integral $I = \int_0^1 \frac{dx}{1+x^2}$ $(=0.785398\ldots)$ using the trapezoid formula and 11 ordinates $(n=10)$. This yields

$x_1=0.1$	$y_1=0.9901$
$x_2=0.2$	$y_2=0.9615$
$x_3=0.3$	$y_3=0.9174$
$x_4=0.4$	$y_4=0.8621$
$x_5=0.5$	$y_5=0.8000$
$x_6=0.6$	$y_6=0.7353$
$x_7=0.7$	$y_7=0.6711$
$x_8=0.8$	$y_8=0.6098$
$x_9=0.9$	$y_9=0.5525$

the sum $\sum_{i=1}^{i=9} y_i = 7.0998$

$x_0=0.0 \quad y_0=1.0000$

$x_{10}=1.0 \quad y_{10}=0.5000$

$y_0+y_{10}=1.5000$

$$I \approx \frac{1}{10}\left(\frac{1.5000}{2}+7.0998\right) = \underline{\underline{0.78498}}$$

The error comes out to roughly 0.0004, as in the case of formula (3), Sec. 330. But there the approximation was in excess, here it is in defect.

332. Simpson's Rule (for Parabolic Trapezoids)

In the notations of Sec. 330, we have

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left[\frac{y_0+y_n}{2}+(y_1+y_2+\ldots+y_{n-1})+\right.$$

$$\left.+2\,(y_{1/2}+y_{3/2}+\ldots+y_{n-1/2})\right] \qquad (1)$$

This is *Simpson's formula.* It gives the total area of the curvilinear trapezoids $x_0M_0M_{1/2}M_1x_1$, $x_1M_1M_{3/2}M_2x_2$, ...

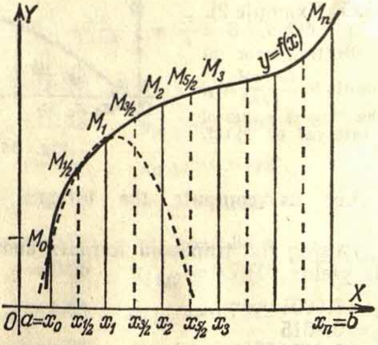

Fig. 356

(Fig. 356), where in place of the arcs $M_0M_{1/2}M_1$, $M_1M_{3/2}M_2$, ... of the given curve $y=f(x)$ we take the same arcs of parabolas with vertical axes. Fig. 356 shows one parabola $M_0M_{1/2}M_1$ (cf. Sec. 329, Example 3).

For the same number of ordinates, Simpson's formula is in most cases much more accurate than the rectangle formula (Sec. 330) and the trapezoid formula (Sec. 331).

Note. The limiting error of formula (1) is

$$\frac{(b-a)^5}{180\,(2n)^4}M_4 \tag{2}$$

Here, M_4 is the largest value of $\left|f^{\mathrm{IV}}(x)\right|$ in the interval (a, b).

Example. Compute the integral

$$I=\int\limits_0^1 \frac{dx}{1+x^2}\ (=0.785398\ldots)$$

by Simpson's formula using five ordinates:

$$\left(n=2,\ \frac{b-a}{3n}=\frac{1}{6}\right)$$

$$\begin{array}{rr}
x_0=0 & \frac{1}{2}\,y_0=0.50000 \\
x_{1/2}=0.25 & 2y_{1/2}=1.88235 \\
x_1=0.50 & y_1=0.80000 \\
x_{3/2}=0.75 & 2y_{3/2}=1.28000 \\
x_2=1.00 & \frac{1}{2}\,y_2=0.25000 \\
\hline
 & \text{the sum}=4.71235
\end{array}$$

$$I\approx\frac{1}{6}\cdot 4.71235=\underline{\underline{0.78539}}$$

The error amounts to approximately 0.00001, which is less by a factor of 40 than in the examples of Secs. 330, 331, though the number of ordinates in the latter instances was twice as great.

It is useful to compare the Simpson formula with the trapezoid
formula. In the first case, we have an extra term $2\,(y_{1/2}+y_{3/2}+\ldots)$
which is about twice the sum of the remaining terms. Which means
that the expression in square brackets in the Simpson formula is ro-
ughly three times the corresponding expression in the trapezoid for-
mula. Accordingly, the factor $\frac{b-a}{3n}$ is three times less than the factor
$\frac{b-a}{n}$.

333. Areas of Figures Referred to Rectangular Coordinates

The area of a curvilinear trapezoid ($aABb$ in Fig. 357) located above the x-axis is expressed (Sec. 316) by the integral

$$S=\int_a^b f(x)\,dx \tag{1}$$

For the trapezoid beneath the x-axis, we have

$$S=-\int_a^b f(x)\,dx \tag{1'}$$

Figures of any other shape are partitioned into trapezoids (or the trapezoids are completed) and then the area is found as

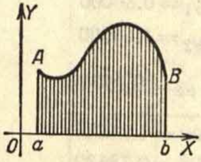

Fig. 357

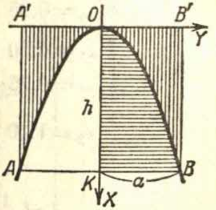

Fig. 358

the sum (or difference) of the areas of the trapezoids. Computation is facilitated by a suitable choice of rectangular system.

Example 1. Find the area of the parabolic segment AOB (Fig. 358) from the base $AB=2a$ and the height $KO=h$.

Let us choose the axes as in Fig. 358. Divide the segment AOB into equal curvilinear trapezoids OKB and OKA:

$$\text{area } OKB=\int_0^h y\,dx \tag{2}$$

The coordinates x, y are connected by the equation

$$y^2=2px \tag{3}$$

The parameter p is found from the condition that the parabola passes through the point $B(h, a)$:

$$a^2=2ph \tag{4}$$

From (3) and (4) we find

$$y=\frac{a}{\sqrt{h}}\sqrt{x} \tag{5}$$

Substituting into (2) we get

$$\text{area } OKB=\frac{a}{\sqrt{h}}\int_0^h \sqrt{x}\,dx=\frac{2}{3}ah,$$

$$\text{area } AOB=2 \text{ area } OKB=\frac{2}{3}(2a)h$$

that is, the area of the parabolic segment comprises $\frac{2}{3}$ of the area of the rectangle $ABB'A'$ having the same base and the same height.

Alternative method. Take the segment AOB and complete the rectangle $AA'B'B$. The area of the complementary trapezoid is $\int_{-a}^{+a} x\,dy$ or by virtue of (5)

$$\frac{h}{a^2}\int_{-a}^{+a} y^2\,dy=\frac{1}{3}\cdot 2ah$$

Hence

$$\text{area } AOB=2ah-\frac{1}{3}\cdot 2ah=\frac{2}{3}\cdot 2ah$$

Example 2. Find the area S of the figure (Fig. 359) lying between the parabolas $y^2=2px$ and $x^2=2py$.

The area S is the difference between the areas of $ONAL$ and $OKAL$. The parabolas intersect at the points $O(0, 0)$ and $A(2p, 2p)$. We have

$$S=\int_0^{2p}\sqrt{2px}\,dx-\int_0^{2p}\frac{x^2}{2p}\,dx=\frac{4}{3}p^2=\frac{(2p)^2}{3}$$

S comprises a third of the area of the square[1] $OLAR$.

---

[1] Using the result of Example 1, the area S may be found in elementary fashion.

Example 3. The area of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$S=2\int_{-a}^{+a}\frac{b}{a}\sqrt{a^2-x^2}\,dx=\pi ab$$

Example 4. The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (Fig. 360):

$$\text{area } AMP=\int_a^x\frac{b}{a}\sqrt{x^2-a^2}\,dx=$$

$$=\frac{b}{2a}x\sqrt{x^2-a^2}-\frac{ab}{2}\ln\frac{x+\sqrt{x^2-a^2}}{a}=\frac{xy}{2}-\frac{ab}{2}\ln\left(\frac{x}{a}+\frac{y}{b}\right)$$

$$\text{area } OAM=\text{area } OPM-\text{area } AMP=\frac{ab}{2}\ln\left(\frac{x}{a}+\frac{y}{b}\right)$$

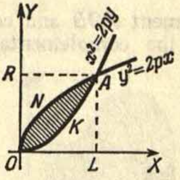

Fig. 359

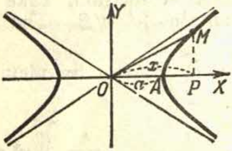

Fig. 360

Example 5. The cycloid (Fig. 361):

$$x=a(t-\sin t),\quad y=a(1-\cos t)$$

$$\text{area } ONALO=\int_0^{2\pi}y\,dx=a^2\int_0^{2\pi}(1-\cos t)^2\,dt=3\pi a^2$$

Hence, the area of the cycloid is three times the area of the generating circle.

334. Scheme for Employing the Definite Integral

The definite integral can be used to express a great variety of geometrical and physical quantities (see Secs. 335-338). In all cases, the following uniform scheme is employed.

1) The unknown quantity U is associated with an interval $(a\ b)$, the range of some argument.

Thus, to express by an integral the area $aABb$ under the line AB (Fig. 362) we associate it with the interval (a, b), the range of the abscissa x.

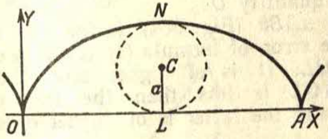

Fig. 361

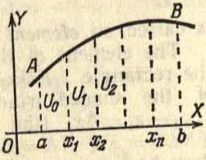

Fig. 362

2) The interval (a, b) is partitioned into subintervals (a, x_1), (x_1, x_2), $\ldots$, (x_n, b) (the number of subintervals will then tend to infinity and their lengths will approach zero).

Let the unknown quantity U be subdivided into parts U_0, U_1, U_2, $\ldots$ (Fig. 362), the sum of which yields U.

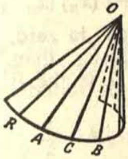

Fig. 363

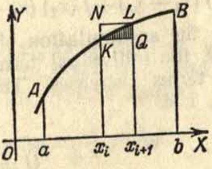

Fig. 364

Quantities having this property are called *additive* quantities (in contrast to nonadditive quantities; for instance, the angle between the generatrices of a conical surface is a nonadditive quantity). The angle AOB (Fig. 363) may be associated with the interval (a, b), where $a=\overset{\frown}{RA}$ and $b=\overset{\frown}{RB}$, which are arcs of the directrix reckoned from some initial point R. But if we partition (a, b) into subintervals (a, c) and (c, b), the corresponding angles AOC and COB do not yield, as a sum, the angle AOB.

An additive quantity can be expressed by an integral, a nonadditive quantity cannot.

(3) One of the subintervals U_i is taken as a typical representative of the parts U_0, U_1, $\ldots$. Depending on the conditions of the problem, it is expressed by an approximate

formula of the form

$$U_i \approx f(x_i)(x_{i+1}-x_i) \tag{1}$$

and the error must be of higher order than $x_{i+1}-x_i$.

The expression $f(x_i)(x_{i+1}-x_i)$ or, briefly,

$$f(x)\,\Delta x \tag{2}$$

is called an *element* of the quantity U.

The element of the area $aABb$ (Fig. 364) is the area of the rectangle x_iKQx_{i+1}; the error of formula (1) is the area of the shaded triangle KQL. It is of higher order than $x_{i+1}-x_i=\Delta x_i$ (the area KQL is less than the area of $KNLQ=KQ\cdot KN=\Delta x_i\Delta y_i$, and the latter is of higher order than Δx_i).

(4) From the approximate equality (1) we get the *exact* equality

$$U=\int_a^b f(x)\,dx \tag{3}$$

Explanation As the number n increases, the error in the sum

$$f(x_0)(x_1-x_0)+f(x_1)(x_2-x_1)+\ldots+f(x_n)(x_{n+1}-x_n) \tag{4}$$

(despite the accumulation of errors) tends to zero, since the errors of the individual terms decrease faster than the number of terms increases. That is why U is the limit of the sum (4), i. e.

$$U=\int_a^b f(x)\,dx$$

335. Areas of Figures Referred to Polar Coordinates

The area S of the sector AOB bounded by the curve AB and the rays OA and OB (Fig. 365) is given by the formula

$$S=\frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2 d\varphi \tag{1}$$

where r is the radius vector of the variable point M of the curve AB, φ is its polar angle.

Explanation. The scheme of Sec. 334 is applied here as follows.

(1) The area of AOB is associated with the interval (φ_1, φ_2) of variation of the polar angle.

(2) The interval (φ_1, φ_2) is partitioned into subintervals, the sector AOB is thus subdivided into subsectors of the form AOM_1, M_1OM_2 and so on; the sum of their areas yields the area of AOB.

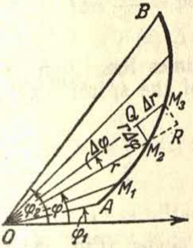

Fig. 365

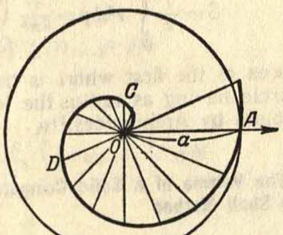

Fig. 366

(3) We take the subsector M_2OM_3 (Fig 365) as a representative of the subsectors AOM_1, M_1OM_2, etc., and replace it by the circular sector M_2OQ, the area of which,

$$\frac{1}{2}OM_2 \cdot M_2Q = \frac{1}{2}r \cdot r\,\Delta\varphi = \frac{1}{2}r^2\,\Delta\varphi$$

is an element of the area of AOB. The error of the approximate formula

$$\text{area } M_2OM_3 \approx \frac{1}{2}r^2\,\Delta\varphi \tag{2}$$

is of higher order than $\Delta\varphi$.

The error is equal to the area of the curvilinear triangle M_2QM_3, and the latter is less, area $QM_2RM_3 = \frac{1}{2}\left(OM_3^2 - OM_2^2\right)\Delta\varphi \approx r\Delta r\Delta\varphi$.

(4) Formula (1) follows from the approximate equation (2).

Example. Find the area of the figure $OCDA$ (Fig. 366) bounded by the first whorl of the spiral of Archimedes (Sec. 75) and the line-segment $OA=a$ (the *lead* of the spiral)

Choosing the polar system as in Fig. 366, we have

$$r=\frac{a}{2\pi}\varphi$$

The beginning O of the whorl and the end A are associated with the values

$$\varphi_1=0,\ \varphi_2=2\pi$$

By formula (1)

$$S=\frac{1}{2}\int_0^{2\pi} r^2 d\varphi=\frac{a^2}{8\pi^2}\int_0^{2\pi}\varphi^2\,d\varphi=\frac{1}{3}\pi a^2 \tag{3}$$

The area of the first whorl is three times less than that of the circle having as radius the lead of the spiral. This result was found by Archimedes.[1]

336. The Volume of a Solid Computed by the Shell Method

We consider a solid of arbitrary shape (Fig. 367). Let the areas $F(x)$ be known of all its cross sections parallel to the plane R (x is the distance of a section from the R plane). Then the volume of the solid is

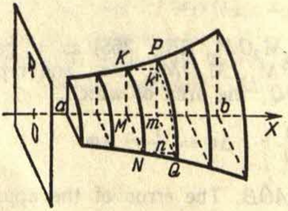

Fig. 367

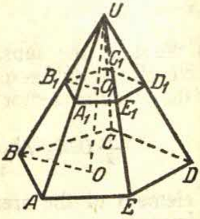

Fig. 368

$$V=\int_{x_1}^{x_2} F(x)\,dx \tag{1}$$

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[1] Though Archimedes did not explicitly introduce either the concept of an integral or that of a limit, his method actually coincides with the method of integral calculus.

Explanation. Divide the solid into parallel slices, or shells; the solid $NMKQmP$ is a representative of these slices. Construct the cylinder $NMKnmk$. Its volume, equal to $F(x)\,\Delta x$, is an element of the volume V. Formula (1) follows therefrom (cf. Secs. 334 and 335, Explanation).

Example 1. Find the volume V of a pyramid $UABCDE$ (Fig. 368) from the area of the base S and the altitude $UO=H$.

Solution. The area $F(x)$ of a section $A_1B_1C_1D_1E_1$ is found from the proportion

$$F(x):S=UO_1^2:UO^2=x^2:H^2$$

By formula (1)

$$V=\int_0^H F(x)\,dx=\frac{S}{H^2}\int_0^H x^2\,dx=\frac{1}{3}SH \tag{2}$$

This formula is a familiar formula of elementary geometry, but the derivation there is much more complicated.

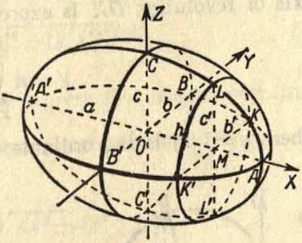


Fig. 369

Example 2. Find the volume of an ellipsoid (Sec. 173) with axes $2a$, $2b$, $2c$.

Solution. The section $KLK'L'$ (Fig. 369) parallel to the principal ellipse $BCB'C'$ and distant $h=OM$ from it is (Sec. 173) the ellipse with semiaxes

$$b'=MK=b\sqrt{1-\frac{h^2}{a^2}},\quad c'=ML=c\sqrt{1-\frac{h^2}{a^2}}$$

The area $F(h)$ of a section is equal (Sec. 333, Example 3) to

$$F(h)=\pi b'c'=\pi bc\left(1-\frac{h^2}{a^2}\right)$$

By formula (1)

$$V=\int_{-a}^{+a} F(h)\,dh=2\int_0^a \pi bc\left(1-\frac{h^2}{a^2}\right)dh=\frac{4}{3}\pi abc \tag{3}$$

A cone with elliptical base $BCB'C'$ and altitude $OA=a$ has volume

$$V_1=\frac{1}{3}Sa$$

(the derivation is the same as in Example 1); that is, $V_1=\frac{1}{3}\pi abc$. Hence, the volume of an ellipsoid is four times that of a cone having as base one of the principal sections and as vertex the opposite vertex of the ellipsoid. This result was found by Archimedes (for an ellipsoid of revolution).

When the ellipsoid becomes a sphere ($a=b=c$), we get the familiar formula $V=\frac{4}{3}\pi r^3$.

337. The Volume of a Solid of Revolution

The volume V of a solid (Fig. 370) bounded by a surface of revolution and two planes P_1, P_2 perpendicular to the axis of revolution OX is expressed by the formula

$$V=\pi\int_{x_1}^{x_2} y^2\,dx \tag{1}$$

where $y=f(x)$ is the ordinate of the meridian AB.

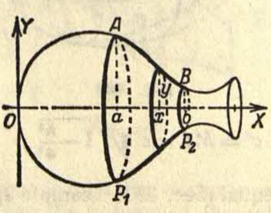


Fig. 370

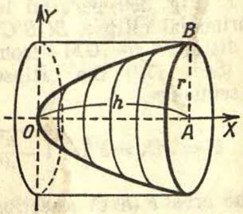


Fig. 371

Note. The quantity πy^2 is the area of a transverse (circular) section (cf. Sec. 336).

Example. Find the volume of a segment of a paraboloid of revolution (Fig. 371) from the radius of the base $AB=r$ and the altitude $OA=h$.

Solution. As in Sec. 333 (Example 1), we find that the meridian (parabola) is given by the equation

$$y^2 = \frac{r^2 x}{h}$$

By formula (1)

$$V = \pi \int_0^h \frac{r^2 x}{h}\, dx = \frac{1}{2}\pi r^2 h$$

That is, a segment of a paraboloid is equal in volume to half the cylinder having the same base and the same altitude.

This result was found by Archimedes.

338. The Arc Length of a Plane Curve

The length s of an arc AB of a plane curve is expressed (in rectangular coordinates) by the formula

$$s = \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \qquad (1)$$

where t is a parameter used to express the current coordinates x, y $(t_2 > t_1)$.

If the parameter has not been chosen, formula (1) is more conveniently written as

$$s = \int_{(A)}^{(B)} \sqrt{dx^2 + dy^2} \qquad (2)$$

The labels (A) and (B) indicate that for the limits of integration we have to take those values of the parameter which correspond to the ends of the arc AB.

In particular, it is often convenient to take the abscissa x as the parameter. Then we have

$$s = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \qquad (3)$$

Explanation. The infinitesimal arc $\overset{\frown}{MN}$ is equivalent to the chord MN (Fig. 372). On the other hand,

$$MN = \sqrt{MQ^2 + QN^2} = \sqrt{\Delta x^2 + \Delta y^2} \approx \sqrt{dx^2 + dy^2}$$

Hence

$$\overset{\smile}{MN} \approx \sqrt{dx^2+dy^2}$$

Thus, the expression $\sqrt{dx^2+dy^2}$ (it is proportional to the increment Δt in the argument t) is an element (differential) of the arc AB. By Item 4 of the scheme given in Sec. 334 we obtain formula (2).

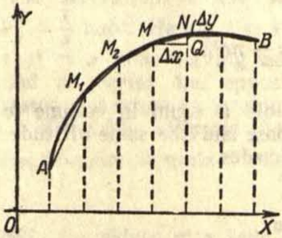


Fig. 372

Finding the arc length is called *rectification* of the arc.

Example. Find the arc length of one branch of the cycloid (Sec. 253) $x=a(t-\sin t)$, $y=a(1-\cos t)$.

Solution.

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=a\sqrt{(1-\cos t)^2+\sin^2 t}=$$

$$=a\sqrt{2(1-\cos t)}=2a\left|\sin\frac{t}{2}\right|, \qquad (4)$$

$$s=\int_0^{2\pi} 2a\sin\frac{t}{2}\,dt=8a \qquad (5)$$

The length of one branch of a cycloid is equal to four times the diameter of the generating circle.

If we compute the total length of two branches of a cycloid using the formula $s=\int_0^{4\pi} 2a\sin\frac{t}{2}\,dt$, we get zero. The error is due to the fact that in the interval $(2\pi, 4\pi)$ we have

$$\left|\sin\frac{t}{2}\right|=-\sin\frac{t}{2}\ \left(\text{and not} +\sin\frac{t}{2}\right)$$

Note. When we measure the length of a curving path by paces or the length of a sinuous river (on a map) using some scale unit, we are implying the equivalence of arc and chord. This property is suggested by practical experience. But if we wish to prove it mathematically, we must define arc length.

Definition. The arc length of a plane or space curve is the limit to which the perimeter of a polygonal line inscribed in the arc tends when the number of segments of the polygonal line increases without bound and the lengths of the segments approach zero. This is the usual definition.

Using this definition, we can now prove that $\overset{\smile}{MN} \approx MN$. Formula (1) is also directly derived from it, so that the scheme of Sec. 334 would appear to be dispensed with completely. Actually, however, the scheme is "hidden" in the definition.

Arc length may also be defined as the limit of a *circumscribed* polygonal curve. This definition is equivalent to the preceding one.

339. Differential of Arc Length

The *differential of arc length* is given (Sec. 338, Explanation) by the formula

$$ds = \sqrt{dx^2 + dy^2} \tag{1}$$

If the argument is x, then (Fig. 373)

$$dx = MQ, \quad dy = QP,$$

$$ds = \sqrt{MQ^2 + QP^2} = MP$$

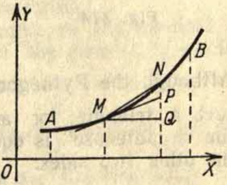


Fig. 373

The differential of the arc expresses the length of a segment of the tangent line from the point of tangency to intersection with the increased ordinate.

Example. The differential of the arc of a cycloid is (cf. Example, Sec. 338)

$$ds = \sqrt{dx^2 + dy^2} = a\sqrt{2(1 - \cos t)}\, dt =$$

$$= 2a \sin\frac{t}{2}\, dt \quad (0 \leqslant t \leqslant 2\pi)$$

340. The Arc Length and Its Differential in Polar Coordinates

The length of the arc $\overset{\smile}{AB}$ (Fig. 374) is given in terms of polar coordinates r, φ by the integral

$$s = \int_{(A)}^{(B)} \sqrt{dr^2 + r^2 d\varphi^2} \tag{1}$$

The differential of the arc is given by the formula

$$ds = \sqrt{dr^2 + r^2 d\varphi^2} \tag{2}$$

Explanation. From point O as centre, draw a circle (Fig. 374) of radius $OM = r$. Its arc $\overset{\frown}{MK}\,(= r\Delta\varphi)$, the segment $KN\,(= \Delta r)$ and the arc $\overset{\frown}{MN}\,(= \Delta s)$ of the curve AB form a curvilinear triangle with right angle at the vertex K.

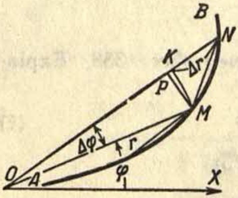


Fig. 374

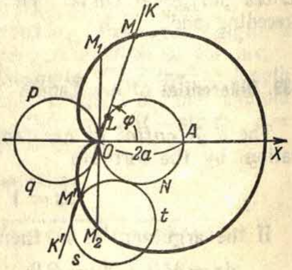


Fig. 375

Although the Pythagorean theorem does not hold exactly for such a triangle, for an infinitesimal arc $\overset{\frown}{MN}$ the square of the "hypotenuse" is equivalent to the sum of the squares of the other two sides:

$$\overset{\frown}{MN} \approx \sqrt{KN^2 + \overset{\frown}{KM}{}^2}$$

or

$$\Delta s \approx \sqrt{\Delta r^2 + r^2\,\Delta\varphi^2} \approx \sqrt{dr^2 + r^2\,d\varphi^2}$$

Hence, the expression $\sqrt{dr^2 + r^2 d\varphi^2}$ is the element (differential) of the arc length s.

Note. Formula (2) of this section may be obtained from (2) of Sec. 338 by the substitution

$$dx = d\,(r\cos\varphi) = \cos\varphi\,dr - r\sin\varphi\,d\varphi,$$
$$dy = d\,(r\sin\varphi) = \sin\varphi\,dr + r\cos\varphi\,d\varphi$$

Example. From point O on a circle of radius a, draw a ray OK (Fig. 375); from point L where the straight line OK again cuts the circle, lay off a segment $LM = 2a$ in the direction of the ray OK.[1] The curve described by the point M as the ray is rotated is called a *cardioid* (which means "heart-shaped"; see Sec. 508 for a detailed discussion of the cardioid). Find its length.

Solution. Choose the polar system as in Fig. 375. We then have

$$\left.\begin{aligned} OL&=OA\cos\varphi=2a\cos\varphi,\\ r&=OL+LM=2a(\cos\varphi+1)\end{aligned}\right\} \tag{3}$$

The cardioid is completely described when φ ranges over the interval $(-\pi, +\pi)$. Its length, by (1), is

$$s=\int\limits_{-\pi}^{+\pi}\sqrt{4a^2(1+\cos\varphi)^2+4a^2\sin^2\varphi}\,d\varphi=$$

$$=2a\int\limits_{-\pi}^{+\pi}\sqrt{2+2\cos\varphi}\,d\varphi=4a\int\limits_{-\pi}^{+\pi}\cos\frac{\varphi}{2}\,d\varphi=16a \tag{4}$$

Note. A cardioid may be obtained as the path of a point on the circumference of circle Opq (Fig. 375) rolling (without sliding) on a circle $ONAL$ of the same radius.

From (4) it is clear that the length of the cardioid is equal to eight times the diameter of the generating circle.

The cardioid may be drawn by varying φ from zero to 2π But if we compute its length from the formula $s=4a\int\limits_0^{2\pi}\cos\frac{\varphi}{2}\,d\varphi$, we get zero. The source of the error is indicated in Sec. 338 (fine print).

341. The Area of a Surface of Revolution

The area S of a surface formed by the revolution of an arc $\overset{\frown}{AB}$ about the x-axis is given by the integral

$$S=\int\limits_{(A)}^{(B)}2\pi y\,ds \tag{1}$$

where y is the ordinate of the meridian AB, $ds=\sqrt{dx^2+dy^2}$ is the differential of its arc (Sec. 339), (A) and (B) are the end-point values of the parameter in terms of which the coordinates are expressed.

---

[1] When the straight line touches the circle at point O, segments equal to $2a$ are laid off in both directions from O ($OM_1=OM_2=2a$).

Explanation. Partition the surface $ABB'A'$ (Fig. 376) into parallel shells, and replace each shell $MNN'M'$ by the lateral surface of a frustum of a cone with the same bases. The areas of these surfaces are equivalent. Therefore,

$$\text{area}\cdot MNN'M' \approx$$
$$\approx \pi(PM+QN)\,MN \qquad (2)$$

Since $PM+QN=2y+\Delta y$, $MN \approx \overset{\frown}{MN}=\Delta s$, it follows that

$$\text{area}\, MNN'M' \approx \pi(2y+\Delta y)\,\Delta s \approx$$
$$\approx 2\pi y\,\Delta s \qquad (3)$$

Whence formula (1) follows.

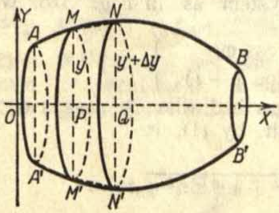


Fig. 376

Example. Find the area of a surface formed by the revolution of a cycloid about its base.

Solution. We have (Secs. 338, 339)

$$ds=2a\sin\frac{t}{2}\,dt\ (0\leqslant t\leqslant 2\pi),$$

$$S=\int_0^{2\pi} 2\pi a(1-\cos t)\cdot 2a\sin\frac{t}{2}\,dt=$$

$$=8\pi a^2\int_0^{2\pi}\sin^3\frac{t}{2}\,dt=\frac{64}{3}\pi a^2$$

By way of comparison, take the area of the axial section (that is, double the area of the cycloid) $6\pi a^2$ (Sec. 333). The area of the surface of revolution exceeds it by a factor of $3\frac{5}{9}$.

Note. In order to prove the equivalence of the area of the shell $MNN'M'$ and the area of the lateral surface of a frustum of a cone, we have to define the concept of "the area of a curved surface". This definition is given in Sec. 459. Due to its complexity, one often introduces the following particular definition which is in agreement with the general definition.

The area of a surface of revolution is the limit to which tends the area of the surface formed by the rotation of a polygonal curve inscribed in the meridian when the number of straight-line segments of the polygonal curve increases without bound and the lengths of the segments tend to zero.

From this definition, it is possible to derive formula (1) directly (cf. Sec. 338, Note 1).

PLANE AND SPACE CURVES (FUNDAMENTALS)

342. Curvature

Suppose, as point M on curve L moves to point M' (Fig. 377), the tangent line which is in the direction of the motion turns through an angle of ω (omega) from position MT to $M'T'$. The ratio of the angle ω to the arc length $\overset{\smile}{MM'}$, $\frac{\omega}{\overset{\smile}{MM'}}$ describes the curvature of the curve L on the segment MM' and is called the *average curvature* of the arc MM'. The angle ω is ordinarily measured in radians.

The average curvature of any segment of a straight line (its tangent coincides with the straight line itself) is equal to zero; the average curvature of any arc of a circle of radius R is equal to $\frac{1}{R}$.

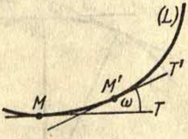


Fig. 377

The dimensions of average curvature are reciprocal to the dimensions of length; that is, when the scale changes, the numerical measure of curvature varies inversely with the numerical measure of segment length.

Definition. The *curvature* of a curve L at a point M is the limit to which the average curvature of an arc $\overset{\smile}{MM'}$ tends when the point M' tends to M. The curvature is denoted by the letter K:

$$K = \lim_{\overset{\smile}{MM'} \to 0} \frac{\omega}{\overset{\smile}{MM'}} \qquad (1)$$

Curvature characterizes the amount of bend of a curve at the point under consideration. The curvature of a straight line is everywhere equal to zero. The curvature of a circle of radius R is everywhere equal to $\frac{1}{R}$. For any other curve, the curvature varies from point to point. It may be zero at some points (called *points of rectification*). Near a point of rectification, a curve resembles a straight line.

Note. We consider curvature to be a positive quantity (if it is not zero). To the curvature of a plane curve we can affix a sign, to the curvature of a space curve we cannot (see Sec. 364).

343. The Centre, Radius and Circle of Curvature of a Plane Curve

Suppose a point M' (Fig. 378) is in motion along a plane curve L and tends to a fixed point M where the curvature K is nonzero. Then the point C', where the fixed normal MN cuts the normal $M'N'$, tends to the point C which is distant $MC=\frac{1}{K}$[1] from M. The ray MC is in the direction of concavity of L.

The line segment MC is called the *radius of curvature*, and the point C, the *centre of curvature* of L (for the point M).

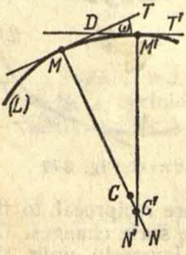


Fig. 378

The radius of curvature is denoted by R or the Greek letter ρ (rho). The quantities R and K are reciprocal quantities, i.e.

$$R=\frac{1}{K} \tag{1}$$

and

$$K=\frac{1}{R} \tag{2}$$

The radius of curvature of a circle is equal to its radius and the centre of curvature coincides with its centre.

A circle described from the centre of curvature C (Fig. 379) with radius $R=MC$ is called the *osculating circle*, or the *circle of curvature*, of the curve L (at the point M).

The curve L is exterior to the circle of curvature in the direction of increasing radius of curvature (in Fig. 379, it is to the right of M), and interior to the circle of curvature in the direction of decreasing radius of curvature (to the left of M in Fig. 379). Therefore, as a rule the circle of curvature is tangent to the curve L and at the same time intersects it.

---

[1] In the triangle $MC'M'$ (Fig. 378) the angle C' is equal to the angle ω (angles with mutually perpendicular sides). $\angle C'M'M=90^\circ-\lambda$, where angle $\lambda=\angle MM'D$ is infinitesimal (it is less than ω). By the sine theorem, $MC'=\frac{\sin(90^\circ-\lambda)}{\sin\omega}MM'=\cos\lambda\frac{MM'}{\sin\omega}$. Passing to the limit and taking into account the fact that $MM'\approx\overset{\frown}{MM'}$, $\sin\omega\approx\omega$ and $\cos\lambda\to 1$, we get

$$MC=\lim\frac{\overset{\frown}{MM'}}{\omega}=1:\lim\frac{\omega}{\overset{\frown}{MM'}}=1:K$$

In exceptional cases when the radius of curvature at the point M has an extremum, the curve L, on both sides of M, is situated inside the circle of curvature (in the case of a maximum, Fig. 380) or outside it (in the case of a minimum, Fig. 381). An instance of the first case is the end of the minor axis of an ellipse, an instance of the second case is the end of the major axis.

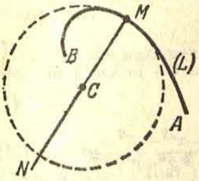


Fig. 379

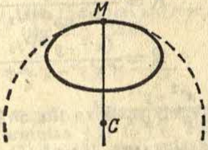


Fig. 380

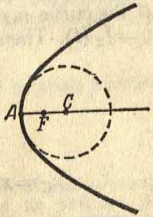


Fig. 381

Note. If at M the curvature of the curve L is zero, then the point of intersection of the normals MN and $M'N'$ recedes from M indefinitely when the point M' tends to M. We then say that the radius of curvature at the point of rectification is infinite, and we write $R=\infty$.

344. Formulas for the Curvature, Radius and Centre of Curvature of a Plane Curve[1]

The curvature of a curve $y=f(x)$ is computed from the formula

$$K=\frac{|y''|}{(1+y'^2)^{3/2}} \tag{1}$$

the radius of curvature, from the formula

$$R=\frac{(1+y'^2)^{3/2}}{|y''|} \tag{2}$$

the coordinates of the centre of curvature C, from the formulas

$$x_C=x-\frac{y'(1+y'^2)}{y''}, \quad y_C=y+\frac{1+y'^2}{y''} \tag{3}$$

---

[1] The corresponding formulas for a space curve are given in Sec. 363.

If $y''=0$, then the curvature is zero, the radius of curvature is infinite and the centre of curvature is absent. That is what always occurs, for example, at points of inflection (cf. Sec. 283).

Formulas (1) to (3) are replaced by symmetric formulas if the curve is represented by the parametric equations $x=f_1(t)$, $y=f_2(t)$. Then

$$K=\frac{|x'y''-y'x''|}{(x'^2+y'^2)^{3/2}}, \qquad \text{(I)}$$

$$R=\frac{(x'^2+y'^2)^{3/2}}{|x'y''-y'x''|}, \qquad \text{(II)}$$

$$x_C=x-\frac{x'^2+y'^2}{x'y''-y'x''}y', \quad y_C=y+\frac{x'^2+y'^2}{x'y''-y'x''}x' \qquad \text{(III)}$$

The primes denote differentiation with respect to the parameter t. Formulas (1) to (3) are obtained from (I) to (III) if we put $x=t$ (then $x'=1$, and $x''=0$). If we put $y=t$ (then $y'=1$, $y''=0$), i.e. if the equation of the curve is taken in the form $x=f(y)$, then in place of (1)-(3) we get the following formulas:

$$K=\frac{|x''|}{(1+x'^2)^{3/2}}, \qquad \text{(1a)}$$

$$R=\frac{(1+x'^2)^{3/2}}{|x''|}, \qquad \text{(2a)}$$

$$x_C=x+\frac{1+x'^2}{x''}, \quad y_C=y-\frac{x'(1+x'^2)}{x''} \qquad \text{(3a)}$$

The existence of the derivatives x', y', x'', y'' at the point A of the given curve ensures the existence of curvature at that point. The converse does not hold: it may happen that the curvature exists at A but the derivatives x', y', x'', y'' (one or several) do not exist. Then formulas (I)–(III) are not valid and this shows that the parameter was not properly chosen. See Example 1 (fine print).

Example 1. Find the curvature, the radius and the centre of curvature C at the vertex A (0, 0) of the parabola $y^2=2px$ (Fig. 381).

Solution. The easiest way is to take the ordinate y for the argument: from the given equation we get

$$x=\frac{y^2}{2p}, \quad x'=\frac{y}{p}, \quad x''=\frac{1}{p} \qquad (4)$$

At the vertex of the parabola we have

$$x'=0, \quad x''=\frac{1}{p} \qquad (5)$$

Using formulas (1a) to (3a) we get

$$K=\frac{1}{p},\quad R=p,\quad x_C=p,\quad y_C=0 \tag{6}$$

The radius of curvature at the vertex of the parabola is equal to its parameter, that is, the focus F bisects the segment AC.

If for the argument we take the abscissa x of the parabola $y^2=2px$, then in place of (4) we get (see Sec. 250)

$$y'=\frac{p}{y},\qquad y''=-\frac{p^2}{y^3} \tag{7}$$

The derivatives y', y'' do not exist at the vertex of the parabola ($x=0$, $y=0$), so that the formulas (1) to (3) cannot be utilized directly However, formulas (1) to (3) are suitable for all the other points of the parabola, and after the substitution (7) they are transformed to

$$\left.\begin{aligned} K&=\frac{p^2}{(y^2+p^2)^{3/2}}, \\ R&=\frac{(y^2+p^2)^{3/2}}{p^2}, \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} x_C&=x+\frac{y^2}{p}+p\ (=3x+p), \\ y_C&=-\frac{y^3}{p^2} \end{aligned}\right\} \tag{9}$$

If we substitute $x=0$, $y=0$ here, then we again get the values (6). The meaning of this computation lies in the fact that we have found the limits to which the quantities K, R, x_C, y_C tend when a point of the parabola approaches the vertex of the parabola.

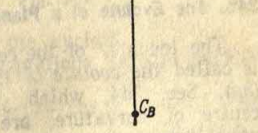


Fig. 382

Example 2. Find the radius and the centre of curvature at the vertices of an ellipse with semiaxes a, b (Fig. 382).

Solution. The simplest way is to use the parametric equations of the ellipse (Sec. 252)

$$x=a\cos t,\quad y=b\sin t$$

From them we find

$$x'=-a\sin t,\quad y'=b\cos t;$$
$$x''=-a\cos t,\quad y''=-b\sin t$$

From formulas (II) and (III) we get

$$R=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab},\tag{10}$$

$$\left.\begin{aligned}x_C&=\frac{(a^2-b^2)\cos^3 t}{a},\\ y_C&=-\frac{(a^2-b^2)\sin^3 t}{b}\end{aligned}\right\}\tag{11}$$

At the vertex $A(a, 0)$, where $t=0$, we have

$$R_a=\frac{b^2}{a},\quad x_C=\frac{a^2-b^2}{a},\quad y_C=0\tag{12}$$

At the vertex $B(0, b)$, where $t=\frac{\pi}{2}$, we have

$$R_b=\frac{a^2}{b},\quad x_C=0,\quad y_C=-\frac{a^2-b^2}{b}\tag{13}$$

Note. Forming the equation of a tangent to the ellipse (Sec. 252)

$$b\cos t\cdot X+a\sin t\cdot Y-ab=0$$

we find that its distance from the centre (Sec. 28) is

$$d=\frac{ab}{\sqrt{a^2\sin^2 t+b^2\cos^2 t}}$$

Comparing with (10) we find

$$R=\frac{a^2b^2}{d^3}$$

Thus, the radius of curvature of the ellipse is inversely proportional to the cube of the distance from the centre to the tangent at the corresponding point. In particular, from (12) and (13) we find that

$$R_a:R_b=b^3:a^3$$

345. The Evolute of a Plane Curve

The locus L' of the centres of curvature of a plane curve L is called the *evolute* of the curve L. Formulas (III), (3) and (3a), Sec. 344, which yield the coordinates x_C, y_C of the centre of curvature, are at the same time the parametric equations of the evolute [in formulas (3) and (3a) the role of parameter is played by x and y, respectively]. Eliminating the parameter, we get an equation relating the coordinates of the evolute.

Example 1. Find the evolute of the parabola

$$y^2=2px \tag{1}$$

Solution. For the parameter let us take the ordinate y. Substituting into formulas (3a), Sec. 344, the expressions $x'=\frac{y}{p}$, $x''=\frac{1}{p}$, we get

$$x_C=\frac{y^2}{2p}+\frac{p^2+y^2}{p}=\frac{3}{2}\frac{y^2}{p}+p \tag{2}$$

$$y_C=y-\frac{y(p^2+y^2)}{p^2}=-\frac{y^3}{p^2} \tag{3}$$

These are the parametric equations of the evolute with parameter y. To eliminate y, represent the system (2)-(3) in the form

$$\frac{2}{3}p(x_C-p)=y^2 \quad p^2y_C=-y^3$$

Cubing the first equation and squaring the second, and equating the left members, we get the equation of the evolute:

$$27py_C^2=8(x_C-p)^3$$

The evolute is a semicubical parabola (Fig. 383).

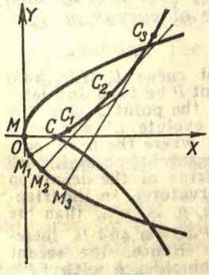


Fig. 383

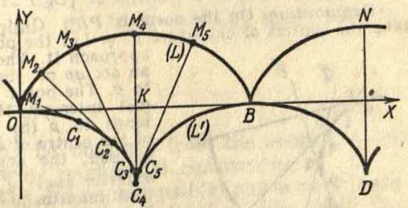


Fig. 384

Example 2. Find the evolute of the cycloid.

Solution From the parametric equations of the cycloid (Sec. 253)

$$x=a(t-\sin t), \quad y=a(1-\cos t) \tag{4}$$

we find, from formulas (III) of Sec. 344,

$$x_C=a(t+\sin t), \quad y_C=-a(1-\cos t) \tag{5}$$

The similarity of equations (4) and (5) is not accidental; if a new parameter t' is introduced with the aid of the relation

$$t = t' + \pi \tag{6}$$

then Eqs. (5) transform to

$$\left.\begin{aligned} x_C &= \pi a + a(t' - \sin t'), \\ y_C &= -2a + a(1 - \cos t') \end{aligned}\right\} \tag{7}$$

Hence, the evolute L' of the cycloid L (Fig. 384) is a cycloid congruent to the given one but displaced along the base OB by half of the base and dropped below the base a distance KC_4 equal to the altitude.

From (6) it is seen that the turns of the generating circles at the corresponding points of the common cycloids differ by 180°; in particular, the vertex of one of the cycloids is associated with the point at which the arches of the other come together.

346. The Properties of the Evolute of a Plane Curve

Property 1. The normal of a curve L is tangent to its evolute L' at the corresponding centre of curvature.

Example 1. The normal M_3C_3 of the cycloid L (Fig. 384) is tangent to the cycloid L' at the centre of curvature C_3 of the first cycloid.

Explanation. On the normals PP', QQ' of curve L (Fig. 385) take the centres of curvature p, q. Let the point P be fixed and let Q approach it. Then the point q describes an arc qp of the evolute L' and tends to p. The point a, where the fixed normal intersects the moving normal, also tends to p (by virtue of the definition of the centre of curvature). In the triangle pqa, the angle p is less than the exterior angle $\angle PaQ = \omega$ and is therefore infinitesimal. Hence, the secant line qp tends to coincidence with PP'; i.e. PP' is tangent to the evolute; the point of tangency is the centre of curvature p corresponding to the point P.

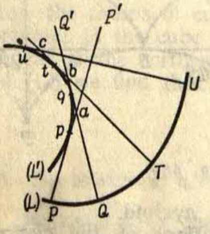


Fig. 385

Property 2. Let the radius of curvature R of the curve L increase as we move from point P to point U (Fig. 385). Then the length of the arc pu of the evolute L' is equal to the increase in the radius of curvature of the curve L:

$$\overset{\smile}{pu} = R_U - R_P$$

Example 2. The radius of curvature of the cycloid L (Fig. 384) at point O is zero; it increases on the arc OM_4 and at M_4 is equal to $M_4C_4=4a$ (see Example 1). By Property 2, the length of the arc OC_4 of the cycloid L' is $4a-0=4a$ (cf. Sec. 345, Example 2).

Explanation. Partition the arc pu of the evolute L' into subarcs pq, qt, etc.; their number will then tend to infinity. Suppose that all the subarcs pq, qt, ... are of the same order of smallness. Then the arcs PQ, QT, ... of the curve L will be of the same order. The differences, however, $Pa-Qa$, $Tb-Qb$, etc. will be of higher order. Since

$$pa+aq=(Pa-Pp)+(Qq-Qa)=Qq-Pp+(Pa-Qa)$$

and similarly for the polygonal lines qbt, tcu, ..., the perimeter of the polygonal line $pqtu$ differs from the quantity

$$(Qq-Pp)+(Tt-Qq)+(Uu-Tt)+\ldots=Uu-Pp$$

by an infinitesimal (that resulting from the accumulation of infinitesimals of higher order).

Hence, the length of the arc pu of the evolute, which is the limit of the length of the circumscribed polygonal line, is equal to $Uu-Pp$.

Note. If between the extremities of the arc of curve L there are points with extremal radius of curvature, then Property 2 breaks down. Thus, at points M_3 and M_5 (Fig. 384) of the cycloid L the radii of curvature are the same, whereas the length of the arc $C_3C_4C_5$ is of course nonzero. Property 2 breaks down because at the point M_4 the radius of curvature has a maximum. The arc $\breve{C_3C_4}$ is equal to $M_4C_4-M_3C_3$, the arc $\breve{C_4C_5}$ is also equal to $M_4C_4-M_3C_3$ (and not to $M_3C_3-M_4C_4$).

347. Involute of a Plane Curve

A plane curve L may be obtained from the evolute L' of the curve by the following mechanical construction.

Take a string (flexible and nonextensible) and wind it onto the evolute L'; coming off the evolute at p (Fig. 385), the string would have a free end at P of curve L. Now if we unwind the string from the evolute, the free end will describe the curve L.

Explanation. The taut string is all the time tangent to L'. When it comes off the evolute at point q, its free part increases by the length of the arc $\breve{pq}$, i. e. (Sec. 346, Property 2) by $Qq-Pp$. The free part becomes equal to $Pp+(Qq-Pp)=Qq$ and the end of the string coincides with point Q

This construction leads to the following geometrical definition.

Definition. On a given curve L' (Fig. 385) choose the direction of increasing arcs (either one is possible, say, from u to p); in this direction, lay off on the tangents the segments uU, tT, Qq, ..., the lengths of which decrease just as much as the arc length increases. The locus L of the end-points of these line-segments is called the *involute* of the given curve.

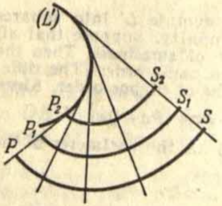


Fig. 386

Every plane curve L' has an infinite number of involutes (PS, P_1S_1, P_2S_2 in Fig. 386). For each one of them the curve L' is the evolute.

The involutes of the curve L' are *orthogonal trajectories* of its tangents (i. e. they intersect all tangents at right angles; cf. Sec. 346, Property 1).

For the involute of a space curve, see Sec. 362, Note 2.

348. Parametric Representation of a Space Curve

A curve in space that is regarded as the intersection of surfaces is given by a system of two equations relating x, y, z (see Sec. 170).

A curve in space regarded as the trace of a moving point is represented by a system of three equations:

$$x=f(t), \quad y=\varphi(t), \quad z=\psi(t) \tag{1}$$

expressing the coordinates of the point in terms of a *parameter* t (in mechanics, time is usually the parameter). Eqs. (1) are called the *parametric equations* of the space curve (cf. Sec. 251).

One of the coordinates is ordinarily taken for the parameter, say x. Then the equations of the curve have the form

$$y=\varphi(x), \quad z=\psi(x) \tag{2}$$

[the first of Eqs. (1) becomes an identity: $x=x$].

Eqs. (2) are not suitable for representing a curve lying in a plane perpendicular to the x-axis (because all the points of such a curve have the same abscissa).

If the equation of a surface becomes an identity after substitution of expressions (1), then the curve (1) lies on that surface.

Any line can be represented parametrically in an infinite number of ways. If one system of parametric equations is known, then we obtain any other system by replacing t by some function of the new parameter t'.

The projection of curve (1) on the plane $z=c$ (in particular, on the coordinate plane XOY) is given by the equations

$$x=f(t), \quad y=\varphi(t), \quad z=c \tag{3}$$

The equation $z=c$ is often simply implied. Similarly, for projections on the planes $x=a$ and $y=b$.

Example. The parametric equations

$$x=-2+t, \quad y=3+2t, \quad z=1-2t \tag{1a}$$

describe a straight line.

If for the parameter we take x, then the same straight line will be represented by the equations

$$x=2x+7, \quad z=-2x-3 \tag{2a}$$

Line (1a) lies on the surface

$$z-\frac{1}{2}=\frac{2x^2}{7}-\frac{y^2}{14} \tag{4}$$

(hyperbolic paraboloid) because equality (4) becomes an identity when we substitute expressions (1a) into it.

The straight line (1a) also lies in the plane

$$y+z-4=0 \tag{5}$$

Hence, the straight line (1a) belongs to the intersection of the surfaces (4) and (5).

From this it does not follow that the surfaces (4) and (5) intersect *only* in the points of the straight line (1a). Plane (5) intersects the paraboloid (4) along two rectilinear generatrices (Sec. 180); one of these is the straight line (1a).

Taking the expression of the parameter t, $t=2+\frac{1}{2}t'$ in terms of the new parameter t', we get other parametric equations of the same straight line:

$$x=\frac{1}{2}t', \quad y=7+t', \quad z=-3-t' \tag{1b}$$

The projection of the straight line (1a) on the xy-plane is given by the parametric equations

$$x=-2+t, \quad y=3+2t \tag{3a}$$

(equation $z=0$ is implied). We obtain the equation of the same projection from (1b) in the form

$$x=\frac{1}{2}t',\quad y=7+t' \tag{3b}$$

and so forth. Eliminating the parameter, we get $y=2x+7$ in both cases.

349. Helix

Let a point M (Fig. 387) be in uniform motion along the generatrix QR of a circular cylinder and let the generatrix itself be in uniform rotation along the surface of the cylinder. Then M describes a curve AMC called a helix. The *radius* of the helix is the radius a of the cylinder on which the helix is drawn.

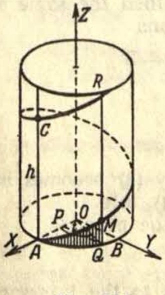
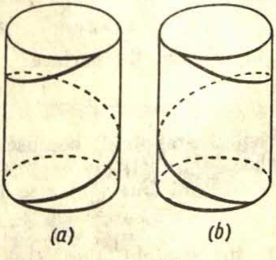


Fig. 387



(a) (b)

Fig. 388

If the motion of the point M is viewed from the base towards which it is moving, then the rotation of the generatrix is either positive (counterclockwise) or negative (clockwise).[1] In the first case, the helix is termed *right-hand* (Fig. 388*a*), in the second, *left-hand* (Fig. 388*b*).

The straight-line path $AC=h$ (Fig. 387) traversed by point M along the generatrix during one complete turn of the latter is called the *lead* (or *pitch*) of the helix. The lead of a right-hand helix is taken to be positive, that of a left-hand helix, negative.

---

[1] If M is in motion along the helix in the opposite direction, we observe it from the other base, but the generatrix too will be revolving in the opposite direction. Hence, positive rotation remains positive and negative rotation, negative.

Right- and left-hand helixes (of one and the same radius and with the same absolute value of lead) cannot be brought to coincidence. They are mirror symmetric.

Note. If we develop a cylindrical surface on a plane, the circle AQB (Fig. 387) becomes a straight line perpendicular to the generatrices. Since the segment QM is proportional to the arc AQ:

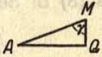


Fig. 389

$$QM : \overset{\smile}{AQ} = h : 2\pi a \qquad (1)$$

it follows that the helix becomes a straight line (AM in Fig. 389). The angle γ of its inclination to the generatrices is determined from the formula

$$\tan\gamma = \frac{AQ}{QM} = \frac{a}{b} \qquad (2)$$

where $b = \frac{h}{2\pi}$.

Parametric equations of a helix. Take the axis of the cylinder for the z-axis (Fig. 387), and take the x-axis towards an arbitrarily chosen point A of the helix. For the parameter t we take the angle of turn of the plane of the axial section $OQMR$ from its initial position OAC. Then

$$x = OP = a\cos t, \quad y = PQ = a\sin t, \quad z = QM = bt \qquad (3)$$

The two equations $y = a\sin t$, $z = bt$ are the projection of the helix on the plane YOZ. This projection is a sine curve. The projection on the plane XOZ is also a sine curve, the projection on the plane XOY is a circle.

350. The Arc Length of a Space Curve

The length of the arc $\overset{\smile}{AB}$ of a space curve is given by the integral

$$s = \int_{(A)}^{(B)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad (1)$$

or

$$s = \int_{(A)}^{(B)} \sqrt{dx^2 + dy^2 + dz^2} \qquad (2)$$

The differential of the arc (cf. Sec. 339)

$$ds=\sqrt{dx^2+dy^2+dz^2}=\sqrt{x'^2+y'^2+z'^2}\,dt \tag{3}$$

Example 1. Find the length s_1 of one turn (revolution) of a helix.

Solution. Formula (2) yields [taking into account formulas (3) of Sec. 349]

$$s_1=\int_0^{2\pi}\sqrt{[d(a\cos t)]^2+[d(a\sin t)]^2+[d(bt)]^2}=$$
$$=\int_0^{2\pi}\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}\,dt=2\pi\sqrt{a^2+b^2} \tag{4}$$

that is the length of one revolution of the helix is equal to the hypotenuse of a triangle, one leg of which ($2\pi a$) is equal to the circumference of the base, and the other ($2\pi b$) is equal to the lead of the helix (cf. Sec. 349, Note).

If the initial point of the arc is fixed and the terminal point varies, the arc length is a function of the parameter t, and, hence, (Sec. 348), can itself be taken as the parameter.

Example 2. Write the equations of a helix, taking for the parameter the arc length reckoned from the initial point $t=0$.

Solution. As in Example 1, we get

$$s=\int_0^t\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}\,dt=\sqrt{a^2+b^2}\,t \tag{5}$$

Expressing t in terms of s and substituting into (2), Sec. 349, we get

$$x=a\cos\frac{s}{\sqrt{a^2+b^2}},\quad y=a\sin\frac{s}{\sqrt{a^2+b^2}},\quad z=\frac{b}{\sqrt{a^2+b^2}}\,s \tag{6}$$

351. A Tangent to a Space Curve

The tangent to a curve (L) at a point $M(x, y, z)$ is the straight line MT to which the secant line MM' tends when the point M' tends to M (cf. Sec. 225).

If the curve (L) is represented parametrically,

$$x=f(t),\quad y=\varphi(t),\quad z=\psi(t) \tag{1}$$

then for the direction vector (Sec. 143) of the tangent we can take the vector[1]

$$\boldsymbol{r}'=\left\{\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}\right\} \tag{2}$$

or the vector collinear with it,

$$\boldsymbol{t}=\left\{\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds}\right\} \tag{3}$$

Its absolute value is equal to unity[2]. The vector $\boldsymbol{t}$ is therefore called the *unit vector of the tangent*.

The coordinates of the vector $\boldsymbol{t}$ are the direction cosines (Sec. 144) of the tangent:

$$\cos\alpha=\frac{dx}{ds},\quad \cos\beta=\frac{dy}{ds},\quad \cos\gamma=\frac{dz}{ds} \tag{4}$$

(in Fig. 390, $\alpha=\angle AMT$, $\beta=\angle BMT$, $\gamma=\angle CMT$).

Explanation. For the direction vector of the secant line we can take the vector $\overrightarrow{MM'}=\{\Delta x,\ \Delta y,\ \Delta z\}$ and, hence, also the vectors $\frac{\overrightarrow{MM'}}{\Delta t}=\left\{\frac{\Delta x}{\Delta t},\ \frac{\Delta y}{\Delta t},\ \frac{\Delta z}{\Delta t}\right\}$ and $\frac{\overrightarrow{MM'}}{\overset{\frown}{MM'}}=\left\{\frac{\Delta x}{\Delta s},\ \frac{\Delta y}{\Delta s},\ \frac{\Delta z}{\Delta s}\right\}$ collinear with it. Formulas (2) and (3) are obtained by proceeding to the limit.

From Fig. 390 we have $\cos\angle CMM'=\frac{MC}{MM'}\approx\frac{\Delta z}{\Delta s}$. Passage to the limit yields $\cos\gamma=\frac{dz}{ds}$. We obtain the other two formulas in (4) in the same way.

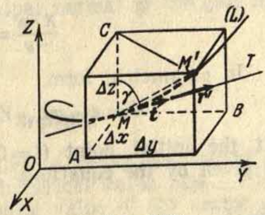


Fig. 390

The symmetric equations of the tangent (Sec. 150) are of the form

$$\frac{X-x}{x'}=\frac{Y-y}{y'}=\frac{Z-z}{z'} \tag{5}$$

The primes denote derivatives with respect to any parameter.

Example. Consider the helix (Sec. 349)

$$x=a\cos t,\quad y=a\sin t,\quad z=bt \tag{1a}$$

---

[1] The vector $\boldsymbol{r}'$ is the derivative of the radius vector $\boldsymbol{r}\{x,\ y,\ z\}$ (see the theorem in Sec. 355).

[2] $\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=\frac{dx^2+dy^2+dz^2}{ds^2}=\frac{ds^2}{ds^2}=1.$

The vector

$$r'=\{-a\sin t,\quad a\cos t,\quad b\}=\{-y,\ x,\ b\} \tag{2a}$$

is the direction vector of the tangent. From equations (6), Sec. 350, we get the unit vector of the tangent:

$$t=\left\{-\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}}\right\} \tag{3a}$$

so that

$$\left.\begin{aligned}\cos\alpha&=-\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}=-\frac{a}{\sqrt{a^2+b^2}}\sin t,\\ \cos\beta&=\frac{a}{\sqrt{a^2+b^2}}\cos t,\quad \cos\gamma=\frac{b}{\sqrt{a^2+b^2}}\end{aligned}\right\} \tag{4a}$$

This formula yields: $\tan\gamma=\frac{a}{b}$ (cf. Sec. 349).

The equations of the tangent have the form

$$\frac{X-a\cos t}{-a\sin t}=\frac{Y-a\sin t}{a\cos t}=\frac{Z-bt}{b} \tag{5a}$$

or

$$\frac{X-x}{-y}=\frac{Y-y}{x}=\frac{Z-z}{b} \tag{5b}$$

In parametric form,

$$X=x-yu,\quad Y=y+xu,\quad Z=z+bu \tag{6}$$

At the initial point ($t=0$, $x=a$, $y=0$, $z=0$) the tangent is given by the equations $X=a$, $Y=au$, $Z=bu$.

352. Normal Planes

A plane P (Fig. 391) passing through a point M of a curve L and perpendicular to the tangent MT is called a *normal plane* to the curve L.

The direction vector of the tangent (Sec. 351), $r'=\{x',\ y',\ z'\}$, is a normal vector to the plane P. The equation of a *normal plane* is of the form (Sec. 123)

$$(X-x)\,x'+(Y-y)\,y'+(Z-z)\,z'=0$$

or, in vector form,

$$(R-r)\,r'=0$$

Example. The equation of a normal plane to the helix

$$x=a\cos t,\quad y=a\sin t,\quad z=bt$$

is of the form

$$(X-a\cos t)(-a\sin t)+(Y-a\sin t)(a\cos t)+(Z-bt)\,b=0$$

or

$$-yX+xY+bZ-bz=0$$

The normal plane at the initial point $(a, 0, 0)$ is given by the equation

$$aY+bZ=0$$

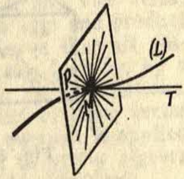


Fig. 391

Any straight line passing through the point M of the space curve L and perpendicular to the tangent MT is called the *normal* to the curve L (at the point M). A space curve has an infinity of normals all of which lie in the normal plane.

If a curve L lies in one plane, then from the infinity of normals one is selected (the *principal normal*) lying in that plane. We can also choose a principal normal in the case of a twisted curve (Sec. 359).

353. The Vector Function of a Scalar Argument

Diflnition. A vector $\boldsymbol{p}$ is called a *vector function* of a scalar argument u if to every numerical value that u can take on there corresponds a definite value of the vector $\boldsymbol{p}$ (i.e. a definite magnitude and a definite direction of the vector).

In contrast to a vector function, a scalar quantity depending on u is called a *scalar function*.

Example 1. Point M is in motion along a curve L (Fig. 392). The velocity $\boldsymbol{v}$ (regarded as a vector) is the vector function of the scalar argument t (the time reckoned from some initial instant) because at every instant the vector $\boldsymbol{v}$ has a definite magnitude and a definite direction (it is collinear with the tangent to the curve L). The vector $\boldsymbol{v}$ may also be regarded as a function of the (scalar) argument s (arc length M_0M). The magnitude of the velocity is a scalar function of the argument t (or s).

Example 2. The radius vector (Sec. 95) $\overrightarrow{OM}$ of point M describing a curve L (Fig. 392) is a vector function of the arc length $s=\overset{\frown}{M_0M}$. The coordinates x, y, z of the vector $\overrightarrow{OM}$ (i.e. the coordinates of the point M) are scalar functions of s (cf. Sec. 350, Example 2).

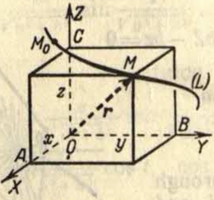


Fig. 392

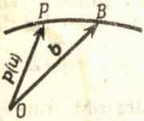


Fig. 393

Note. If the terminal point of a vector $\boldsymbol{p}$ is moving (as in Example 1), then we can choose some fixed point O (Fig. 393) and take it for the origin of the vector $\overrightarrow{OP}$ equal to the vector $\boldsymbol{p}$. The locus of the terminus P (as a rule, this is a curve) is called the *hodograph* of the vector function $\boldsymbol{p}$.

Notation of a vector function. The notation

$$\boldsymbol{p}=\boldsymbol{p}(u)$$

signifies that $\boldsymbol{p}$ is a vector function of a scalar argument u.

354. The Limit of a Vector Function

Definition. A constant vector $\boldsymbol{b}$ is called the *limit of a vector function* $\boldsymbol{p}(u)$ as $u \longrightarrow a$ (or as $u \longrightarrow \infty$) if the absolute value of the difference of the vectors $\boldsymbol{p}(u)$ and $\boldsymbol{b}$ is infinitesimal as $u \longrightarrow a$ (as $u \longrightarrow \infty$).

Notation:

$$\lim_{u \to a} \boldsymbol{p}(u)=\boldsymbol{b} \tag{1}$$

Explanation. Let us refer the variable vector $\boldsymbol{p}(u)$ to the fixed origin O (Fig. 393). If as $u \longrightarrow a$ the moving terminus P tends to coincidence with the fixed point B, then the vector $\overrightarrow{OB}=\boldsymbol{b}$ is the limit of the vector $\boldsymbol{p}(u)$. The difference $\boldsymbol{p}(u)-\boldsymbol{b}$ is the vector $\overrightarrow{BP}$, and the absolute value of the latter is infinitesimal.

Note 1. If the absolute value of a vector function $\boldsymbol{p}(t)$ is infinitesimal, then the vector $\boldsymbol{p}$ itself is infinitesimal. The *order of smallness* of a vector is the order of smallness of its absolute value.

Note 2. The continuity of a vector function is defined in the same way as that of a scalar function (Sec. 218). Pictorially, the continuity of a vector function is expressed by the fact that its hodograph is an unknown curve. If a vector $\boldsymbol{p}$ is a continuous function of t, then its coordinates are likewise continuous (scalar) functions of t, and vice versa.

Note 3. The theorems on the limit of a sum and product are also extended to vector functions; observe that all possible products may be considered (a scalar function by a vector function, a scalar product by two vector functions, their vector product, and a mixed product of three vector functions). The theorem on the limit of a quotient is applied to the only type of division considered in vector algebra (the division of a vector function by a scalar function).

355. The Derivative Vector Function

Definition. The derivative of a vector function $\boldsymbol{p}(u)$ is the vector

$$\boldsymbol{p}' = \lim_{\Delta u \to 0} \frac{\boldsymbol{p}(u+\Delta u) - \boldsymbol{p}(u)}{\Delta u} \tag{1}$$

The vector $\boldsymbol{p}'$ is itself a vector function of the argument u. Whence the name, *derivative vector function*, and the designation: $\boldsymbol{p}'(u)$.

Geometrical interpretation. Let the moving terminus of the vector $\overrightarrow{OM} = \boldsymbol{r}(u)$ (Fig. 394) describe a curve L [the hodograph of the vector function $\boldsymbol{r}(u)$]. Then the vector $\boldsymbol{r}'(u)$ is directed along the tangent MT towards increasing values of the parameter u; its length $|\boldsymbol{r}'(u)|$ is equal to $\left|\frac{ds}{du}\right|$ (see Example 1). If we take s for the argument, then the derivative vector function is equal to unity (see Example 2).

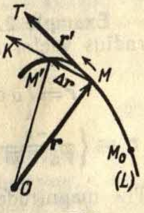


Fig. 394

Explanation. As the vector $\boldsymbol{r}(u)$ moves from point $M(u)$ to $M'(u+\Delta u)$, it receives the increment

$$\Delta \boldsymbol{r} = \boldsymbol{r}(u+\Delta u) - \boldsymbol{r}(u) = \overrightarrow{OM'} - \overrightarrow{OM} = \overrightarrow{MM'}$$

The vector $\frac{\Delta \boldsymbol{r}}{\Delta u}=\frac{\overrightarrow{MM'}}{\Delta u}$ is directed along the secant MM'; its length is equal to $\frac{MM'}{|\Delta u|} \approx \frac{\overset{\frown}{MM'}}{|\Delta u|}=\left|\frac{\Delta s}{\Delta u}\right|$. As $\Delta u \to 0$, the secant MM' tends to coincidence with the tangent, and the ratio $\frac{\Delta s}{\Delta u}$ tends to the limit $\frac{ds}{du}$.

The coordinates of the derivative $\boldsymbol{p}'(u)$ of the vector $\boldsymbol{p}(u)$ are respectively equal to the derivatives of the coordinates of the vector $\boldsymbol{p}(u)$; that is,

$$[x(u)\,\boldsymbol{i}+y(u)\,\boldsymbol{j}+z(u)\,\boldsymbol{k}]'=x'(u)\,\boldsymbol{i}+y'(u)\,\boldsymbol{j}+z'(u)\,\boldsymbol{k} \tag{2}$$

or, in an alternative notation,

$$\{x, y, z\}'=\{x', y', z'\} \tag{3}$$

Example 1. Using the notations of Sec. 349, we express the radius vector $\boldsymbol{r}$ of a helix in terms of a parameter t as follows:

$$\boldsymbol{r}=\{a\cos t,\ a\sin t,\ bt\}$$

By (3),

$$\boldsymbol{r}'=\{-a\sin t,\ a\cos t,\ b\}$$

The vector $\boldsymbol{r}'$ is directed along the tangent to the helix [cf. Sec. 351, formula (2a)]; its length $\sqrt{a^2+b^2}$ is equal to $\frac{ds}{dt}$ [cf. (5), Sec. 350].

Example 2. If we take the arc s for the argument of the radius vector $\boldsymbol{r}$ of the helix, then (Sec. 350, Example 2)

$$\boldsymbol{r}=\left\{a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}}s\right\},$$

$$\boldsymbol{r}'=\left\{\frac{-a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}}\right\}$$

The magnitude of the vector $\boldsymbol{r}'$ is

$$\frac{a^2}{a^2+b^2}\sin^2\frac{s}{\sqrt{a^2+b^2}}+\frac{a^2}{\sqrt{a^2+b^2}}\cos^2\frac{s}{\sqrt{a^2+b^2}}+\frac{b^2}{a^2+b^2}=1$$

Higher-order derivatives. They are defined in the same way as for scalar functions, and are denoted $\boldsymbol{p}''(u)$, $\boldsymbol{p}'''(u)$, etc. The expressions of derivatives in terms of differentials are given in Sec. 356.

Mechanical interpretation of derivatives. Let $\boldsymbol{r}(t)$ be a vector function expressing the radius vector of a moving point in terms of the time t. Then $\boldsymbol{r}'(t)$ is the vector of the velocity of point M and $\boldsymbol{r}''(t)$ is the vector of its acceleration.

356. The Differential of a Vector Function

The differential of a vector function $\boldsymbol{p}(u)$ is defined in the same way as for a scalar function (Sec. 228) and is denoted $d\boldsymbol{p}$.

The differential of a vector function $\boldsymbol{p}(u)$ is a vector; it is equal to the product of the derivative of the vector function $\boldsymbol{p}'(u)$ by the increment in the argument:

$$d\boldsymbol{p} = \boldsymbol{p}'(u)\,\Delta u \tag{1}$$

or

$$d\boldsymbol{p} = \boldsymbol{p}'(u)\,du \tag{2}$$

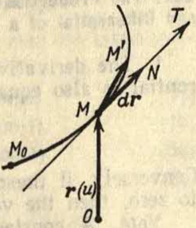


Fig. 395

Geometrical interpretation. The differential $d\boldsymbol{r}(u)$ is a vector $\overrightarrow{MN}$ (Fig. 395) directed along the tangent MT; the coordinates of the vector $d\boldsymbol{r}$ are the differentials of the coordinates x, y, z of point M:

$$d\boldsymbol{r} = \{dx,\ dy,\ dz\} \tag{3}$$

The length of the vector $d\boldsymbol{r}$ is equal to the differential of the arc $s = \overset{\frown}{M_0M}$:

$$|d\boldsymbol{r}| = \sqrt{dx^2 + dy^2 + dz^2} = ds \tag{4}$$

or

$$d\boldsymbol{r}^2 = ds^2 \tag{5}$$

If the arc s is the argument of the vector function $\boldsymbol{r}(s)$, then $|d\boldsymbol{r}| = \Delta s = \overset{\frown}{MM'}$. In the general case, however, $|\Delta\boldsymbol{r}|$ differs from the arc $\overset{\frown}{MM'}$ (and also from the chord MM') by an infinitesimal of higher order than Δu.

The invariance of expression (2). Formula (2) also holds true when u is regarded as a function of some argument. Formula (1) does not possess this property (cf. Sec. 234).

Differentials of higher order. They are defined in the same way as for scalar functions (Sec. 258) and are denoted by $d^2\boldsymbol{p}$, $d^3\boldsymbol{p}$, etc.

Expressing derivatives in terms of differentials:

$$\mathbf{p}'(u)=\frac{d\mathbf{p}}{du} \tag{6}$$

$$\mathbf{p}''(u)=\frac{d^2\mathbf{p}}{du^2}, \quad \mathbf{p}'''(u)=\frac{d^3\mathbf{p}}{du^3}, \ \ldots \tag{7}$$

In formula (6), u may be either an independent variable or a dependent variable; formulas (7) are valid when u is an independent variable, otherwise they do not, as a rule, hold true (cf. Sec. 259).

357. The Properties of the Derivative and Differential of a Vector Function

1. The derivative of a constant vector $\mathbf{a}$ is zero; the differential is also equal to zero:

$$\frac{d\mathbf{a}}{du}=0, \quad d\mathbf{a}=0 \tag{1}$$

Conversely, if the derivative of a vector is identically equal to zero, then the vector is constant.

Note. A constant vector not only has constant length, but constant direction as well. The derivative $\frac{d\mathbf{p}}{du}$ of a variable vector $\mathbf{p}$ of a constant is not equal to zero (it is perpendicular to the vector $\mathbf{p}$; see Property 6).

2. The differential of a sum of several vectors is equal to the sum of their differentials. The property is the same for derivative:

$$d[\mathbf{p}(u)+\mathbf{q}(u)-\mathbf{r}(u)]=d\mathbf{p}(u)+d\mathbf{q}(u)-d\mathbf{r}(u), \tag{2}$$

$$\frac{d}{du}[\mathbf{p}+\mathbf{q}-\mathbf{r}]=\frac{d\mathbf{p}}{du}+\frac{d\mathbf{q}}{du}-\frac{d\mathbf{r}}{du} \tag{2a}$$

3. Formulas of differentiation similar to those of Sec. 239 hold true for all types of multiplication of vectors, with the sole difference that a strict order of the factors is observed in vector and mixed products (cf. Sec. 112, Item 2, Sec. 117, Item 1):

$$d(m\mathbf{p})=m\,d\mathbf{p}+\mathbf{p}\,dm, \tag{3}$$

$$d(\mathbf{p}\times\mathbf{q})=\mathbf{p}\times d\mathbf{q}+d\mathbf{p}\times\mathbf{q}, \tag{4}$$

$$d(\mathbf{p}\mathbf{q})=\mathbf{p}\,d\mathbf{q}+\mathbf{q}\,d\mathbf{p}, \tag{5}$$

$$d(\mathbf{p}\mathbf{q}\mathbf{r})=d\mathbf{p}\,\mathbf{q}\mathbf{r}+\mathbf{p}\,d\mathbf{q}\mathbf{r}+\mathbf{p}\mathbf{q}\,d\mathbf{r} \tag{6}$$

The corresponding formulas for derivatives are:

$$\frac{d}{du}(mp) = m\frac{dp}{du} + p\frac{dm}{du}, \tag{3a}$$

$$\frac{d}{du}(\mathbf{p}\times\mathbf{q}) = \mathbf{p}\times\frac{d\mathbf{q}}{du} + \frac{d\mathbf{p}}{du}\times\mathbf{q}, \tag{4a}$$

$$\frac{d}{du}(\mathbf{pq}) = \mathbf{p}\frac{d\mathbf{q}}{du} + \mathbf{q}\frac{d\mathbf{p}}{du}, \tag{5a}$$

$$\frac{d}{du}(\mathbf{pqr}) = \frac{d\mathbf{p}}{du}\mathbf{qr} + \mathbf{p}\frac{d\mathbf{q}}{du}\mathbf{r} + \mathbf{pq}\frac{d\mathbf{r}}{du} \tag{6a}$$

4. As a particular case of formulas (5) and (5a), we have

$$d(\mathbf{p}^2) = 2\mathbf{p}\,d\mathbf{p}, \quad \frac{d}{du}(\mathbf{p}^2) = 2\mathbf{p}\frac{d\mathbf{p}}{du} \tag{7}$$

5. A constant factor (scalar or vector) may be taken outside the sign of the differential (derivative):

$$d(a\mathbf{p}) = a\,d\mathbf{p} \qquad (a = \text{const}), \tag{3b}$$

$$d(\mathbf{a}\times\mathbf{q}) = \mathbf{a}\times d\mathbf{q} \qquad (\mathbf{a} = \text{const}), \tag{4b}$$

$$d(\mathbf{aq}) = \mathbf{a}\,d\mathbf{q} \qquad (\mathbf{a} = \text{const}), \tag{5b}$$

$$d(\mathbf{aqr}) = \mathbf{a}d(\mathbf{q}\times\mathbf{r}) \qquad (\mathbf{a} = \text{const}) \tag{6b}$$

This follows from Properties 1 and 3.

6. If the vector $\mathbf{p}(u)$ maintains constant length, then it is perpendicular to the vector $\mathbf{p}'(u)$, and also to the vector $d\mathbf{p}(u)$, i.e. if

$$\mathbf{p}^2 = \text{const} \tag{8}$$

then (cf. Item 4)

$$\mathbf{pp}' = 0, \ \mathbf{p}\,d\mathbf{p} = 0 \tag{9}$$

This follows from (7).

Geometrically, the hodograph of the vector $\mathbf{p}(u)$ is a spherical line; its tangent is perpendicular to the radius of the sphere.

358. Osculating Plane

Definition. The *osculating plane* of a curve L at a point M is the plane P, with which the plane KMK' (Fig. 396) tends to come to coincidence when two (noncoincident) points K and K' approach (along L) the point M.

Note 1. For the curve L lying in the plane Q, the osculating plane coincides with the plane Q. For a straight line, the osculating plane remains indeterminate.

Explanation. Label three points M, K, K' on a wire model of the curve L. If they are not too far away from each other, the arc KMK' will practically lie in the plane KMK' (although the arc will depart considerably from rectilinear form). The osculating plane is an abstract image of the plane KMK'. If, using the model, we put a sheet of paper so that it practically coincides with the osculating plane, then, despite the inclination, it will retain its equilibrium (due to friction on the section KMK'). In all other positions, the sheet of paper will fall away from the model.

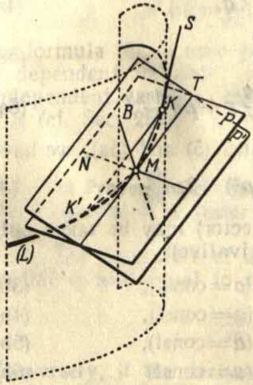


Fig. 396

The equation of the osculating plane. The "velocity vector" $\boldsymbol{r}'(u)$ and the "acceleration vector" $\boldsymbol{r}''(u)$ both lie in the osculating plane. If they are not collinear, then the vector product

$$\boldsymbol{B}=\boldsymbol{r}'\times\boldsymbol{r}'' \tag{1}$$

is the normal vector to the osculating plane [1] and the equation of the latter is

$$(\boldsymbol{R}-\boldsymbol{r})\,\boldsymbol{r}'\boldsymbol{r}''=0 \tag{2}$$

or, in coordinate form,

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}=0 \tag{3}$$

Example. Find the osculating plane of the helix

$$\boldsymbol{r}=\{a\cos u,\ a\sin u,\ bu\}$$

Solution. We find

$$\begin{aligned}\boldsymbol{r}'(u)&=\{-a\sin u,\ a\cos u,\ b\},\\ \boldsymbol{r}''(u)&=\{-a\cos u,\ -a\sin u,\ 0\},\\ \boldsymbol{r}'(u)\times\boldsymbol{r}''(u)&=\{ab\sin u,\ -ab\cos u,\ a^2\}=\\ &=a\,\{b\sin u,\ -b\cos u,\ a\}\end{aligned}$$

---

[1] If $\boldsymbol{r}'$ and $\boldsymbol{r}''$ are collinear and if $\boldsymbol{r}^{(k)}$ is the first of the derivative vectors not collinear with $\boldsymbol{r}'$, then for the normal vector to the osculating plane we can take $\boldsymbol{r}'\times\boldsymbol{r}^{(k)}$.

By virtue of (3) the equation of the osculating plane is

$$(X-a\cos u)\,b\sin u-(Y-a\sin u)\,b\cos u+(Z-bu)\,a=0$$

or

$$b\sin uX-b\cos uY+aZ=abu$$

The angle φ formed by the osculating plane and the axis of the helix is found (Sec. 146) from the formula

$$\sin\varphi=\frac{a}{\sqrt{b^2\sin^2 u+b^2\cos^2 u+a^2}}=\frac{a}{\sqrt{a^2+b^2}}$$

Whence $\tan\varphi=\frac{a}{b}$, that is, the osculating plane forms with the axis of the helix the very same constant angle as the tangent (Sec. 351, Example).

The osculating plane has the following properties.

(1) The plane TMK (Fig. 396) which passes through the tangent MT and the point K of curve L tends to coincidence with the osculating plane P when the point K tends to M.

(2) The plane P' (Fig. 396) which passes through the tangent MT and is parallel to the tangent KS also tends to coincidence with the P plane when the point K tends to M.

Note. Either of these properties may be taken as a difinition of the osculating plane.

359. Principal Normal. The Moving Trihedron

The normal MN to curve L (Fig. 396) lying in the osculating plane P is called the *principal normal*; the normal MB perpendicular to the osculating plane is called the *binormal*. The plane TMB which passes through the tangent and binormal is called the *rectifying plane*.

The three mutually perpendicular planes TMN (osculating), NMB (normal) and BMT (rectifying) form a *moving trihedron*; the three mutually perpendicular straight lines MT, MN, MB (*edges* of the moving trihedron) are frequently taken as the coordinate axes (the tangent MT as the axis of abscissas, or x-axis, the principal normal MN as the axis of ordinates, or y-axis, and the binormal MB as the z-axis). See Sec. 361 on the choice of positive directions.

It is convenient in the general case to compute the direction vectors of the edges in the following order:

$T=r'$ (vector of tangent, see Sec. 351), (1)

$B=r'\times r''$ (vector of binormal, see Sec. 358), (2)

$N=B\times T=$
$=(r'\times r'')\times r'$ (vector of principal normal) (3)

Expression (3) of the vector N is simplified when the arc s of curve L is taken as the parameter. Namely,

$$N=\frac{d^2r}{ds^2}\text{ [1]} \tag{4}$$

Example. Find the moving trihedron of the helix

$$r=\{a\cos u,\ a\sin u,\ bu\}$$

Solution. The vector of the tangent (Sec. 355, Example 1) is

$$T=r'=\{-a\sin u,\quad a\cos u,\ b\}$$

The vector of the binormal (Sec. 358, Example) is

$$B=r'\times r''=\{ab\sin u,\quad -ab\cos u,\ a^2\}$$

The vector of the principal normal is

$$N=B\times T=\{-a(a^2+b^2)\cos u,\ -a(a^2+b^2)\sin u,\ 0\}$$

The equations of the principal normal have the form

$$\frac{X-a\cos u}{\cos u}=\frac{Y-a\sin u}{\sin u}=\frac{Z-bu}{0}$$

We can see that the principal normal is perpendicular to the axis of the helix and intersects this axis at the point (0, 0, bu). Hence, the principal normal goes along the radius of the cylinder that carries the helix. The rectifying plane coincides with the tangent plane of the cylinder.

---

[1] Using the formula of the vector triple product (Sec. 122), we get

$$N=(r'\times r'')\times r'=r''(r'^2)-r'(r'r'')$$

Since in the given case $r'^2=1$ and $r'r''=0$, it follows that $N=r''$.

Geometrically, the vector of acceleration $\frac{d^2r}{ds^2}$ is in the osculating plane and is perpendicular to the vector of the tangent $\frac{dr}{ds}$. Hence, it is directed along the principal normal.

360. Mutual Positions of a Curve and a Plane

1. If a plane Q passing through a point M does not pass through the tangent MT of curve L, then near the point M this curve lies on both sides of the plane.

In particular, a normal plane always cuts the curve L.

In the case under consideration, the distance d from a neighbouring point M' of curve L to the plane Q is of the same order as the arc $\overset{\frown}{MM'}$.

2. If the plane Q contains the tangent MT but is not the osculating plane, then L, as a rule, lies on one side of the plane near point M (the *direction of concavity* of curve L). An exception is possible only if the vectors $\boldsymbol{r}'$, $\boldsymbol{r}''$ are collinear.

In particular, as a rule, curve L lies to one side of the rectifying plane.

The distance d is of second order with respect to the arc $\overset{\frown}{MM'}$ in the general case under consideration.

3. If plane Q is the osculating plane, then curve L near point M, as a rule, lies on both sides of the plane. The only exception is for coplanarity of the vectors $\boldsymbol{r}'$, $\boldsymbol{r}''$, $\boldsymbol{r}'''$.

The distance d, here, is as a rule of third order with respect to $\overset{\frown}{MM'}$. Only in the indicated exceptional case is the order of d above third.

361. The Base Vectors of the Moving Trihedron

For the positive directions, on the edges of the moving trihedron we take the directions of the following unit vectors (which play the part of the vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ in the rectangular coordinate system).

1. The base vector of the tangent $\boldsymbol{t}$ is directed along the tangent in the direction of increasing values of the parameter:

$$\boldsymbol{t}=\frac{\boldsymbol{T}}{\sqrt{\boldsymbol{T}^2}}=\frac{\boldsymbol{r}'(u)}{\sqrt{\boldsymbol{r}'^2(u)}} \tag{1}$$

If we take the arc s of curve L for the parameter, then

$$\boldsymbol{t}=\frac{d\boldsymbol{r}}{ds} \tag{1a}$$

2. The base vector of the principal normal $\boldsymbol{n}$ is directed along the principal normal towards concavity of the curve L:

$$\boldsymbol{n}=\frac{\boldsymbol{N}}{\sqrt{\boldsymbol{N}^2}}=\frac{(\boldsymbol{r}'\times\boldsymbol{r}'')\times\boldsymbol{r}'}{\sqrt{(\boldsymbol{r}'\times\boldsymbol{r}'')^2}\,\sqrt{\boldsymbol{r}'^2}} \tag{2}$$

If we take the arc s for the parameter, this expression can be substantially simplified:

$$\boldsymbol{n}=\frac{\dfrac{d^2\boldsymbol{r}}{ds^2}}{\sqrt{\left(\dfrac{d^2\boldsymbol{r}}{ds^2}\right)^2}} \tag{2a}$$

3. The base vector of the binormal $\boldsymbol{b}$ is directed along the binormal so that the triad of vectors $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ is right-handed:

$$\boldsymbol{b}=\boldsymbol{t}\times\boldsymbol{n}=\frac{\boldsymbol{r}'\times\boldsymbol{r}''}{\sqrt{(\boldsymbol{r}'\times\boldsymbol{r}'')^2}} \tag{3}$$

For the parameter s we have

$$\boldsymbol{b}=\frac{\dfrac{d\boldsymbol{r}}{ds}\times\dfrac{d^2\boldsymbol{r}}{ds^2}}{\sqrt{\dfrac{d^2\boldsymbol{r}}{ds^2}}} \tag{3a}$$

Note. The direction of the base vector of the principal normal does not depend on the choice of parameter, i.e. it has objective geometrical meaning. The base vector of the tangent can have either one of two opposite directions, depending on parametrization. If for instance we take the time as the parameter, then the direction of the vector $\boldsymbol{t}$ coincides with the direction of motion of point M along the curve L. If the arc is the parameter, then the direction of the vector $\boldsymbol{t}$ coincides with the direction of reckoning positive arcs. Thus, the direction of the vector $\boldsymbol{t}$ does not have objective geometrical meaning. When the direction of the vector $\boldsymbol{t}$ has been established, the direction of the vector $\boldsymbol{b}$ is fully defined.

362. The Centre, Axis and Radius of Curvature of a Space Curve

In Fig. 397, let the point M' moving along a space curve L tend to a fixed point M where the curvature K is not equal to zero. Then the straight line $A'B'$, along which the fixed normal plane Q intersects the moving normal plane Q', tends to coincidence with the straight line AB, which is perpendicular to the osculating plane P and distant $MC=\frac{1}{K}$ from point M. Here, the ray MC is in the direction of concavity of curve L.

The straight line AB is called the *axis of curvature*, the point C where AB intersects the osculating plane P is the *centre of curvature*, and the segment MC is the *radius of curvature*.

The radius of curvature is denoted by ρ; the quantities ρ and K are reciprocal:

$$\rho=\frac{1}{K}, \quad K=\frac{1}{\rho} \tag{1}$$

For a plane curve (its plane is the osculating plane) the centre and radius of curvature may be obtained by a construction as indicated in Sec. 343.

A circle with radius $CM=\rho$ described from the centre of curvature C is called the *osculating circle* or the *circle of curvature* of the curve L at the point M.

Note 1. If the curvature of L at M is zero, then we say that the radius of curvature is infinite and we write $\rho=\infty$ (cf. Sec. 343, Note).

Note 2. The definition of an involute given in Sec. 347 refers both to plane and twisted curves. The twisted curve L' also has an infinity of involutes (all twisted). But in contrast to the case of the plane curve (cf. Sec. 347), the centre of curvature of each one of the involutes L describes a curve which does not coincide with L'. For this reason, the locus of the centres of curvature of a twisted curve is not given the name evolute.

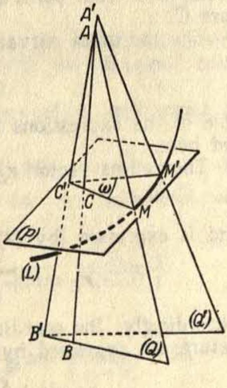


Fig. 397

363. Formulas for the Curvature, and the Radius and Centre of Curvature of a Space Curve

The curvature K is given by the formula

$$K=\frac{\sqrt{(\mathbf{r}'\times\mathbf{r}'')^2}}{\sqrt{(\mathbf{r}'^2)^3}} \tag{1}$$

In coordinate form,

$$K=\frac{\sqrt{(y'z''-z'y'')^2+(z'x''-x'z'')^2+(x'y''-y'x'')^2}}{\sqrt{(x'^2+y'^2+z'^2)^3}} \tag{2}$$

If the arc is taken for the parameter, formulas (1) and (2) are simplified:

$$K=\sqrt{\left(\frac{d^2r}{ds^2}\right)^2}=\left|\frac{d^2r}{ds^2}\right|, \tag{1a}$$

$$K=\sqrt{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2} \tag{2a}$$

In accordance with formula (1a), the vector $\frac{d^2r}{ds^2}$ is called the *curvature vector*. This vector is in the same direction as vector $\overrightarrow{MC}$ from point M of curve L to the centre of curvature C.

The radius of curvature ρ is found from the formula

$$\rho=\frac{1}{K} \tag{3}$$

One of the expressions (1), (2), (1a), (2a) has to be substituted here.

The radius vector r_C of the centre of curvature is

$$r_C=r+n\rho \tag{4}$$

and is expressed [by (2), Sec. 361] by the following formula:

$$r_C=r+\frac{r'^2}{(r'\times r'')^2}[(r'\times r'')\times r'] \tag{5}$$

Accordingly, the coordinates x_C, y_C, z_C of the centre of curvature are expressed by the formulas

$$\left.\begin{aligned} x_C&=x+\frac{x'^2+y'^2+z'^2}{A^2+B^2+C^2}(Bz'-Cy'),\\ y_C&=y+\frac{x'^2+y'^2+z'^2}{A^2+B^2+C^2}(Cx'-Az'),\\ z_C&=z+\frac{x'^2+y'^2+z'^2}{A^2+B^2+C^2}(Ay'-Bx') \end{aligned}\right\} \tag{6}$$

For the sake of brevity we use the following notation:

$$A=y'z''-z'y'',\quad B=z'x''-x'z'',\quad C=x'y''-y'x'' \tag{7}$$

If for the parameter we take the arc, then formulas (5) and (6) take the form (after simplifications):

$$r_C=r+\frac{\frac{d^2r}{ds^2}}{\left(\frac{d^2r}{ds^2}\right)^2}=r+\rho^2\frac{d^2r}{ds^2} \tag{5a}$$

$$\left.\begin{aligned}
x_C&=x+\frac{\dfrac{d^2x}{ds^2}}{\left(\dfrac{d^2x}{ds^2}\right)^2+\left(\dfrac{d^2y}{ds^2}\right)^2+\left(\dfrac{d^2z}{ds^2}\right)^2}=x+\rho^2\frac{d^2x}{ds^2},\\
y_C&=y+\frac{\dfrac{d^2y}{ds^2}}{\left(\dfrac{d^2x}{ds^2}\right)^2+\left(\dfrac{d^2y}{ds^2}\right)^2+\left(\dfrac{d^2z}{ds^2}\right)^2}=y+\rho^2\frac{d^2y}{ds^2},\\
z_C&=z+\frac{\dfrac{d^2z}{ds^2}}{\left(\dfrac{d^2x}{ds^2}\right)^2+\left(\dfrac{d^2y}{ds^2}\right)^2+\left(\dfrac{d^2z}{ds^2}\right)^2}=z+\rho^2\frac{d^2z}{ds^2}
\end{aligned}\right\}\tag{6a}$$

Note. The formulas for the curvature, the radius and centre of curvature of a plane curve (Sec. 344) are obtained from the foregoing if we put $z=z'=z''=0$.

Example. Find the curvature, the radius and centre of curvature of the helix L:

$$\boldsymbol{r}=\{a\cos t,\ a\sin t,\ bt\}\tag{8}$$

Solution. Taking the arc length for the parameter, we have (Sec. 350)

$$\boldsymbol{r}=\left\{a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right\}$$

Differentiating twice, we get

$$\boldsymbol{r}''=\left\{\frac{-a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}},\ -\frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}},\ 0\right\}$$

Formulas (2a) and (3) yield

$$K=\frac{a}{a^2+b^2},\quad \rho=\frac{a^2+b^2}{a}=a+\frac{b^2}{a}\tag{9}$$

That is, the curvature and the radius of curvature are constant. Formulas (6a) yield

$$\left.\begin{aligned}
x_C&=a\cos\frac{s}{\sqrt{a^2+b^2}}-\frac{a^2+b^2}{a}\cos\frac{s}{\sqrt{a^2+b^2}}=\\
&=-\frac{b^2}{a}\cos\frac{s}{\sqrt{a^2+b^2}}=-\frac{b^2}{a^2}x,\\
y_C&=-\frac{b^2}{a}\sin\frac{s}{\sqrt{a^2+b^2}}=-\frac{b^2}{a^2}y,\\
z_C&=\frac{bs}{\sqrt{a^2+b^2}}=z
\end{aligned}\right\}\tag{10}$$

From (10) it is evident that to construct the centre of curvature it is necessary to produce the radius of the cylinder (carrying the helix) beyond the axis of the cylinder to a constant distance $\frac{b^2}{a}$. Thus, the centre of curvature of the helix L will describe a helical curve L_1 (with the same lead $h=2\pi b$) plotted on a cylinder of radius $a_1=\frac{b^2}{a}$ (with the same axis). The symmetry of the relation $aa_1=b^2$ shows that the curves L and L_1 are reciprocal, that is, the centre of curvature of L_1 will describe L.

364. On the Sign of the Curvature

We attach sign in the following manner to the curvature of plane curves lying in the same plane. If the vector of the tangent rotates counterclockwise as point M moves towards increasing values of the parameter u, then the curvature is considered positive; if clockwise, it is negative.

The sign of the curvature is reversed if the parameter u is replaced by another parameter u' which decreases when u increases. When the abscissa is taken for the parameter, the increase in the parameter is associated with displacement of the point M to the right. In this case, the curvature is positive when the curve is concave up, and negative when the curve is concave down (Sec. 282).

Formulas (1) and (I), Sec. 344, are replaced by the following:

$$K=\frac{y''}{(x+y'^2)^{3/2}}, \tag{1}$$

$$K=\frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}} \tag{I}$$

Example. The curvature of the circle

$$x=a\cos u, \quad y=a\sin u$$

computed from formula (I) is $\frac{1}{a}$ (increasing parameter is associated with counterclockwise traversal; the vector of the tangent rotates in the same direction) If this circle is represented by the equations

$$x=a\cos u', \quad y=-a\sin u'$$

then formula (I) yields $K=-\frac{1}{a}$.

If the circle is given by the equation

$$x^2+y^2=a^2$$

and we apply formula (1), then for the upper semicircle we get $K=-\frac{1}{a}$ (traversal is counterclockwise and concavity is down), for the lower semicircle we get $K=\frac{1}{a}$.

This example shows us that in itself the sign of the curvature is devoid of any geometrical meaning. It is only the change in sign when passing through some point (point of inflection) or, on the contrary, preservation of sign on some segment that is of any significance.

Sign cannot be attached to the curvature of space curves (plane curves included) because in space there is no clockwise or counterclockwise rotation. These two directions are distinguished for lines in the same plane, because we can choose the front of a plane and have in mind an observer on that side. Now if (by some criterion) we were to distinguish the front and back sides on the osculating planes of an arbitrary curve in space, there would not be any position from which an observer could perceive all planes from the front side.

365. Torsion

The torsion of a space curve characterizes the degree of deviation of a curve from the plane form (just as the curvature characterizes the degree of deviation from the rectilinear form).

Definition. The *torsion* of a curve L at a point M is a quantity defined as follows: it is equal in absolute value to the limit to which the ratio of the angle ω' formed by the binormals MB and $M'B'$ to the arc $\overset{\smile}{MM'}$ tends when the point M' tends to M on L. The sign of the torsion (and also the sign of the angle ω') is considered positive when the pair of binormals MB, $M'B'$ is right-handed (see Sec. 165a) and negative when this pair is left-handed. The symbol for torsion is σ:

$$\sigma=\lim\frac{\omega'}{\overset{\smile}{MM'}}$$

Note. The binormal of a plane curve preserves constant direction so that the torsion of a plane curve is everywhere zero. Conversely, if the torsion of a curve is everywhere zero, then the curve is plane. The torsion of a twisted curve can equal zero only at special points called *planar points*.

Radius of torsion. The quantity $\tau=\frac{1}{\sigma}$, the reciprocal of the torsion, is called the *radius of torsion*, by analogy with the radius of curvature, $\rho=\frac{1}{K}$. But this analogy is incomplete: the process, similar to the construction of a centre of curvature, does not yield any "centre of torsion".

Torsion is expressed by the formula

$$\sigma=\frac{\mathbf{r}'\mathbf{r}''\mathbf{r}'''}{(\mathbf{r}'\times\mathbf{r}'')^2} \qquad (1)$$

or, in coordinate form,

$$\sigma=\frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{(y'z''-z'y'')^2+(z'x''-x'z'')^2+(x'y''-y'x'')^2} \tag{2}$$

If we take the arc s for the parameter, then formulas (1) and (2) are somewhat simplified:

$$\sigma=\frac{\frac{d\mathbf{r}}{ds}\,\frac{d^2\mathbf{r}}{ds^2}\,\frac{d^3\mathbf{r}}{ds^3}}{\left(\frac{d^2\mathbf{r}}{ds^2}\right)^2}=\rho^2\left(\frac{d\mathbf{r}}{ds}\,\frac{d^2\mathbf{r}}{ds^2}\,\frac{d^3\mathbf{r}}{ds^3}\right) \tag{1a}$$

or, in coordinate form,

$$\sigma=\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \\ \frac{d^3x}{ds^3} & \frac{d^3y}{ds^3} & \frac{d^3z}{ds^3} \end{vmatrix} : \left[\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right] \tag{2a}$$

Example. Find the torsion of the helix

$$x=a\cos u,\quad y=a\sin u,\quad z=bu$$

Solution. We have

$$\mathbf{r}'\mathbf{r}''\mathbf{r}'''=\begin{vmatrix} -a\sin u & a\cos u & b \\ -a\cos u & -a\sin u & 0 \\ a\sin u & -a\cos u & 0 \end{vmatrix}=a^2b$$

$$(\mathbf{r}'\times\mathbf{r}'')^2=a^2(a^2+b^2)$$

By formula (1) we obtain

$$\sigma=\frac{b}{a^2+b^2}$$

From this we see that the torsion of a right-handed helix ($b>0$) is positive, that of a left-handed helix is negative.

SERIES

366. Introductory Remarks

In order to overcome the difficulties involved in integration, Newton and Leibniz expressed the integrand function in the form of a polynomial with an infinite number of terms (see Sec. 270). Applying to such expressions the ordinary rules of algebra, mathematicians of the 18th century made a host of remarkable discoveries. It was observed, however, that if one applied the rules of algebra to infinite sums without restriction, errors would inevitably crop up. It became necessary to state in precise form the basic concepts of infinite series and prove in rigorous fashion the properties of infinite series. This problem was resolved by mathematicians of the 19th century, some of whom we will encounter in the following sections.

367. The Definition of a Series

Let there be given a sequence of numbers

$$u_1,\ u_2,\ u_3,\ \ldots,\ u_n,\ \ldots \tag{1}$$

Let us add these numbers together in the order they are given. We obtain a new sequence $s_1, s_2, s_3, \ldots, s_n, \ldots$ where

$$\left.\begin{array}{l} s_1 = u_1, \\ s_2 = u_1 + u_2, \\ s_3 = u_1 + u_2 + u_3, \\ \ldots\ldots\ldots\ldots\ldots \\ s_n = u_1 + u_2 + u_3 + \ldots + u_n, \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\} \tag{2}$$

The process of its formation is given, in abbreviated fashion, as follows:

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \tag{3}$$

This is called an *infinite series* or, simply, a *series*. The numbers $u_1, u_2, u_3, \ldots$ are called the *terms* of the series. The sum

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n$$

is called the *partial* sum of the series ($s_1 = u_1$ is the first partial sum, $s_2=u_1+u_2$ the second partial sum, $s_3=u_1+u_2+u_3$, the third partial sum, and so on).

Example 1. The expression

$$1+(-1)+1+(-1)+\ldots+(-1)^{n+1}+\ldots \tag{4}$$

or, as it is usually written,

$$1-1+1-1+\ldots \tag{4a}$$

is a series. The meaning of expression (4) is that from the terms

$$1,\ -1,\ +1,\ -1,\ \ldots,\ (-1)^{n+1},\ \ldots$$

we form the partial sums

$$s_1=1,\ s_2=1-1=0,\ s_3=1-1+1=1,\ \ldots$$

$$\ldots,\ s_n=1-1+\ldots+(-1)^{n+1}=\frac{1+(-1)^{n+1}}{2},\ \ldots \tag{5}$$

Example 2. The expression

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\left(\frac{1}{2}\right)^{n-1}+\ldots \tag{6}$$

is a series. It states that the terms

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \ldots,\ \left(\frac{1}{2}\right)^{n-1},\ \ldots$$

go to make up the partial sums

$$s_1=1,\ s_2=1\frac{1}{2},\ s_3=1\frac{3}{4},\ \ldots,\ s_n=2-\left(\frac{1}{2}\right)^{n-1},\ \ldots \tag{7}$$

368. Convergent and Divergent Series

Definition. A series is called *convergent* if the sequence of its partial sums has a finite limit. This limit is termed the *sum* of the convergent series.

If a sequence of partial sums has no finite limit, then the series is called *divergent*. A divergent series has no sum.[1]

Example 1. The series

$$1+2+3+4+\ldots+n+\ldots \tag{1}$$

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[1] The word sum is understood in the sense established by the definition. The concept of a sum of a series may be extended and then some divergent series will also have sums (in the extended sense).

is divergent because the sequence of its partial sums

$$s_1=1, \quad s_2=3, \quad s_3=6, \quad \ldots, \quad s_n=\frac{n(n+1)}{2}, \ \ldots \tag{2}$$

has the limit infinity.

Example 2. The series

$$1-1+1-1+\ldots+(-1)^{n+1}+\ldots \tag{3}$$

is divergent because the sequence of its partial sums

$$s_1=1, \quad s_2=0, \quad s_3=1, \ \ldots, \quad s_n=\frac{1+(-1)^{n+1}}{2}, \ \ldots \tag{4}$$

(cf. Sec. 367, Example 1) has no limit at all

Note 1. When the sequence $s_1, s_2, s_3, \ldots$ does not have any limit, the divergent series is called *indeterminate*.

Example 3. The series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\left(\frac{1}{2}\right)^{n-1}+\ldots \tag{5}$$

is a convergent series because the sequence

$$s_1=1, \quad s_2=1\frac{1}{2}, \quad s_3=1\frac{3}{4}, \ \ldots, \quad s_n=2-\left(\frac{1}{2}\right)^{n-1}, \ \ldots \tag{6}$$

has a limit equal to 2:

$$\lim_{n\to\infty} s_n=2$$

The number 2 is the sum of the series (5).

Note 2. The notation

$$u_1+u_2+\ldots+u_n\ldots=S \tag{7}$$

means that the series $u_1+u_2+\ldots+u_n+\ldots$ converges and that its sum is equal to S, i.e. the notation (7) is equivalent to the notation

$$\lim_{n\to\infty}(u_1+u_2+\ldots+u_n)=S$$

Example 4. The notation

$$1-\frac{1}{2}+\frac{1}{4}-\ldots+\left(-\frac{1}{2}\right)^{n-1}+\ldots=\frac{2}{3}$$

means that the sequence of partial sums

$$s_1=1, \ s_2=\frac{1}{2}, \ s_3=\frac{3}{4}, \ \ldots, \quad s_n=\frac{2}{3}\left[1-\left(-\frac{1}{2}\right)^n\right], \ \ldots$$

has a limit equal to $\frac{2}{3}$, that is, that

$$\lim_{n\to\infty}\left[1-\frac{1}{2}+\ldots+\left(-\frac{1}{2}\right)^{n-1}\right]=\frac{2}{3}$$

369. A Necessary Condition for Convergence of a Series

The series

$$u_1+u_2+\ldots+u_n+\ldots \tag{1}$$

can converge only when the term u_n (the *general term* of the series) tends to zero:

$$\lim_{n\to\infty} u_n=0 \tag{2}$$

To put it differently, if the general term u_n does not tend to zero, then the series diverges.

Example 1. The series

$$0.4+0.44+0.444+0.4444+\ldots \tag{3}$$

definitely diverges because the general term $\left(\text{it has the limit } \frac{4}{9}\right)$ does not tend to zero.

Example 2. The series

$$1-1+1-1+\ldots \tag{4}$$

definitely diverges because the general term does not tend to zero (and has no limit at all).

Caution. Condition (2) *is not sufficient* for convergence of a series: a series whose general term tends to zero may converge but may also diverge (see Examples 3 and 4).

Example 3. The so-called *harmonic*[1] series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots \tag{5}$$

diverges although its general term tends to zero. To convince yourself of the convergence of the series, consider the partial sums

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[1] The name stems from the fact that when a string is divided into 2, 3, 4, . . . even parts, the sounds emitted are in harmony with the fundamental tone.

$$s_2=1+\frac{1}{2}=3\cdot\frac{1}{2},$$

$$s_4=s_2+\left(\frac{1}{3}+\frac{1}{4}\right)>3\cdot\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)=4\cdot\frac{1}{2},$$

$$s_8=s_4+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)>4\cdot\frac{1}{2}+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)=$$
$$=5\cdot\frac{1}{2},$$

$$s_{16}=s_8+\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)>6\cdot\frac{1}{2}\text{ and so on.}$$

We see that the partial sum increases without bound, i.e. series (5) diverges.

Example 4. The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots \tag{6}$$

which is obtained from the harmonic series by reversing the sign of even-numbered terms converges. To see this, take a number scale (Fig. 398) and plot points representing the partial sums $s_1=1$, $s_2=\frac{1}{2}$, $s_3=\frac{5}{6}$, $s_4=\frac{7}{12}$, $s_5=\frac{47}{60}$, $s_6=\frac{37}{60}$. Each of the odd points s_1, s_3, s_5 will be more to left than the preceding one, and each of the even points s_2, s_4, s_6 will be to the right; that is, the even and odd points move towards each other. It can be proved that this law holds true 1) and that the points s_{2n}, s_{2n+1} come closer together indefinitely. 2) This means that both even and odd points

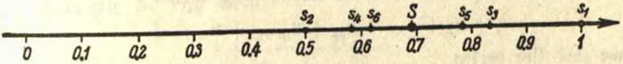

Fig. 398

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1) The difference $s_{2n+1}-s_{2n-1}=\left(1-\frac{1}{2}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}+\frac{1}{2n+1}\right)-\left(1-\frac{1}{2}+\ldots+\frac{1}{2n-1}\right)=-\frac{1}{2n}+\frac{1}{2n+1}$ is negative, the difference $s_{2n+2}-s_{2n}=\frac{1}{2n+1}-\frac{1}{2n+2}$ is positive.

2) The difference $s_{2n+1}-s_{2n}=\frac{1}{2n+1}$ tends to zero as $n\to\infty$.

tend to some point S (even from the right, odd from the left). Hence, the sequence of partial sums of the series (6) has as its limit the number S, i.e. series (6) converges and S is its sum.

The partial sums s_1, s_3, s_5 yield approximations of S that are too large, while s_2, s_4, s_6 yield approximations that are too small. Computing $s_9=0.745$ and $s_{10}=0.645$, we get for S one correct digit: $S=0.7$. Computing s_{999} and s_{1000}, we would find that $S=0.693$ with three correct digits. The exact value of S is $\ln 2$:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\ln 2 \tag{7}$$

Formula (7) is obtained from the expansion

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

for $x=1$ (cf. Sec. 270, Item 4 and Sec. 272, Example 2).

370. The Remainder of a Series

If we discard the first m terms of a series

$$u_1+u_2+\ldots+u_m+u_{m+1}+u_{m+2}+\ldots \tag{1}$$

we get the series

$$u_{m+1}+u_{m+2}+\ldots \tag{2}$$

which converges (or diverges) if the series (1) converges (or diverges). Therefore, when investigating the convergence of a series we can disregard a few of the first terms.

When the series (1) converges, the sum

$$R_m=u_{m+1}+u_{m+2}+\ldots \tag{3}$$

of series (2) is called the *remainder* (or *remainder term*) of the first series ($R_1=u_2+u_3+\ldots$ is the first remainder, $R_2=u_3+u_4+\ldots$ is the second, etc.). The remainder R_m is the error committed by substituting the partial sum s_m for the sum S of the series (1). The sum S of the series and the remainder R_m are connected by the relation

$$S=s_m+R_m \tag{4}$$

As $m \to \infty$ the remainder term of the series approaches zero. It is of practical importance that this approach be "sufficiently rapid", that is, that the remainder R_m should become less than the permissible error, for m not too great. Then we say that the series (1) converges *rapidly*, otherwise the series is said to converge *slowly*. Quite naturally, the speed of convergence is a relative notion.

Example 1. The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots \qquad (5)$$

converges very slowly. Summing the first twenty terms, we get the value of the sum of the series only to within $0.5\cdot 10^{-1}$; to attain an accuracy of up to $0.5\cdot 10^{-4}$, we have to take at least 19,999 terms (see Example 4, Sec. 369).

Example 2. The series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots \qquad (6)$$

(a geometric progression) converges much more rapidly than series (5); its fifteenth remainder $-\frac{1}{2^{15}}+\frac{1}{2^{16}}-\frac{1}{2^{17}}+\dots$ is already less, in absolute value, than $\left(\frac{1}{2}\right)^{15}<0.5\cdot 10^{-4}$ so that $0.5\cdot 10^{-4}$ accuracy is ensured by fifteen terms of the series.

Example 3. The series

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$$

(its sum is equal to e; cf. Sec. 272, Example 1) converges still faster; $0.5\cdot 10^{-4}$ accuracy is ensured by eight terms of the series.

371. Elementary Operations on Series

1. **Termwise multiplication by a number.** If the series

$$u_1+u_2+\dots+u_n+\dots \qquad (1)$$

converges and its sum is S, then the series

$$wu_1+wu_2+\dots.wu_n+\dots \qquad (2)$$

obtained by term-by-term multiplication of (1) by one and the same number w also converges, and its sum is equal to wS, i.e.

$$wu_1+wu_2+\dots+wu_n+\dots=w(u_1+u_2+\dots+u_n+\dots) \qquad (3)$$

Example 1. The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots \tag{4}$$

converges and its sum is equal to $0.693\ldots=\ln 2$ (Sec. 369, Example 4). Consequently, the series

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\ldots \tag{5}$$

converges and its sum is $0.346\ldots=\frac{1}{2}\ln 2$.

2. **Termwise addition and subtraction.** If the series

$$u_1+u_2+\ldots+u_n+\ldots, \tag{6}$$

$$v_1+v_2+\ldots+v_n+\ldots \tag{7}$$

converge and their sums are respectively equal to U and V, then the series

$$(u_1\pm v_1)+(u_2\pm v_2)+\ldots+(u_n\pm v_n)+\ldots \tag{8}$$

obtained by means of termwise addition (or subtraction) is also convergent and its sum is equal to $U+V$ (or $U-V$), i.e.

$$(u_1\pm v_1)+(u_2\pm v_2)+\ldots=(u_1+u_2+\ldots)\pm(v_1+v_2+\ldots) \tag{9}$$

Example 2. The series

$$0.11+0.0101+0.001001+\ \ldots$$

converges and its sum is $\frac{12}{99}$. Indeed, this series is obtained by termwise addition of the convergent series $0.1+0.1^2+0.1^3+\ldots$ and $0.01+0.01^2+0.01^3+\ldots$, and the sums of these series are, respectively, $\frac{1}{9}$ and $\frac{1}{99}$.

Caution. Not all properties of finite sums hold true for convergent series. Thus, if the terms of a convergent series are rearranged, the sum may be different and the series may even become divergent. To illustrate, rearrange the terms of the convergent series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\ldots= \\ =0.693\ldots \tag{10}$$

so that one negative term follows two positive ones (the order of the positive terms remains unchanged; this goes for the negative terms as well). We get a series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15}-\\ -\frac{1}{8}+\ldots \qquad (11)$$

which converges but whose sum is one and a half times greater than that of the original series. Indeed, we have (see Example 1)

$$0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+0+\frac{1}{10}+\ldots=\frac{1}{2}\cdot 0.693 \qquad (12)$$

(inserting zeros does not alter the sum of the series!). Adding the series (10) and (12) termwise (Item 2), we get

$$1+0+\frac{1}{3}-\frac{2}{4}+\frac{1}{5}+0+\frac{1}{7}-\frac{2}{8}+\frac{1}{9}+0+\ldots=\\ =\frac{3}{2}\cdot 0.693\ldots$$

Reducing fractions and deleting zeros, we get series (11) on the left.

372. Positive Series

A *positive series* (that is, a series all terms of which are positive) cannot be indeterminate (Sec. 368, Note 1). Its partial sums always have a limit—finite or infinite. In the former case, the series converges, in the latter it diverges.

A positive convergent series remains convergent upon rearragement of terms and *its sum does not change* (cf. Sec. 371, Caution); a divergent positive series remains divergent.

373. Comparing Positive Series

To test a given positive series

$$u_0+u_1+u_2+\ldots \qquad (1)$$

for convergence, it is frequently compared with another positive series

$$v_0+v_1+v_2+\ldots \qquad (2)$$

about which it is known that it converges or diverges.

If the series (2) converges and its sum is equal to V, while the terms of the given series do not exceed the corresponding terms of the series (2), then the given series converges, and its sum does not exceed V. Likewise, the remainder of the given series does not exceed the remainder of the series (2).

If the series (2) diverges, and the terms of the given series are not less than the corresponding terms of (2), then the given series diverges.

Example 1. Test for convergence the series

$$1+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 5^2}+\ldots+\frac{1}{n\cdot 5^{n-1}}+\ldots \tag{3}$$

and if it converges find four significant digits of its sum S.

Solution. Compare the given series with the geometric progression

$$1+\frac{1}{5}+\frac{1}{5^2}+\ldots+\frac{1}{5^{n-1}}+\ldots \tag{4}$$

Series (4) converges; its sum is 1.25. The terms of the given series do not exceed the corresponding terms of (4). Hence, the given series converges, and $S<1.25$. The remainder

$$R_n=\frac{1}{(n+1)\,5^n}+\frac{1}{(n+2)\,5^{n+1}}+\frac{1}{(n+3)\,5^{n+2}}+\ldots \tag{5}$$

of series (3) is less than the nth remainder of the series (4), that is

$$R_n<\frac{1}{5^n}+\frac{1}{5^{n+1}}+\frac{1}{5^{n+2}}+\ldots=\frac{1}{4\cdot 5^{n-1}}$$

For a better estimate, compare the remainder (5) with the series

$$\frac{1}{(n+1)\,5^n}+\frac{1}{(n+1)\,5^{n+1}}+\frac{1}{(n+1)\,5^{n+2}}+\ldots=\frac{1}{(n+1)\cdot 4\cdot 5^{n-1}} \tag{6}$$

Reasoning as before, we get the inequality

$$R_n<\frac{1}{4\,(n+1)\,5^{n-1}} \tag{7}$$

Putting $n=1, 2, 3, \ldots$ in succession, we find that the expression $\frac{1}{4\,(n+1)\,5^{n-1}}$ becomes less than 0.0005 for $n=4$. Sum four terms of the given series. This yields the following approximation (too small)

$$S\approx 1+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 5^2}+\frac{1}{4\cdot 5^3}=1.115$$

(to within $0.5\cdot 10^{-3}$).

Example 2. To investigate the convergence of the series

$$\frac{1}{1}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\dots \tag{8}$$

we compare it with the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots \tag{9}$$

This series diverges (Sec. 369, Example 3), and the terms of the given series are not less than the corresponding terms of the series (9). Hence, series (8) diverges.

Example 3. To investigate the convergence of the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots+\frac{1}{n^2}+\dots \tag{10}$$

compare it with the series

$$1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{(n-1)\,n}+\dots \tag{11}$$

the terms of which, from the second onwards, are greater than the corresponding terms of the series (10). The series (11) converges and its sum is $S=2$ because the nth partial sum can be represented in the form

$$s_n=1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\dots+ \\ +\left(\frac{1}{n-1}-\frac{1}{n}\right)=1+1-\frac{1}{n} \tag{12}$$

The series (10) most definitely converges and its sum is less than 2. The remainder R_n of (11) is equal (Sec. 370) to

$$R_n=S-s_n=\frac{1}{n}$$

The remainder of (10) is only slightly less, so that the series (10) converges slowly: to find four significant digits of the sum, it is necessary to add about 2000 terms. The exact value of the sum of the series (10) is $\frac{\pi^2}{6}$ (see Sec. 417, Example 3).

374. D'Alembert's Test for a Positive Series

Theorem. In the positive series

$$u_1+u_2+\ldots+u_n+\ldots \tag{1}$$

let the ratio $\frac{u_{n+1}}{u_n}$ of a following term to a preceding term have the limit q as $n \to \infty$. Three cases are possible:

Case 1. $q<1$. The series converges.

Case 2. $q>1$. The series diverges.[1]

Case 3. $q=1$. The series can either converge or diverge. This theorem is called the *D'Alembert's test*.[2]

Example 1. Consider the positive series

$$2\cdot 0.8+3\cdot 0.8^2+4\cdot 0.8^3+\ldots+(n+1)\cdot 0.8^n+\ldots$$

At first we observe an increase in the terms[3] ($a_1=1.6$, $a_2=1.92$, $a_3=2.048$, ...). However the series converges because $a_{n+1}: a_n=0.8\left(1+\frac{1}{n+1}\right)$ and the limit of this ratio is 0.8, which is less than 1.

Explanation. Suppose that for some positive series a_1+ $+a_2+a_3+\ldots+a_n+\ldots$ the limit of the ratio $u_{n+1}:u_n$ is 0.8. Then from some term N onwards the ratio $u_{n+1}:u_n$ will differ from 0.8 by less than $\pm$ 0.1. Hence, it will remain less than 0.9 so that

$$\left.\begin{aligned} &u_{N+1}<0.9u_N,\\ &u_{N+2}<0.9u_{N+1}<0.9^2u_N,\\ &u_{N+3}<0.9u_{N+2}<0.9^3u_N \end{aligned}\right\} \tag{2}$$

and so on. A comparison of the series $u_{N+1}+u_{N+2}+u_{N+3}+\cdots$ with the series $0.9u_N+0.9^2u_N+0.9^3u_N+\ldots$ (decreasing geometric progression) shows (Sec. 373) that the given series converges.

In place of 0.9 we can take any number between 0.8 and 1. (It is meaningless to take unity or a number greater than unity.)

The same scheme is used in the general proof of the theorem for the case $q<1$.

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[1] Included here is the case when $\lim u_{n+1}:u_n=\infty$.

[2] This is a misnomer because the theorem was first stated and proved by Cauchy.

[3] A decrease sets in later.

Example 2. Consider the positive series

$$1.1+\frac{1.1^2}{2}+\frac{1.1^3}{3}+\ldots+\frac{1.1^n}{n}+\ldots \tag{3}$$

The initial terms decrease, but the series diverges because the limit of the ratio

$$u_{n+1}:u_n=\frac{1.1^{n+1}}{n+1}:\frac{1.1^n}{n}=\left(1-\frac{1}{n+1}\right)1.1$$

is equal to 1.1, which is greater than 1.

Explanation. Since $\lim (u_{n+1}:u_n)=1.1$, it follows that after some term N, the ratio $u_{n+1}:u_n$ is greater than 1.09. Comparing the series $u_{N+1}+u_{N+2}+u_{N+3}+\ldots$ with the divergent series $1.09u_N+1.09^2u_N+1.09^3u_N+\ldots$ and reasoning as in the preceding explanation, we prove (Sec. 373) that the given series diverges.

In place of 1.09 we can take any number between 1.1 and 1 (but not unity).

The same scheme is used in the general proof for the case $q>1$.

Example 3. Consider the series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots \tag{4}$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots \tag{5}$$

For both we have

$$q=\lim_{n\to\infty}(u_{n+1}:u_n)=1$$

But series (4) diverges (Sec. 369) while (5) converges (Sec. 373).

Note. In Case 1 ($q<1$) the convergence is the faster, the smaller q. In Case 2 ($q>1$) the divergence is the faster, the greater q. In Case 3 ($q=1$), the series converges slowly if it converges at all, and for this reason it is poorly suited for computations.

375. The Integral Test for Convergence

If every term of a positive series

$$u_1+u_2+\ldots+u_n+\ldots \tag{1}$$

is less than the preceding term, then in testing for convergence we can consider the improper integral

$$\int_1^{\infty} f(n)\,dn \tag{2}$$

where $f(n)$ is a continuous decreasing function of n, which takes on the values $u_1, u_2, u_3, \ldots$ for $n=1, 2, 3, \ldots$.

The series (1) converges or diverges depending on whether the improper integral (2) converges or diverges. In the case of convergence, the remainder R_n of the series (1) satisfies the inequalities

$$\int_{n+1}^{\infty} f(n)\,dn < R_n < \int_{n}^{\infty} f(n)\,dn \tag{3}$$

Note. The integral test is convenient in cases when the term u_n is given by an expression that is meaningful not only for integral values of n but for all n greater than unity.

Example 1. Let us investigate the convergence of the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots \tag{4}$$

This series is positive; each term is less than the preceding one, and the general term u_n is given by the expression $\frac{1}{n}$, which is meaningful for all values of n (except zero). The function $f(n)=\frac{1}{n}$ in the interval $(1, \infty)$ is continuous and is decreasing. Consider the improper integral $\int_1^{\infty} \frac{dn}{n}$. It diverges because it has an infinite value:

$$\lim_{x \to \infty} \int_1^{x} \frac{dn}{n} = \lim_{x \to \infty} \ln x = \infty$$

Hence, series (4) diverges too (cf. Sec. 369, Example 3).

Example 2. Test for convergence the series of inverse squares

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots \tag{5}$$

Here, $f(n)=\frac{1}{n^2}$. The corresponding improper integral

$$\int_1^{\infty}\frac{dn}{n^2}=\lim_{x\to\infty}\int_1^{x}\frac{dn}{n^2}=1$$

converges. Hence, so does the series (5). Taking 10 terms, we find $S_{10}=1.5498$. The remainder R_{10} satisfies the inequality

$$\int_{11}^{\infty}\frac{dn}{n^2}<R_{10}<\int_{10}^{\infty}\frac{dn}{n^2}, \quad \text{i.e. } \frac{1}{11}<R_{10}<\frac{1}{10}$$

Hence, the error in the approximate equality

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\approx 1.5498$$

does not exceed 0.1.

Explanation. The graph of Fig. 399 depicts a function $f(n)$; the terms $u_1, u_2, \ldots$ are indicated by the ordinates $P_1M_1, P_2M_2, \ldots$; the latter are numerically equal to the areas of the rectangles $P_1M_1N_2P_2$, $P_2M_2N_3P_3, \ldots$

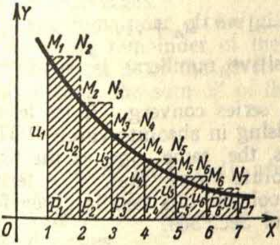

Fig. 399

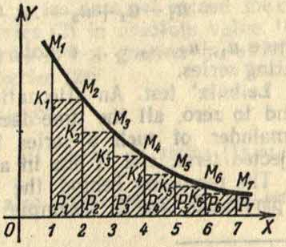

Fig. 400

The divergence of the integral $\int_1^{\infty} f(n)\,dn$ signifies that the area of the strip under the curve $M_1M_2M_3 \ldots$ is infinitely great. The area of the circumscribed step-like figure is all the more so infinite, that is, the series $u_1+u_2+\ldots$ diverges.

Now if the integral $\int\limits_1^\infty f(n)\,dn$ converges, then the area of the strip under $M_1M_2M_3$ ... is finite. The area of the inscribed figure shown shaded in Fig. 400 is all the more so finite, that is, the series $u_2+u_3+\ldots$ converges. Hence, the series $u_1+u_2+u_3+\ldots$ converges as well.

Inequality (3) can be explained by an example in which $n=2$. The remainder $R_2=u_3+u_4+\ldots$ is numerically equal to the area of the circumscribed figure $XP_3M_3N_4M_4N_5\ldots$ (Fig. 399); hence, $R_2 > \int\limits_3^\infty f(n)\,dn$ (by hypothesis, this integral converges). The same remainder is equal to the area of the figure $XP_2K_2M_3K_3M_4$ ... (Fig. 400); hence,

$$R_2 < \int\limits_2^\infty f(n)\,dn$$

376. Alternating Series. Leibniz' Test

A series is called *alternating* if its terms are alternately of opposite sign. The series

$$u_1-u_2+u_3-\ldots+(-1)^{n-1}u_n+\ldots \qquad (1)$$

where $u_1, u_2, u_3, \ldots$ denote positive numbers, is an alternating series.

Leibniz' test. An alternating series converges if its terms tend to zero, all the time decreasing in absolute value.[1] The remainder of such a series has the same sign as the first rejected term, and is less in absolute value than this term.

The reasoning behind the proof of this test is given for a particular case in Example 4 of Sec. 369.

---

[1] The terms of an alternating series may tend to zero but not decrease all the time. Then there is no guarantee that the series converges. Thus, the series

$$-\frac{1}{2}+\frac{2}{2}-\frac{1}{3}+\frac{2}{3}-\frac{1}{4}+\frac{2}{4}-\frac{1}{5}+\frac{2}{5}-\ldots$$

the terms of which tend to zero, but do not decrease all the time, diverges. Indeed, grouping the terms in pairs, we find that $s_{2n}=\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ so that (Sec. 369, Example 3) $\lim\limits_{n\to\infty} s_{2n}=\infty$.

Example. The alternating series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots \tag{2}$$

converges because its terms tend to zero, all the time decreasing in absolute value. The fifteenth remainder

$$R_{15}=-\frac{1}{16}+\frac{1}{17}-\frac{1}{18}+\dots$$

is negative, so that the partial sum s_{15} yields an approximation in excess for the sum of the series (2). The remainder is less than $\frac{1}{16}$ in absolute value.

377. Absolute and Conditional Convergence

Theorem. The series

$$u_1+u_2+\dots+u_n+\dots \tag{1}$$

definitely converges if the positive series

$$|u_1|+|u_2|+\dots+|u_n|+\dots \tag{2}$$

composed of the absolute values of the terms of the given series converges.

The remainder of the given series does not exceed the corresponding remainder of the series (2) in absolute value.[1]

The sum S of the given series is not greater in absolute value than the sum S' of the series (2):

$$|S|\le S'$$

The equality holds only when all terms of series (1) are of the same sign.

Note 1. The series (1) can also converge when the series (2) diverges.

Example 1. The series

$$1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\dots \tag{3}$$

every third term of which is negative converges because the following series converges (Sec. 373, Example 3):

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\dots \tag{4}$$

---

[1] The proof is given below (see Explanation).

This series is composed of the absolute values of the terms of the given series. The sum S of (3) is less than the sum S' of (4).[1]

Example 2. The alternating series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots$$

converges (Sec. 369, Example 4) despite the fact that the series composed of the absolute values of the terms of the given series diverges (Sec. 369, Example 3).

Definition 1. A series is called *absolutely convergent* if the series composed of the absolute values of its terms converges (in this case, the given series converges as well; cf. Example 1).

Definition 2. A series is called *conditionally convergent* if it converges while the series composed of the absolute values of its terms diverges (cf. Example 2).

Note 2. A convergent series in which all the terms are positive or all the terms are negative is an absolutely convergent series.

Explanation of Example 1. Retain the positive terms in (3) and replace the negative terms by zeros. We get a convergent positive series:

$$1+\frac{1}{2^2}+0+\frac{1}{4^2}+\frac{1}{5^2}+0+\frac{1}{7^2}+\frac{1}{8^2}+0+\dots=U \tag{5}$$

[its convergence follows from a comparison with series (10), Sec. 373]. Now replace the positive terms of (3) with zeros and reverse the sign of the negative terms. This yields the following convergent series of positive terms:

$$0+0+\frac{1}{3^2}+0+0+\frac{1}{6^2}+0+0+\frac{1}{9^2}+\dots=V \tag{6}$$

Subtract series (6) from (5) term-by-term. This yields series (3). By virtue of Sec. 371 (Item 2), it converges and its sum S is

$$S=U-V \tag{7}$$

Each of the positive numbers U, V is less (Sec. 373) than the sum S' of the series (4). Therefore

$$S<S'$$

The general theorem is proved in exactly the same way.

Note. Adding (5) and (6), we get

$$S'=U+V \tag{8}$$

---

[1] Here, $S=\frac{7}{9}S'$ (see Explanation).

In the given example we also have (Sec. 371, Item 1)

$$V=\frac{1}{3^2}+\frac{1}{6^2}+\frac{1}{9^2}+\ldots=\frac{1}{3^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)=\frac{1}{9}S' \qquad (9)$$

From (7), (8) and (9) it follows that

$$S=\frac{7}{9}S'$$

378. D'Alembert's Test for an Arbitrary Series

Suppose that in the series

$$u_1+u_2+\ldots+u_n+\ldots \qquad (1)$$

we have some negative terms (or all the terms are negative). Let the absolute value of the ratio $u_{n+1}:u_n$ have the limit q:

$$\lim_{n\to\infty}|u_{n+1}:u_n|=q$$

Then for $q<1$ the series converges, for $q>1$ it diverges, and for $q=1$ it can either converge or diverge.

This follows from Secs. 374 and 377.

Example. The series

$$1+\frac{1}{1!}-\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}-\frac{1}{6!}-\frac{1}{7!}+\frac{1}{8!}+\ldots \qquad (2)$$

where any two negative terms are followed by two positive terms which are followed by two negative terms, converges because $|u_{n+1}:u_n|=\frac{1}{n!}:\frac{1}{(n-1)!}=\frac{1}{n}$; hence

$$q=\lim_{n\to\infty}|u_{n+1}:u_n|=0, \text{ i.e. } q<1$$

379. Rearranging the Terms of a Series

In an absolutely convergent series it is possible to rearrange the terms in any fashion whatsoever: the absolute convergence of the series will not be upset thereby and the sum will remain unchanged (in particular, the sum of a convergent positive series is independent of the order of the terms).

Contrariwise, in a conditionally convergent series not every rearrangement of the terms is admissible because the sum can change and even the convergence can break down.

Example 1. The series

$$\left(\frac{1}{2}\right)^2-\frac{1}{2}+\left(\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^6-\left(\frac{1}{2}\right)^5+\ldots \qquad (1)$$

obtained by rearrangement of the terms of the absolutely convergent series

$$-\frac{1}{2}+\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^4+\ldots \qquad (2)$$

also converges and has the same sum S as the geometric progression (2). Hence

$$S=\frac{-\frac{1}{2}}{1+\frac{1}{2}}=-\frac{1}{3} \qquad (3)$$

Formula (3) can be verified by considering the partial sum s_{2n} of the series (1) as the sum of terms of a geometric progression with first term $\left(\frac{1}{2}\right)^2-\frac{1}{2}=-\frac{1}{4}$ and common ratio $\left(\frac{1}{2}\right)^2=\frac{1}{4}$.

Example 2. The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\ldots \qquad (4)$$

converges conditionally (Sec. 377). The series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\ldots \qquad (5)$$

obtained by rearranging the terms of (4) converges but its sum is one and a half times the sum of the given series (Sec. 371, Caution).

Note. In every conditionally convergent series it is possible to rearrange the terms in such a manner that the new series will have as sum *any preassigned number* (it is also possible to make the series diverge).

380. Grouping the Terms of a Series

In contrast to the commutative property (which holds only for absolutely convergent series; cf. Sec. 379), *all* convergent series possess the property of associativity.

Namely, in any convergent series it is possible to group the terms in any way, *without changing the order of the terms*. Combining the terms inside all groups, we get a new series, which also converges and whose sum is the same.

Example 1. In the convergent (by Leibniz' test) series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots \qquad (1)$$

we can group the terms as follows

$$\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\ldots \qquad (2)$$

Combining terms inside the groups, we get

$$\frac{2}{2^2-1}+\frac{2}{6^2-1}+\frac{2}{10^2-1}+\dots \tag{3}$$

This positive series has the same sum[1] as the alternating series (1).

Example 2. In the series (1) we can group the second term and the third, the fourth and fifth, etc. The resulting convergent series is

$$1-\frac{2}{4^2-1}-\frac{2}{8^2-1}-\frac{2}{12^2-1}-\dots \tag{4}$$

which has the same sum.

Note. The inverse operation (removal of brackets) is definitely admissible only if *after such a removal* a convergent series is obtained (then the given series is definitely convergent). However, it is also possible for the given series to converge and the series obtained after removal of brackets to be divergent.

Example 3. The series

$$(1-0.9)+(1-0.99)+(1-0.999)+\dots \tag{5}$$

(the geometric progression $0.1+0.01+0.001+\dots$) converges and has the sum $\frac{1}{9}$.

Removing brackets, we get the series

$$1-0.9+1-0.99+1-0.999+\dots \tag{5'}$$

which diverges because the even-numbered partial sums have the original limit of $\frac{1}{9}$, while the odd-numbered ones have the limit $1\frac{1}{9}$.

Example 4. Consider the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\dots \tag{6}$$

It can be represented as

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots \tag{7}$$

Here the brackets may be removed because the resulting series

$$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\dots \tag{8}$$

converges. Indeed, any partial sum s_{2n-1} is equal to unity, while the partial sum

$$s_{2n}=1-\frac{1}{n+1}$$

tends to unity. The sum $S=1$ of the series (8) is also the sum of (6):

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\dots=1$$

---

[1] It is equal to $\frac{\pi}{4}$; see Sec. 398, Example 3.

381. Multiplication of Series

Theorem. Two absolutely convergent series

$$u_1+u_2+u_3+\ldots=U, \tag{1}$$

$$v_1+v_2+v_3+\ldots=V \tag{2}$$

can be multiplied together like polynomials. Each term of the series (1) is multiplied by each term of (2) and the products are added in any order whatsoever. We get an absolutely convergent series whose sum is UV:

$$u_1v_1+u_1v_2+u_2v_1+u_1v_3+u_2v_2+u_3v_1+\ldots=UV \tag{3}$$

Note 1. So that in (3) no term is repeated twice or omitted, it is advisable to group terms u_iv_k with the same sum of the subscripts i, k (this sum is called the *weight* of the term u_iv_k). Then the series becomes

$$w_1+w_2+w_3+\ldots \tag{4}$$

where

$$\begin{aligned} w_1&=u_1v_1,\\ w_2&=u_2v_1+u_1v_2,\\ w_3&=u_3v_1+u_2v_2+u_1v_3,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ w_n&=u_nv_1+u_{n-1}v_2+u_{n-2}v_3+\ldots+u_1v_n \end{aligned} \tag{5}$$

This grouping corresponds to multiplication via the scheme

$$\begin{array}{ccccc} u_1+u_2 & +u_3 & +u_4 & + \ldots & \\ v_1+v_2 & +v_3 & +v_4 & + \ldots & \\ \hline u_1v_1+u_2v_1 & +u_3v_1 & +u_4v_1 & + \ldots & \\ u_1v_2 & +u_2v_2 & +u_3v_2 & + \ldots & \\ & u_1v_3 & +u_2v_3 & + \ldots & \\ & & u_1v_4 & + \ldots & \\ \hline w_1+w_2 & +w_3 & +w_4 & + \ldots & \end{array} \tag{6}$$

Example 1. Consider two absolutely convergent series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^{n-1}}+\ldots, \tag{7}$$

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n-1}}{2^{n-1}}+\ldots \tag{8}$$

Multiplying them by scheme (6), we get

$$\begin{array}{rcccccccccl}
1 & + & \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + & \ldots \\
 & - & \frac{1}{2} & - & \frac{1}{4} & - & \frac{1}{8} & - & \frac{1}{16} & - & \ldots \\
 & & & & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + & \ldots \\
 & & & & & - & \frac{1}{8} & - & \frac{1}{16} & - & \ldots \\
 & & & & & & & + & \frac{1}{16} & + & \ldots \\
\hline
1 & + & 0 & + & \frac{1}{4} & + & 0 & + & \frac{1}{16} & + & \ldots
\end{array} \qquad (9)$$

The law of formation of the resulting series is expressed by the formulas [1]

$$w_{2n-1}=\frac{1}{2^{2n-2}}, \quad w_{2n}=0$$

Dropping zeros, we obtain an absolutely convergent series:

$$1+\frac{1}{4}+\frac{1}{16}+\ldots+\frac{1}{4^{n-1}}+\ldots \qquad (10)$$

Its sum is the product of the sum of the series (7) and (8). This is easy to verify since the sum of the progression (7) is 2, the sum of (8) is $\frac{2}{3}$, and the sum of (10) is $\frac{4}{3}$.

Example 2. The series

$$1+\frac{2}{7}+\frac{3}{7^2}+\frac{4}{7^3}+\ldots+\frac{n}{7^{n-1}}+\ldots \qquad (11)$$

converges absolutely (by the d'Alembert test). Find its sum.

Solution. The required sum is the product of the sums of two identical absolutely convergent series

$$1+\frac{1}{7}+\frac{1}{7^2}+\ldots+\frac{1}{7^{n-1}}+\ldots=\frac{7}{6}, \qquad (12)$$

$$1+\frac{1}{7}+\frac{1}{7^2}+\ldots+\frac{1}{7^{n-1}}+\ldots=\frac{7}{6} \qquad (13)$$

---

[1] The terms in each column of (9) have the same absolute value but alternate in sign. The even-numbered column (where the number of the terms is even) yields zero. In the odd-numbered column $2n-1$, the first term is $\frac{1}{2^{2n-2}}$, the remaining ones cancel out.

Indeed, by scheme (6), we get

$$\begin{array}{ccccccccc}
1 & + & \frac{1}{7} & + & \frac{1}{7^2} & + & \frac{1}{7^3} & + & \dots \\
 & & \frac{1}{7} & + & \frac{1}{7^2} & + & \frac{1}{7^3} & + & \dots \\
 & & & & \frac{1}{7^2} & + & \frac{1}{7^3} & + & \dots \\
 & & & & & & \frac{1}{7^3} & + & \dots \\
\hline
1 & + & \frac{2}{7} & + & \frac{3}{7^2} & + & \frac{4}{7^3} & + & \dots
\end{array}$$

Hence, the sum of the series (11) is

$$\frac{7}{6}\cdot\frac{7}{6}=\frac{49}{36}$$

Note 2. If one of the series (1), (2) converges absolutely and the other conditionally, then the series (4) found by ([illegible]) is convergent, and its sum remains equal to UV. But it may prove to be conditionally convergent; then not every rearrangement of terms is permissible (Sec. 379).

If both series (1) and (2) converge conditionally, then (4) *may prove to be divergent*. [1] But if it converges, then its sum is equal to UV.

---

[1] The series

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\dots\pm\frac{1}{\sqrt{n}}\mp\dots \tag{U}$$

is convergent (by the Leibniz test, Sec. 376), but it converges conditionally; that is, the positive series

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\dots+\frac{1}{\sqrt{n}}+\dots$$

diverges (Sec. 373, Example 2). If we apply formulas (4) and (5) to two series, each of which coincides with (U), then we obtain the series

$$w_1+w_2+w_3+\dots \tag{W}$$

in which each term is greater in absolute value than unity. Indeed, formula (5) yields

$$|w_n|=\frac{1}{\sqrt{1\cdot n}}+\frac{1}{\sqrt{2(n-1)}}+\dots+\frac{1}{\sqrt{n\cdot 1}}$$

Here each of the n terms is greater than $\frac{1}{\sqrt{n\cdot n}}=\frac{1}{n}$ Hence w_n does not tend to zero, and the series (W) diverges (Sec. 369).

382. Division of Series

Theorem. Suppose we have two convergent series:

$$u_1+u_2+\ldots+u_n+\ldots=U, \tag{1}$$

$$v_1+v_2+\ldots+v_n+\ldots=V \tag{2}$$

Applying to them the scheme of division of the polynomial $u_1+u_2+\ldots+u_n$ by the polynomial $v_1+v_2+\ldots+v_n$, we get the series

$$w_1+w_2+\ldots+w_n+\ldots \tag{3}$$

If the series (3) is convergent,[1] then its sum W is equal to $U:V$.

Example. Let us apply to the convergent series

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}+\ldots=U, \tag{1a}$$

$$\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\ldots+\frac{(-1)^{n-1}}{2^n}+\ldots=V \tag{2a}$$

the scheme of division of a polynomial by a polynomial. We have

$$
\begin{array}{r|l}
\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\ldots & \frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}-\ldots \\ \cline{2-2}
-\;\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}-\ldots & 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots \\ \cline{1-1}
\multicolumn{1}{r}{\frac{1}{2}+0+\frac{1}{2^3}+0+\frac{1}{2^5}+\ldots} & \\
\multicolumn{1}{r}{-\;\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}-\ldots} & \\ \cline{1-1}
\multicolumn{1}{r}{\frac{1}{2^2}+0+\frac{1}{2^4}+0+\ldots} & \\
\multicolumn{1}{r}{-\;\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}-\frac{1}{2^5}+\ldots} & \\ \cline{1-1}
\multicolumn{1}{r}{\frac{1}{2^3}+0+\frac{1}{2^5}+\ldots} & \\
\multicolumn{1}{r}{-\;\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}-\ldots} & \\ \cline{1-1}
\multicolumn{1}{r}{\frac{1}{2^4}+0+\ldots} &
\end{array}
$$

---

[1] It may prove divergent even in the case of absolute convergence of the series (1) and (2). Thus, using the indicated scheme, if we divide the series $1+0+0+0+\ldots$ (all terms zero except the first) by the series $1+1+0+0+0+\ldots$ (all terms zero, except the first two), we get the divergent series $1-1+1-1+\ldots$

In the given example, the terms of series (3) are formed by the law

$$w_1=1,\ w_2=\frac{1}{2^0},\ w_3=\frac{1}{2^1},$$

$$w_4=\frac{1}{2^2},\ \ldots,\ w_n=\frac{1}{2^{n-2}},\ \ldots\ (n\geqslant 2)$$

Indeed, the second remainder is obtained by termwise multiplication of the first remainder by $\frac{1}{2}$. Consequently, the third term of (3) is obtained from the second term by multiplication by $\frac{1}{2}$. In the third subtraction, all corresponding terms in both the diminuend and the subtrahend are one half those in the second subtraction. Hence the third remainder is obtained from the second by multiplication by $\frac{1}{2}$. The fourth term of the series (3) is obtained from the third by multiplication by $\frac{1}{2}$, etc.

Thus, the terms of the series (3), starting with the second, form a geometric progression with common ratio $\frac{1}{2}$. Hence, the series (3) converges. Its sum W is equal to $U:V$.

Indeed, we have

$$U=1,\quad V=\frac{1}{3},\quad W=1+\frac{1}{1-\frac{1}{2}}=3$$

so that

$$U:V=W$$

383. Functional Series

A *functional series* (i.e. a series of functions) is an expression

$$u_1(x)+u_2(x)+\ldots+u_n(x)\ldots \tag{1}$$

where $u_1(x)$, $u_2(x)$, ... (terms of the series) are the functions of one and the same argument x which are defined in some interval (a, b).

The meaning of expression (1) is explained in Sec. 367. The difference is that the terms of the series are now functions, whereas in Sec. 367 we considered a series whose terms were numbers. That was a *numerical* series, in contrast to the functional series which we shall now discuss. The *partial* sums of a functional series are determined in the same way as for a numerical series.

If in series (1) the argument x is assigned a value [in the interval (a, b)], then the functional series gives rise to a numerical series.

384. The Domain of Convergence of a Functional Series

It may happen that for any value of x taken in the interval (a, b), the functional series converges. It may also happen that the series diverges for any value of x. Typically, a functional series converges for certain values of x and diverges for other values. The totality of values of x for which the series converges is called the *domain of convergence* of the functional series.

In the domain of convergence, each value of x is associated with a definite sum of the series so that the sum is a function, of the argument x, defined in the domain of convergence. Outside this domain the functional series has no sum.

Example 1. Consider the functional series

$$1\cdot x+1\cdot 2x^2+1\cdot 2\cdot 3x^3+\ldots+1\cdot 2\ldots nx^n+\ldots \tag{1}$$

Its terms are the functions

$$u_1(x)=x,\quad u_2(x)=2x^2,\quad u_3(x)=6x^3,\ \ldots \tag{2}$$

defined in the interval $(-\infty, +\infty)$. However, only for $x=0$ does the series (1) converge; for any other value of x, the series diverges. Indeed, assign to x some value x_0 not equal to zero. This yields the numerical series

$$1\cdot x_0+1\cdot 2x_0^2+\ldots+1\cdot 2\ldots nx_0^n+\ldots \tag{3}$$

The ratio

$$|u_{n+1}:u_n|=|(n+1)!\,x_0^{n+1}:n!\,x_0^n|=(n+1)\,|x_0|$$

has an infinite limit as $n\to\infty$. Hence (Sec. 378), the series (3) diverges for $x\neq 0$. The domain of convergence consists of a single point $x=0$.

Example 2. The functional series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots \tag{4}$$

[its terms are functions defined in the interval $(-\infty, +\infty)$]

converges for any value $x=x_0$. Indeed, the ratio

$$|u_{n+1}:u_n|=\frac{|x_0|}{n+1}$$

tends to zero as $n\to\infty$ (Sec. 378). The domain of convergence is the entire interval $(-\infty, +\infty)$. The sum of the series (4) is a function defined in this interval (the function is equal to e^x; cf. Sec. 272, Example 1).

Example 3. Find the domain of convergence and the expression for the sum of the series

$$2+\frac{1}{2}x(1-x)+\frac{1}{2}x^2(1-x)+\ldots+\frac{1}{2}x^{n-1}(1-x)+\ldots \tag{5}$$

Solution. Write the partial sum of (5) in the form

$$s_n=2+\frac{1}{2}x-\frac{1}{2}x^2+\frac{1}{2}x^2-\ldots-\frac{1}{2}x^{n-1}+\frac{1}{2}x^{n-1}-\frac{1}{2}x^n=$$
$$=2+\frac{1}{2}x-\frac{1}{2}x^n \tag{6}$$

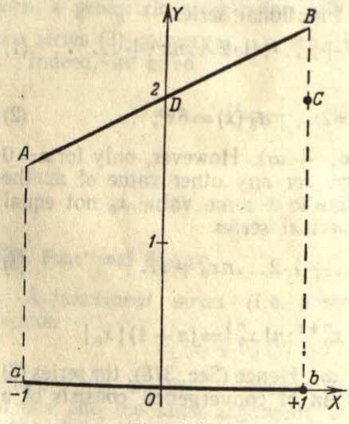

Fig. 401

If $|x|>1$, then s_n does not have a finite limit as $n\to\infty$ $\left(\text{the term} -\frac{1}{2}x^n \text{ is infinitely large}\right)$; that is, the series (5) diverges. For $x=-1$ the series also diverges, because then

$$s_n=2-\frac{1}{2}-\frac{1}{2}(-1)^n=$$
$$=\frac{3}{2}+\frac{(-1)^{n+1}}{2}$$

From this we see that s_n alternately takes on the values 2 and 1.

The series (5) converges for the other values of x (i. e. for $-1<x\leqslant 1$). Indeed, if $x=1$, then all terms of (5), except the first, vanish, and we have

$$S(1)=2 \tag{7}$$

But if $|x|<1$, then in formula (6) the term $-\frac{1}{2}x^n$ tends to zero as $n \to \infty$ for fixed x, as well so that

$$S(x) = \lim_{n \to \infty} \left(2 + \frac{1}{2}x - \frac{1}{2}x^n\right) = 2 + \frac{1}{2}x \tag{8}$$

The domain of convergence of the series (5) is the interval $(-1, +1)$, from which the extremity $x = -1$ is eliminated (in Fig. 401 the segment *ab* with the point *a* deleted). In this domain, the sum S of the series (5) is a function of x defined by the following equations:

$$\left.\begin{aligned} S(x) &= 2 + \frac{1}{2}x \quad && \text{for } -1 < x < 1, \\ S(x) &= 2 && \text{for } x = 1 \end{aligned}\right\} \tag{9}$$

The function $S(x)$ is discontinuous at $x = 1$ and is continuous at all other points of the domain of convergence. Outside the domain $-1 < x \leqslant 1$ the function $S(x)$ is not defined at all. Its graph (see Fig. 401) is the segment AB, the extremities of which are removed and to which is adjoined point C (in place of B).

385. On Uniform and Nonuniform Convergence[1]

Let the functional series

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \tag{1}$$

converge at every point of a (closed or open) interval (a, b)[2] and let it be required to find the approximate sum S of (1) to within ε (in other words, the remainder R_n should not exceed, in absolute value, the positive number ε). This requirement is satisfied for every definite value of x, starting with some index $n = N$. As a rule, the quantity N depends on x and it may happen that there is no number n that can ensure the required accuracy for all x at once. Then we say that the series (1) is nonuniformly convergent in the interval (a, b). If the required degree of accuracy can always be ensured at once for all x starting from one and the same number N, then we say that the series (1) is uniformly convergent in the interval (a, b).

Example 1. The functional series

$$2 + \frac{1}{2}x(1-x) + \frac{1}{2}x^2(1-x) + \ldots + \frac{1}{2}x^{n-1}(1-x) + \ldots \tag{2}$$

---

[1] For definition, see Sec. 386.

[2] The series can also converge at points exterior to the interval (a, b), but such points are disregarded.

(see Sec. 384, Example 3) converges at every point of the closed interval (0, 1). We shall show that it is nonuniformly convergent in this interval.

We require that the partial sum

$$s_n = 2 + \frac{1}{2}x - \frac{1}{2}x^n \tag{3}$$

yield the sum of the series (2) to within $\frac{1}{2} \cdot 0.1$. For $x=1$ and also for $x=0$ this requirement is satisfied by all partial sums (the exact value is $S=2$). For the other values of x the sum of the series is

$$S = 2 + \frac{1}{2}x \tag{4}$$

so that the remainder of the series is

$$R_n = S - s_n = \frac{1}{2}x^n \tag{5}$$

For $x=0.1$ or for $x=0.2$ or for $x=0.3$, the required accuracy is ensured beginning with $N=2$; for example, for $x=0.3$ we have

$$|R_2| = \frac{1}{2} \cdot 0.3^2 < \frac{1}{2} \cdot 0.1$$

But for $x=0.4$ two terms do not suffice; we have to take at least three. Then

$$|R_3| = \frac{1}{2} \cdot 0.4^3 \approx \frac{1}{2} \cdot 0.06 < \frac{1}{2} \cdot 0.1$$

Further trials will show that for $x=0.5$ the required accuracy is ensured only from $N=4$, for $x=0.6$, beginning with $N=5$, and for $x=0.8$ we have to take $N=11$. As x approaches 1, the number N increases without bound so that for all values of x at once there is no number N that is able to ensure an accuracy up to 0.1 (greater accuracy all the more so). Hence in the interval (0, 1) the series (2) is nonuniformly convergent.

Fig. 402 depicts the graphs of the partial sums

$$s_1(x) = 2, \quad s_2(x) = 2 + \frac{1}{2}x - \frac{1}{2}x^2$$

$$s_3(x) = 2 + \frac{1}{2}x - \frac{1}{2}x^3, \quad s_4(x) = 2 + \frac{1}{2}x - \frac{1}{2}x^4$$

The remainder is depicted for $x \neq 1$ as segments of ordinates between the corresponding graph and the straight line $y=2+$

$+\frac{1}{2}x$ [which represents the sum of the series (2) for all values of x except $x=1$].

The convergence of the series (2) is seen from the fact that the graphs of the partial sums hug more closely the straight line DB over an ever increasing portion of it. The nonuniformity of convergence is evident from the fact that near B each one of the graphs s_n departs from DB. But as n increases, perceptible departures occur on an ever smaller section.

Example 2. Let us show that the same series (2) converges uniformly in the interval (0, 0.5).

We will require an accuracy of up to $\frac{1}{2}\cdot 0.1$. For $x=0.5$ this accuracy is ensured beginning with $N=4$, because

$$|R_4|=\frac{1}{2}\cdot 0.5^4=$$
$$=\frac{1}{2}\cdot 0.0625<\frac{1}{2}\cdot 0.1$$

For any other value of x in the interval (0, 0.5) the required accuracy is definitely ensured from $N=4$ onwards.

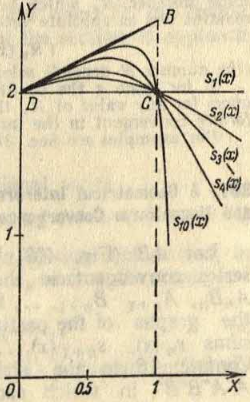

Fig. 402

Let us require an accuracy up to $\frac{1}{2}\cdot 0.01$. Then for $x=0.5$, it suffices to take $N=7$ because

$$|R_7|=\frac{1}{2}\cdot 0.5^7\approx\frac{1}{2}\cdot 0.0078<\frac{1}{2}\cdot 0.01$$

For any other value of x in the interval (0, 0.5) seven terms is all the more so sufficient. Generally, that number N which ensures an accuracy up to ε for $x=0.5$ will always ensure the same accuracy at once for all values of x in the interval (0, 0.5). Hence series (2) converges uniformly in this interval.

In Fig. 402 the uniformity of convergence is seen from the fact that in the interval (0, 0.5) the greatest departure of the graph s_n from the straight line DB tends to zero with increasing n. In the interval (0, 1) this does not occur.

386. Uniform and Nonuniform Convergence Defined[1]

A functional series

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \tag{1}$$

convergent in an (open or closed) interval (a, b) is called *uniformly convergent* in that interval if the remainder $R_n(x)$, beginning with some number N, which is the *same for all values of* x considered, remains less in absolute value than any preassigned positive number ε:

$$|R_n(x)| < \varepsilon \quad \text{for } n \geqslant N(\varepsilon) \tag{2}$$

(the number N depends solely on ε).

If for some ε the condition (2) cannot be satisfied (for all x at once) for any value of N, then we say that the series (1) is *nonuniformly* convergent in the interval (a, b).

For examples see Sec. 385.

387. A Geometrical Interpretation of Uniform and Nonuniform Convergence

Let AB (Fig. 403) be the graph of the sum $S(x)$ of a series convergent in the interval (a, b), and let the lines A_nB_n, $A_{n+1}\ B_{n+1}$, ... be the graphs of the partial sums $s_n(x)$, $s_{n+1}(x)$, ... Confine AB to the strip $A'A''B''B'$, in which each of the boundaries $A'B'$ and $A''B''$ is a constant distance ε from AB (vertically). In the case of uniform conver-

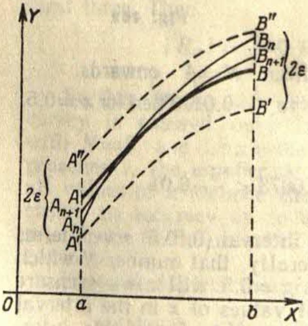

Fig. 403

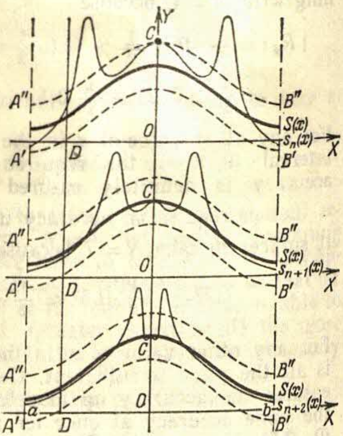

Fig. 404

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[1] It is advisable to read Sec. 385 first.

gence of the series, all the lines A_nB_n, beginning with some number $N(\varepsilon)$, are located entirely inside this strip (within the limits of the interval under consideration).

This does not occur in the case of nonuniform convergence. A vivid explanation is given in Fig. 404; here, all the graphs $s_n(x)$ have two "tongues" each that shoot up from the strip $A'A''B''B'$ (they move towards the point C as n increases). Yet, after the tongue has passed the point D, the graphs $s_n(x)$ above every separate point D of the section ab approach the graph $S(x)$ without bound.

388. A Test for Uniform Convergence. Regular Series

If every term $u_n(x)$ of the functional series

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \tag{1}$$

for any x taken in the interval (a, b) does not exceed, in absolute value, the positive number A_n and if the numerical series

$$A_1+A_2+\ldots+A_n+\ldots \tag{2}$$

converges, then the functional series (1) converges uniformly in this interval.

Explanation. The convergence of the series (1) follows from Secs 377 and 373. The remainder of the series (1) does not exceed, in absolute value, the remainder of (2), starting from some number N, which ensures accuracy up to ε for the series (2) and all the more so for all x at once.

Example. The functional series

$$-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots+(-1)^n\frac{\cos nx}{n^2}+\ldots \tag{3}$$

converges uniformly in the interval $(-\infty, +\infty)$ because its terms, for any x, do not exceed in absolute value the corresponding terms of the positive number series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots \tag{4}$$

This series converges (Sec. 373, Example 3). Accuracy up to 0.1 is ensured for the series (4) beginning with $n=10$. For (3), the same accuracy is all the more so ensured, starting with the tenth partial sum.

In the closed interval $(-\pi, \pi)$ the sum of the series (3) is

$$\frac{1}{4}\left(x^2-\frac{\pi^2}{3}\right)$$

For $x=\pm\pi$ the functional series (3) becomes a numerical series (4), the sum of which is equal to $\frac{\pi^2}{6}$ (see Sec. 417, Example 3).

Note. A functional series which meets the criterion of this section is called a *regular series.* Every regular series converges uniformly. Nonregular series converge uniformly in some cases and nonuniformly in others.

389. Continuity of the Sum of a Series

Theorem. If all terms of a functional series

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \tag{1}$$

uniformly convergent in the interval (a, b) are (in that interval) continuous functions, then the sum of (1) is also a continuous function in (a, b).

Example 1. All the terms of the series

$$-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots+(-1)^n\frac{\cos nx}{n^2}+\ldots \tag{2}$$

which is uniformly convergent in the interval $(-\infty, +\infty)$ (Sec. 388) are continuous functions. Hence, the sum of (2) is a function continuous for any value of x.

Note. The sum of a nonuniformly convergent series of continuous functions is continuous in some cases and discontinuous in others.

Example 2. All the terms of the series

$$2+\frac{1}{2}x(1-x)+\frac{1}{2}x^2(1-x)+\ldots+\frac{1}{2}x^{n-1}(1-x)+\ldots \tag{3}$$

which is nonuniformly convergent in the closed interval $(0, 1)$ (Sec. 385) are continuous functions. But the sum of the series is a function that is discontinuous for $x=1$ (see Sec. 384, Example 2).

Example 3. The series

$$(x-x^2)+[(x^2-x^4)-(x-x^2)]+[(x^3-x^6)-(x^2-x^4)]+\ldots \tag{4}$$

with general term

$$u_n(x)=(x^n-x^{2n})-(x^{n-1}-x^{2n-2}) \tag{5}$$

converges nonuniformly in the closed interval (0, 1), but it has a continuous sum $S(x)$ identically equal to zero.

Indeed, we have $s_n(x)=x^n-x^{2n}$ and for every separate value of x in the interval (0, 1) this expression tends to zero so that the series converges and has the sum $S(x)=0$.

But the remainder $R_n(x)=S(x)-s_n(x)=x^n-x^{2n}$ of the series cannot be made less than $\frac{1}{4}$ at once for all values of x considered because, no matter what n, the remainder is equal to $\frac{1}{4}$ for $x=\sqrt[n]{\frac{1}{2}}$.

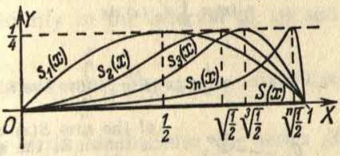

Fig. 405

Consequently (Sec. 385), the convergence of the series (4) is nonuniform. Yet its sum $S(x)=0$ is a continuous function.

Geometrically, the graphs of all partial sums s_n (Fig. 405) have humps on the straight line $y=\frac{1}{4}$ so that no graph lies entirely within the strip between the straight lines $y=\pm\frac{1}{4}$. This does not prevent the sum of the series (depicted by the heavy segment on the x-axis) from being a continuous function.

390. Integration of Series

Theorem. If a convergent series

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots=S(x) \tag{1}$$

made up of functions continuous in an interval (a, b) converges in that interval uniformly, then it may be integrated term by term. The series

$$\int_a^x u_1(x)\,dx+\int_a^x u_2(x)\,dx+\ldots+\int_a^x u_n(x)\,dx+\ldots \tag{2}$$

converges uniformly in the interval (a, b) and its sum is equal to the integral $\int_a^x S(x)\,dx$ of the sum of the series (1):

$$\int_a^x u_1(x)\,dx+\int_a^x u_2(x)\,dx+\ldots+\int_a^x u_n(x)\,dx+\ldots=\int_a^x S(x)\,dx \quad (3)$$

Explanation. The partial sum $s'_n(x)$ of the series (2) is the integral of the partial sum $s_n(x)$ of the series (1)

$$s'_n(x)=\int_a^x s_n(x)\,dx$$

and is depicted as the area aA_nC_nx (Fig. 406). The integral $\int_a^x S(x)\,dx$ of the sum $S(x)$ of the series (1) is shown as the area $aACx$.

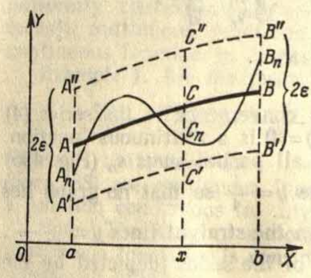

Fig. 406

The theorem asserts firstly that the series (2) converges and that its sum is equal to $\int_a^x S(x)\,dx$.

Geometrically, the area $aACx$ (Fig. 406) is the limit of the area aA_nC_nx as $n\to\infty$.

Indeed, in the case of uniform convergence of (1), the graph aA_nC_nx lies inside the strip $A'A''C''C'$ (Sec. 387). Thus, the area aA_nC_nx lies between the areas $aA'C'x$ and $aA''C''x$. And for both the limit is the area $aACx$.

The theorem secondly asserts that the series (2) converges uniformly. *Geometrically*, it is at once possible, for all positions of the ordinate xC'', to make the quantity

$$|\text{area } aACx-\text{area } aA_nC_nx| \quad (4)$$

starting with some index n, less than any preassigned given area E. Indeed, the strip $A'A''B''A'$ may be made so narrow that its area is less than E. Then the area of $A'A''C''C'$ is definitely less than E and the quantity (4) is still less.

Example 1. The series

$$1+2x+3x^2+\ldots+nx^{n-1}+\ldots \quad (5)$$

in the interval $(0, q)$, where q is a proper fraction, converges uniformly (by the test of Sec. 388) because its terms do not exceed the corresponding terms of the convergent (Sec. 374) positive-term series

$$1+2q+3q^2+\ldots+nq^{n-1}+\ldots \tag{6}$$

Here [1]

$$S(x)=1+2x+\ldots+nx^{n-1}+\ldots=\frac{1}{(1-x)^2} \tag{7}$$

According to the theorem of this section, the series

$$\int_0^x dx+\int_0^x 2x\,dx+\ldots+\int_0^x nx^{n-1}\,dx+\ldots \tag{8}$$

converges uniformly in the interval $(0, q)$ and its sum is

$$\int_0^x S(x)\,dx=\int_0^x \frac{dx}{(1-x)^2}=\frac{1}{1-x}-1 \quad (0\leqslant x\leqslant q) \tag{9}$$

This is readily verifiable because the series (8) is a progression:

$$x+x^2+x^3+\ldots$$

Note. If the series (1) converges nonuniformly, then term-by-term integration in certain cases is permissible but not in others (see Examples 2 and 3).

Example 2. The series

$$(x-x^2)+[(x^2-x^4)-(x-x^2)]+[(x^3-x^6)-(x^2-x^4)]+\ldots=0 \tag{10}$$

nonuniformly convergent in the interval $(0, 1)$ (see Sec. 389, Example 3) may be integrated term-by-term between the limits 0 and 1:

$$\int_0^1 (x-x)^2\,dx+\int_0^1 [(x^2-x^4)-(x-x^2)]\,dx+\ldots=\int_0^1 0\cdot dx=0 \tag{11}$$

Indeed, the partial sum s_n' of the series (11) is

$$s_n'=\int_0^1 (x^n-x^{2n})\,dx=\frac{n}{(n+1)(2n+1)} \tag{12}$$

It tends to zero as $n\to\infty$.

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[1] Formula (7) may be obtained by termwise multiplication of the series $1+x+x^2+\ldots=\frac{1}{1-x}\left(0\leqslant x\leqslant\frac{1}{2}\right)$ by itself (cf. Sec 381, Example 2).

Geometrically, the area bounded by the graph $s_n(x)$ (Fig. 405) and the interval (0, 1) tends to zero despite the presence of a hump. (The hump tapers indefinitely as n increases, but its altitude remains constant.)

Example 3. The series

$$(x-x^2)+[2(x^2-x^4)-(x-x^2)]+[3(x^3-x^6)-2(x^2-x^4)]+\ldots \tag{13}$$

with general term

$$u_n(x)=n(x^n-x^{2n})-(n-1)(x^{n-1}-x^{2n-2}) \tag{14}$$

converges in the interval (0, 1) and has a continuous sum $S(x)=0$ [this is proved in the same way as for the series (10)]. Hence

$$\int_0^1 S(x)\,dx=0 \tag{15}$$

Yet termwise integration from 0 to 1 yields $\frac{1}{2}$, not zero. Indeed, we get the series

$$\int_0^1 (x-x^2)\,dx+\left[2\int_0^1 (x^2-x^4)\,dx-\int_0^1 (x-x^2)\,dx\right]+\ldots$$
$$+\left[n\int_0^1 (x^n-x^{2n})\,dx-(n-1)\int_0^1 (x^{n-1}-x^{2n-2})\,dx\right]+\ldots \tag{16}$$

with partial sum

$$s'_n=n\int_0^1 (x^n-x^{2n})\,dx=$$
$$=\frac{n^2}{(n+1)(2n+1)} \tag{17}$$

Consequently, the sum S' is

$$S'=\lim_{n\to\infty} s'_n=\frac{1}{2} \tag{18}$$

The discrepancy between (15) and (18) is due to nonuniform convergence of the series (13) (nonuniformity is proved as in Example 3, Sec. 389).

Geometrically, the graph of s_n (Fig. 407) tends to coincidence with the axis of abscissas over any section of the interval (0, 1) which does not contain the point $x=1$. But near this point a hump is formed. It approaches the extremity $x=1$ without bound, becoming narrower horizontally but continuing to grow upwards.[1] Because of

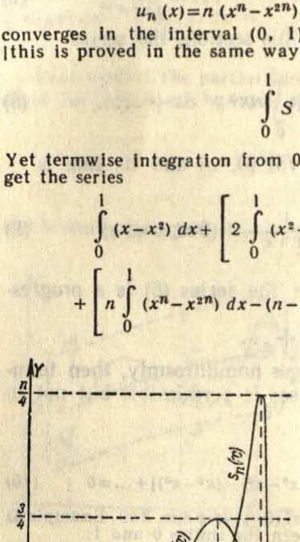

Fig. 407

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[1] The graphs in Fig. 407 are obtained from the similar graphs of Fig. 405 by stretching along the vertical in the ratio $n:1$.

compensation, the area between s_n and the interval (0, 1) tends to $\frac{1}{2}$ and not to zero.

Note. If the series (13) is modified by taking a series with general term

$$u_n(x)=n^2(x^n-x^{2n})-(n-1)^2(x^{n-1}-x^{2n-2}) \tag{14a}$$

we will still have

$$\int_0^1 S(x)\,dx=0 \tag{16a}$$

but after termwise integration we get a series with the partial sum

$$s_n'=\frac{n^3}{(n+1)(2n+1)} \tag{18a}$$

It will be divergent since $\lim\limits_{n\to\infty} s_n'=\infty$ (the hump will grow upwards faster than it will taper horizontally).

391. Differentiation of Series

Even in the case of uniform convergence of a series, it is not always permissible to differentiate it termwise. The following theorem offers a criterion for ensuring the possibility of term-by-term differentiation.

Theorem. If a functional series

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \tag{1}$$

converges in the interval (a, b) and the derivatives of its terms are continuous in this interval, then the series (1) may be differentiated term-by-term *provided that the resulting series*

$$u_1'(x)+u_2'(x)+\ldots+u_n'(x)+\ldots \tag{2}$$

converges uniformly in the given interval. The sum of series (2) will be the derivative of the sum of series (1).

The proof is based on the reciprocal nature of differentiation and integration and rests on the theorem of Sec. 390.

Example. The series

$$x+x^2+\ldots+x^n+\ldots \tag{3}$$

converges in the interval $(0, q)$, where q is a proper fraction. Here

$$x+x^2+\ldots+x^n+\ldots=\frac{x}{1-x}\ (0\leqslant x\leqslant q) \tag{4}$$

The derivatives of the terms are continuous in the interval $(0, q)$, and the series

$$1+2x+\ldots+nx^{n-1}+\ldots \tag{5}$$

made up of them converges uniformly in that interval (Sec. 390, Example 1). Hence the sum of the series (5) is a derivative of the sum $\frac{x}{1-x}$ of series (3):

$$1+2x+\ldots+nx^{n-1}+\ldots=\frac{d}{dx}\left(\frac{x}{1-x}\right)=\frac{1}{(1-x)^2} \tag{6}$$

Note 1. The theorem does not actually require that the series (1) converge uniformly. Under the hypotheses of the theorem this requirement is fulfilled of itself (by virtue of the theorem of Sec. 390).

Note 2. Even in the case of uniform convergence of (1) and continuity of the derivatives of $u_n(x)$ the series (2) may prove to be nonuniformly convergent and then its sum is sometimes equal and sometimes not equal to the derivative of the sum of the series (1). What is more, series (2) may prove to be divergent. Thus, the series

$$\sin x+\frac{\sin 2^4x}{2^2}+\ldots+\frac{\sin n^4x}{n^2}+\ldots \tag{7}$$

converges uniformly on the entire real line (cf. Example in Sec. 388), whereas the series of derivatives

$$\cos x+2^2\cos 2^4x+\ldots+n^2\cos n^4x+\ldots \tag{8}$$

diverges for $x=0$ (and also for an infinity of values of x).

392. Power Series

In practical applications the most important of the functional series are power series (see Sec. 270 for a discussion of their origin). A *power series* is a series of the form

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \tag{1}$$

and also a series of the more general form

$$a_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots+a_n(x-x_0)^n+\ldots \tag{2}$$

where x_0 is a constant. Of the series (1) we say that it is a *series in powers of* x, about (2), that it is a *series in powers of* $x-x_0$.

The constants $a_0, a_1, \ldots, a_n, \ldots$ are called the *coefficients* of the power series.

If we denote $x-x_0$ by z, then (2) is a series in powers of z, which is the same as (1). Therefore, from now on a power series will be understood to be a series of type (1), unless otherwise stated. A power series always converges for $x=0$. As to its convergence at other points, three cases are possible; they are considered in Sec. 393.

393. The Interval and Radius of Convergence of a Power Series

1. It may happen that a power series diverges at all points except $x=0$. Such, for example, is the series

$$1^1x+2^2x^2+3^3x^3+\ldots+n^nx^n+\ldots$$

where the general term $n^nx^n=(nx)^n$ increases in absolute value without bound beyond the point where nx becomes greater than unity. Such power series are of no practical significance.

2. A power series can converge at all points. Such, for instance, is the series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^{n-1}}{(n-1)!}+\ldots$$

the sum of which for all values of x is e^x (Sec. 272, Example 1).

3. Typically, a power series converges at some points and diverges at others.

Example 1. The geometric progression

$$1+x+x^2+\ldots+x^n+\ldots \qquad (1)$$

converges for $|x|<1$ and diverges for $|x|\geqslant 1$. Here the domain of convergence (Sec. 384) is the interval $(-1, +1)$, both end-points of which, $x=+1$ and $x=-1$, are excluded. The sum of the series (1) (in the domain of convergence) is

$$\frac{1}{1-x}$$

Example 2. The power series

$$1+\frac{x}{1^2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{n^2}+\ldots \qquad (2)$$

converges for $|x|\leqslant 1$ and diverges for $|x|>1$ (cf. Sec. 374, Example 2). The domain of convergence is the interval $(-1, +1)$ with both extremities, $x=+1$ and $x=-1$, included. The sum of the series (2) is not expressible in terms of elementary functions.

Example 3. The power series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n+1}\frac{x^n}{n}+\ldots \tag{3}$$

converges for $|x|<1$ and diverges for $|x|>1$. For $x=-1$ it also diverges (Sec. 369, Example 3), for $x=1$ it converges (Sec. 369, Example 4). The domain of convergence is the interval $(-1, +1)$, including the point $x=1$; the point $x=-1$ is excluded.

The sum of the series (3) (in the domain of convergence) is $\ln(1+x)$ (Sec. 272, Example 2). The series (3) is obtained by termwise integration of the series

$$1-x+x^2-x^3+\ldots=\frac{1}{1+x}$$

Theorem. The domain of convergence of the power series

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \tag{4}$$

is some interval $(-R, R)$, symmetric about the point $x=0$. Sometimes both extremities, $x=R$ and $x=-R$, have to be included, sometimes only one, and at yet other times both extremities have to be excluded.

The interval $(-R, R)$ is called the *interval of convergence*, and the positive number R is called the *radius of convergence* of the power series. If a power series converges only at the point $x=0$, then $R=0$. In Examples 1 to 3 the radius of convergence is unity. If the series converges at all points, then we say that the radius of convergence is infinite ($R=\infty$).

(394. Finding the Radius of Convergence

Theorem. The radius R of convergence of the power series

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \tag{1}$$

is equal to the limit of the ratio $|a_n|:|a_{n+1}|$ provided that this limit (finite or infinite) exists:

$$R=\lim_{n\to\infty}|a_n:a_{n+1}| \tag{2}$$

Example 1. Find the radius and domain of convergence of the series

$$\frac{0.1x}{1}-\frac{0.01x^2}{2}+\frac{0.001x^3}{3}-\ldots+\frac{(-0.1)^nx^n}{n}+\ldots \tag{3}$$

Solution. Here $a_n = \frac{(-0.1)^n}{n}$. We have

$$|a_n|:|a_{n+1}| = \frac{0.1^n}{n} : \frac{0.1^{n+1}}{n+1} = 10\,\frac{n+1}{n},$$

$$R = \lim_{n \to \infty} |a_n : a_{n+1}| = 10 \tag{4}$$

The radius of convergence is 10, the interval of convergence is (— 10, 10). The series (3) converges inside the interval and diverges outside it. For $x=10$, the series (3) takes the form

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{(-1)^n}{n} + \ldots \tag{5}$$

This series converges (Sec. 369, Example 4). For $x = -10$ we get a divergent series (Sec. 369, Example 3):

$$-\frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots$$

Hence the domain of convergence is the interval $(-10, +10)$ with the extremity $x = +10$ included; the other extremity is excluded.

Explanation. We will regard x as a given number and will apply to (3) the d'Alembert test (Sec. 378). We have

$$u_n = \frac{(-0.1)^n x^n}{n}.$$

$$\lim_{n \to \infty} |u_{n+1} : u_n| = \lim_{n \to \infty} \left(|x| \cdot 0.1\,\frac{n}{n+1} \right) = |x| \cdot 0.1$$

By d'Alembert's test, (3) converges when $|x| \cdot 0.1 < 1$, i.e. when $|x| < 10$, and diverges when $|x| \cdot 0.1 > 1$; that is, when $|x| > 10$.

Repeating this reasoning literally with regard to (1), we get formula (2).

Note 1. The sum of the series (3) (in the domain of convergence) is equal to $\ln(1+0.1x)$ (cf. Sec. 393, Example 3).

Example 2. Find the radius of convergence of the series

$$1 - \frac{x}{1} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^n}{n!} + \ldots \tag{6}$$

Solution. Here $a_n = \frac{(-1)^n}{n!}$. By formula (2) we get

$$R = \lim_{n \to \infty} \left[\frac{1}{n!} : \frac{1}{(n+1)!} \right] = \lim_{n \to \infty} (n+1) = \infty \tag{7}$$

The series (6) converges at all points. Its sum is e^{-x} (cf. Sec. 272, Example 1).

Note 2. If series (1) contains an infinity of coefficients equal to zero, then the ratio $|a_n|:|a_{n+1}|$ has no limit and *formula* (2) *cannot be employed* even if we discard the zero coefficients and renumber the remaining ones in a sequence.

Example 3. Find the radius of convergence of the series

$$\frac{0.1z^2}{1}-\frac{0.01z^4}{2}+\frac{0.001z^6}{3}-\dots \tag{8}$$

obtained from (3) by the substitution $x=z^2$.

Solution. Since the series (3) converges for $|x|<10$ and diverges for $|x|>10$, it follows that series (8) converges for $|z|<\sqrt{10}$ and diverges for $|z|>\sqrt{10}$. Hence, the radius of convergence of (8) is $\sqrt{10}$. Formula (2) is inapplicable: if we take into account the zero coefficients of odd powers of z, then the ratio $|a_n|:|a_{n+1}|$ is meaningless for even n; but if we discard the zero coefficients and number the remaining ones in a sequence, then the limit of the ratio $|a_n|:|a_{n+1}|$ will be 10 and will not yield a radius of convergence.

The sum of the series (8) (in the domain of convergence) is $\ln(1+z^2)$.

395. The Domain of Convergence of a Series Arranged in Powers of $x-x_0$

The domain of convergence of the power series

$$a_0+a_1(x-x_0)+a_2(x-x_0)^2+\dots+a_n(x-x_0)^n+\dots \tag{1}$$

is some interval $(x_0-R,\ x_0+R)$ symmetric about the point x_0. Sometimes, both extremities must be included, sometimes only one, and sometimes both extremities must be excluded.

The interval $(x_0-R,\ x_0+R)$ is called the *interval of convergence*, the positive number R is the *radius of convergence* of the series (1). If the series converges at all points, then the radius of convergence is infinite ($R=\infty$).

If the ratio $|a_n|:|a_{n+1}|$ has a limit (finite or infinite), then the radius of convergence is found from the formula

$$R=\lim_{n\to\infty}|a_n:a_{n+1}| \tag{2}$$

Example. Find the radius and domain of convergence of the series

$$\frac{x+0.2}{1}+\frac{(x+0.2)^2}{2}+\dots+\frac{(x+0.2)^n}{n}+\dots \tag{3}$$

Here $x_0 = -0.2$, $a_n = \frac{1}{n}$. By formula (2) we have

$$R = \lim_{n\to\infty}\left|\frac{1}{n} : \frac{1}{n+1}\right| = 1$$

The domain of convergence is the interval $(-1.2, 0.8)$, one extremity of which, $x = 0.8$, is excluded. The sum of the series (3) (in the domain of convergence) is

$$-\ln[1-(x+0.2)] = \ln\frac{1}{0.8-x}.$$

396. Abel's Theorem[1]

Theorem. If a power series

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \tag{1}$$

converges (absolutely or conditionally) at some point x_0, then it converges *absolutely and uniformly* in any closed interval (a, b) *interior to* the interval $(-|x_0|, +|x_0|)$.

Note 1. The word "interior to" is to be understood in the narrow meaning of the word, that is by the hypothesis of the theorem neither of the end-points of the interval (a, b) coincides with the point $|x_0|$ or the point $-|x_0|$.

Example. The series

$$\frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^n}{n} + \ldots \tag{2}$$

converges (conditionally) at the point $x = -1$, turning into the series (Sec. 369, Example 4)

$$-\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$$

By Abel's theorem, series (2) converges absolutely and uniformly in any closed interval interior to $(-1, 1)$, say in the closed interval $(-0.99, 0.99)$.

If for the left extremity of the interval (a, b) we take the point $x_0 = -1$, then the absolute convergence is upset (at the point -1 itself). If for the right extremity of (a, b) we take the point $x_0 = 1$, then series (2) becomes divergent.

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[1] Niels Abel (1802-1829), Norwegian mathematician. He lived only 27 years but produced works of fundamental importance. The assertion of uniform convergence of series (1) is a later contribution (the distinction between uniform and nonuniform convergence was made by Weierstrass late in the 1840's).

Note 2. If for one of the extremities of the closed interval (a, b) we take x_0, then the convergence in such an interval remains uniform. The same goes for the point $-x_0$ *if series* (1) *converges at that point.*

397. Operations on Power Series

Suppose we have two power series:

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots=S_1(x), \tag{1}$$

$$b_0+b_1x+b_2x^2+\ldots+b_nx^n+\ldots=S_2(x) \tag{2}$$

Let A be the radius of convergence of series (1) and B the radius of convergence of series (2). Denote by r the smaller one (if they are equal, then r is common to both).

If we add, subtract or multiply termwise (by the scheme of multiplying a polynomial by a polynomial, cf. Sec. 381) the series (1) and (2), we get new power series, whose radii of convergence are at worst equal to r, but may exceed r. Their sums are, respectively, equal to $S_1(x)+S_2(x)$, $S_1(x)-S_2(x)$, $S_1(x)\,S_2(x)$ (cf. Secs. 371, 381, 396).

Termwise division of series (1) by series (2) may be carried out by the scheme given in Sec. 382, provided that $b_0 \neq 0$. If $r \neq 0$, then the radius of convergence r_1 of the derived series differs from zero but does not exceed A; it may even happen that r_1 is less than either of the radii A, B [see Example 4 and the note on formula (4), Sec. 401]. The sum of the new series (in the interval of its convergence) is equal to $S_1(x):S_2(x)$.

If $b_0=0$, then termwise division is impossible for $a\neq 0$ (because the quotient of $S_1(x):S_2(x)$ is infinitely great as $x\to 0$ and it cannot be represented by a series in powers of x). But if $b_0=0$ and $a_0=0$, then termwise division is impossible when the lowest power of the dividend is less than the lowest power of the divisor (for the same reason), otherwise division is possible, and the new series has the sum $S_1(x):S_2(x)$ 1) in the interval $(-r_1, r_1)$.

Example 1. In the interval $(-1, +1)$ we have

$$1+x+x^2+x^3+\ldots=\frac{1}{1-x}, \tag{3}$$

$$1-x+x^2-x^3+\ldots=\frac{1}{1+x} \tag{4}$$

Adding term by term, we get

$$2+2x^2+2x^4+\ldots=\frac{2}{1-x^2} \tag{5}$$

---

1) For $x=0$ this sum (that is, the absolute term of the new series) is the limit of the quotient $S_1(x):S_2(x)$ as $x\to 0$.

Subtracting (4) from (3) term by term, we get

$$2x+2x^3+2x^5+\ldots=\frac{2x}{1-x^2} \tag{6}$$

Multiplying term by term (cf. Sec. 381, Example 1), we get

$$1+x^2+x^4+\ldots=\frac{1}{1-x^2} \tag{7}$$

Dividing the series (3) by the series (4) termwise (cf. Sec. 382, Example), we find

$$1+2x+2x^2+2x^3+\ldots=\frac{1+x}{1-x} \tag{8}$$

By the theorem of Sec. 394, the series (5) to (8) have radius of convergence $R=1$, like (3) and (4). Formulas (5)-(8) are readily verifiable: their left sides are geometric progressions [in (8) from the second term onwards].

Example 2. In the interval $(-\infty, +\infty)$ we have (Sec. 272, Example 1)

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots=e^x \tag{9}$$

Replacing x by $-x$, we get

$$1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^n}{n!}+\ldots=e^{-x} \tag{10}$$

Since $e^x \cdot e^{-x}=1$, it follows that for termwise multiplication, all the terms, except the absolute term, must cancel out, which is what actually happens.

Example 3. Termwise division of (9) by (10) yields the series

$$1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4+\ldots \tag{11}$$

The law of formation of coefficients is not immediately perceivable, but, knowing that (11) converges in some interval and has the sum $e^x:e^{-x}=e^{2x}$ there, it is possible to visualize the series (11) in the form

$$1+\frac{2}{1!}x+\frac{2^2}{2!}x^2+\frac{2^3}{3!}x^3+\frac{2^4}{4!}x^4+\ldots+\frac{2^n}{n!}x^n+\ldots \tag{12}$$

Like (9) and (10), the series (12) has (by the theorem of Sec. 394) an infinite radius of convergence.

Example 4. We will consider the binomials $1+x$ and $1-x$ as power series, the coefficients of all terms of which, except the first two, are equal to zero. The radii of convergence A and B of these series are infinite. Termwise division of $1+x$ by $1-x$ yields the power series

$$1+2x+2x^2+2x^3+2x^4+\dots \tag{13}$$

Its terms, starting with the second, form a geometric progression with common ratio x. The sum of the series (13) in the interval of its convergence is $(1+x):(1-x)$, but the radius of convergence r_1 is not infinite; it is equal to unity.

398. Differentiation and Integration of a Power Series

Theorem 1. If a power series has radius of convergence R and sum $S(x)$,

$$a_0+a_1x+a_2x^2+\dots+a_nx^n+\dots=S(x) \tag{1}$$

then the series derived by termwise differentiation has the same radius of convergence R and its sum is the derivative function of $S(x)$:

$$a_1+2a_2x+3a_3x^2+\dots+na_nx^{n-1}+\dots=S'(x) \tag{2}$$

Hence, the sum of a power series is a differentiable function, and it has derivatives of all orders [because we can again apply Theorem 1 to series (2), etc.].

Note 1. If series (1) diverges at any end-point of the interval $(-R, R)$, then the series (2) also diverges at that end-point. The convergence of (1) at the end-point of the interval $(-R, R)$ may be preserved in (2) or may not.

Note 2. The convergence of (2) is somewhat worse than that of (1) (because na_n is greater in absolute value than a_n).

Example 1. Differentiating successively the series

$$1+x+x^2+\dots+x^n+\dots=\frac{1}{1-x}\ (-1<x<+1) \tag{3}$$

in which $R=1$, we get series with the same radius of convergence. Their sums are the successive derivatives of $\frac{1}{1-x}$:

$$1+2x+3x^2+4x^3+\dots+nx^{n-1}+\dots=\frac{1}{(1-x)^2}, \tag{4}$$

$$2+6x+12x^2+\dots+n(n-1)x^{n-2}+\dots=\frac{1\cdot 2}{(1-x)^3}, \tag{5}$$

$$6+24x+\dots+n(n-1)(n-2)x^{n-3}+\dots=\frac{1\cdot 2\cdot 3}{(1-x)^4} \tag{6}$$

The series (3) diverges at both end-points of the interval of convergence; the series (4) to (6) do likewise.

Example 2. The series (3) is obtained by differentiating the series

$$x+\frac{x^2}{2}+\ldots+\frac{x^{n+1}}{n+1}+\ldots=-\ln(1-x) \qquad (7)$$

Series (7) diverges for $x=1$ and converges for $x=-1$, but after differentiation the convergence at the end-point $x=-1$ breaks down.

Theorem 2. The series derived by termwise integration of series (1) from zero to x has the same radius of convergence and its sum is $\int\limits_0^x S(x)\,dx$:

$$a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3+\ldots+\frac{a_n}{n+1}x^{n+1}+\ldots=\int\limits_0^x S(x)\,dx \qquad (8)$$

Note 3. If series (1) converges at one of the end-points of the interval $(-R, R)$, then (8) converges there too, and formula (8) holds true. However, divergence of (1) at an end-point of $(-R, R)$ may or may not be preserved in (8). The convergence of series (8) is somewhat better than that of (1).

Example 3. The radius of convergence of the geometric progression

$$1-x^2+x^4-x^6+\ldots+(-1)^{n-1}x^{2n}+\ldots=\frac{1}{1+x^2} \qquad (9)$$

is unity. Integrating term by term we get (for $|x|<1$)

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+(-1)^{n-1}\frac{x^{2n+1}}{2n+1}+\ldots=$$

$$=\int\limits_0^x\frac{dx}{1+x^2}=\arctan x \qquad (10)$$

The radius of convergence of series (10) is also unity. At the end-point $x=1$, series (9) diverges and series (10) converges (by the Leibniz test), and we have[1]

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+(-1)^{n-1}\frac{1}{2n+1}+\ldots=\arctan 1=\frac{\pi}{4}$$

---

[1] This result was obtained by Leibniz.

At the end-point $x=-1$, the series (10), like (9), diverges (by the integral test).

Example 4. Integrating termwise the series

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots=\sin x \tag{11}$$

(Sec. 272, Example 2), for which $R=\infty$, we get

$$\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\ldots=\int\limits_0^x \sin x\,dx=1-\cos x$$

where x is any number. Whence we find the expansion of the function $\cos x$:

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \tag{12}$$

Here too, $R=\infty$.

399. Taylor's Series 1)

Definition. The *Taylor series* (in powers of $x-x_0$) of the function $f(x)$ is the power series

$$f(x_0)+\frac{f'(x_0)}{1}(x-x_0)+$$
$$+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots \tag{1}$$

For $x_0=0$ the Taylor series (in powers of x) is of the form

$$f(0)+\frac{f'(0)}{1}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots \tag{2}$$

Example 1. Form the Taylor series for the function $f(x)=\frac{1}{5-x}$ in powers of $x-2$.

Solution. Compute the values of the function $f(x)$ and its successive derivatives for $x=2$. This yields

$$\left.\begin{aligned}&f(2)=\frac{1}{3},\quad f'(2)=\frac{1}{(5-x)^2}\bigg|_{x=2}=\frac{1}{3^2},\\&f''(2)=\frac{1\cdot 2}{(5-x)^3}\bigg|_{x=2}=\frac{1\cdot 2}{3^3},\ \ldots,\ f^{(n)}(2)=\\&=\frac{n!}{(5-x)^{n+1}}\bigg|_{x=2}=\frac{n!}{3^{n+1}},\ \ldots\end{aligned}\right\} \tag{3}$$

---

1) It is highly advisable to read Sec. 270.

The required series is

$$\frac{1}{3}+\frac{1}{3^2}(x-2)+\frac{1}{3^3}(x-2)^2+\ldots+\frac{1}{3^{n+1}}(x-2)^n+\ldots \quad (4)$$

Example 2. Construct the Taylor series of the same function in powers of x.

Solution. As in Example 1, we find

$$f(0)=\frac{1}{5},\ f'(0)=\frac{1}{5^2},\ f''(0)=\frac{2!}{5^3},\ \ldots,\ f^{(n)}(0)=\frac{n!}{5^{n+1}},\ \ldots \quad (5)$$

The required series is of the form

$$\frac{1}{5}+\frac{1}{5^2}x+\frac{1}{5^3}x^2+\ldots+\frac{1}{5^{n+1}}x^n+\ldots \quad (6)$$

Example 3. The function $\frac{1}{x-5}$ does not have a Taylor series in powers of $x-5$ because the function is not defined at the point $x=5$ (it becomes infinite).

Example 4. The function $f(x)=\sqrt[3]{x}$ has no Taylor series in powers of x because the derivative $f'(0)$ is infinite. But it has a Taylor series in powers of $x-1$, which is of the form

$$1+\frac{1}{3}(x-1)-\frac{1}{2!}\frac{2}{3^2}(x-1)^2+\frac{1}{3!}\frac{2\cdot 5}{3^3}(x-1)^3+$$
$$+\frac{1}{4!}\frac{2\cdot 5\cdot 8}{3^4}(x-1)^4+\ldots$$

400. Expansion of a Function in a Power Series

To expand a function $f(x)$ in a series in powers of $x-x_0$ means to construct a series of the form

$$a_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots+a_n(x-x_0)^n+\ldots \quad (1)$$

with nonzero radius of convergence and with sum identically equal to the given function everywhere within the interval of convergence.

Theorem. If a function $f(x)$ is expanded in the power series (1), the expansion is unique and the series (1) coincides with the Taylor series in powers of $x-x_0$.

Explanation. By hypothesis, we have identically, in the interval of convergence,

$$f(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots+a_n(x-x_0)^n+\ldots \quad (2)$$

Hence (Sec. 398, Theorem 1), the function $f(x)$ has derivatives of all orders, and at all points of the interval of convergence we have

$$\left.\begin{aligned} f'(x)&=a_1+2a_2(x-x_0)+3a_3(x-x_0)^2+4a_4(x-x_0)^3+\ldots,\\ f''(x)&=\qquad 2a_2+2\cdot 3a_3(x-x_0)+3\cdot 4a_4(x-x_0)^2+\ldots,\\ f'''(x)&=\qquad\qquad 2\cdot 3a_3+2\cdot 3\cdot 4a_4(x-x_0)+\ldots \end{aligned}\right\} \tag{3}$$

and so on. For $x=x_0$, formulas (2) and (3) yield

$$a_0=f(x_0),\quad a_1=f'(x_0),\quad a_2=\frac{f''(x_0)}{2!},\quad a_3=\frac{f'''(x_0)}{3!},\ \ldots \tag{4}$$

That is, the expansion (2) is unique and coincides with Taylor's series for the function $f(x)$.

Example 1. Find the value of the fifth derivative of the function $f(x)=\frac{x}{1-x^2}$ for $x=0$.

Direct computation is tiresome. But the function $f(x)$ can easily be expanded in a series of powers of x by performing the division $x:(1-x^2)$ (Sec. 397). We get the expansion

$$\frac{x}{1-x^2}=x+x^3+x^5+x^7+\ldots \tag{5}$$

in the interval $(-1, +1)$. But (5) is Taylor's series of the function $f(x)$ in powers of x. Hence, the coefficient $a_5=1$ gives the value $\frac{f^{V}(0)}{5!}$, or $f^{V}(0)=5!=120$. Similarly, we find

$$f^{(2n+1)}(0)=(2n+1)!,\quad f^{(2n)}(0)=0 \tag{6}$$

Definition. A function $f(x)$ that can be expanded in a power series in $x-x_0$ is called *analytic at the point* x_0.

Example 2. The function $\sqrt[3]{x}$ is not analytic at the point $x=0$ (Sec. 399, Example 4); the same function is analytic at the point $x=1$ (and at any point $x_0\neq 0$).

Note. The function $f(x)$ defined at the point $x=x_0$ may be non-analytic at that point for one of three reasons:

(1) For $x=x_0$ it may not have a finite derivative of any order; thus, the function $\sqrt[3]{x}$ is not analytic at $x=0$ because the first derivative here is infinite.

(2) The Taylor series of the function $f(x)$ may have a nonzero radius of convergence and a sum not equal to $f(x)$.

(3) The radius of convergence of the Taylor series of the function $f(x)$ may be equal to zero.

Only the first type is of practical importance. In Example 3 we considered a function of the second type.

The presently known examples of functions of the third type are too involved.

Example 3. Let us define the function $\varphi(x)$ (Fig. 408) by the formula $\varphi(x)=e^{-\frac{1}{x^2}}$ (for $x \neq 0$). For $x=0$, we put $\varphi(0)=0$. All the derivatives of this function are zero at the point $x=0$.[1] Hence, for the function $f(x)=e^x+\varphi(x)$ all the derivatives $f'(0)$, $f''(0)$, ..., $f^{(n)}(0)$...

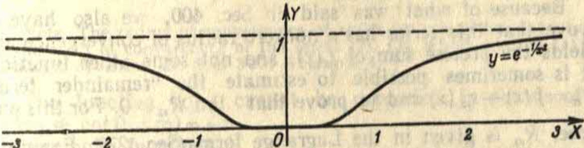

Fig. 408

will have the same values as the corresponding derivatives of e^x; that is, the Taylor series of the function $f(x)$ will be

$$1+\frac{1}{1!}x+\frac{1}{2!}x^2+\ldots+\frac{1}{n!}x^n+\ldots$$

This series has a nonzero radius of convergence ($R=\infty$), but its sum (it is e^x) is not equal to $f(x)$.

401. Power-Series Expansions of Elementary Functions

Preliminary remarks. In order to expand a function $f(x)$ in a series of powers of $x-x_0$ we can seek the successive derivatives $f'(x_0)$, $f''(x_0)$, ..., $f^{(n)}(x_0)$, ... If they exist

---

[1] For $x \neq 0$ we have

$$\varphi'(x)=\frac{2}{x^3}e^{-\frac{1}{x^2}}.$$

For $x=0$ this expression is not suitable; here

$$\varphi'(0)=\lim_{h\to 0}\frac{\varphi(h)-\varphi(0)}{h}=\lim_{h\to 0}\frac{e^{-\frac{1}{h}}}{h}=0$$

(by l'Hospital's rule). Further,

$$\varphi''(0)=\lim_{h\to 0}\frac{\varphi'(h)-\varphi'(0)}{h}=\lim_{h\to 0}\frac{2}{h^3}e^{-\frac{1}{h^2}}=0 \text{ and so on.}$$

and are finite, we get a Taylor series:

$$f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots$$
$$\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots$$

Because of what was said in Sec. 400, we also have to prove that this series has a nonzero radius of convergence and yields the precise sum of $f(x)$, and not some other function. It is sometimes possible to estimate the "remainder term" $R_n=f(x)-s_n(x)$ and to prove that $\lim\limits_{n\to\infty} R_n=0$. For this purpose, R_n is given in the Lagrange form (Sec. 272, Examples 1 and 2) or in other forms.

In most cases this is difficult to do (or practically impossible). Then we can obtain the expansion in other ways, bypassing the computation of the derivatives of $f(x_0)$, $f'(x_0)$, ... (the derivatives are obtained automatically from the expansion, as in Example 1 of Sec. 400).

Below we give the power-series expansions of the simplest functions in x. The general term, when its form is easily recognizable, is omitted.

Exponential functions.

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\quad (R=\infty), \tag{1}$$

$$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\quad (R=\infty) \tag{1a}$$

Both expansions may be obtained by estimating the remainder term (Sec. 272, Example 1). Formula (1a) is obtained from (1) by replacing x by $-x$.

Trigonometric functions.

$$\sin x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\quad (R=\infty), \tag{2}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\quad (R=\infty) \tag{3}$$

Both expansions may be obtained by estimating the remainder term (Sec. 272, Example 2). One of them may be obtained from the other by termwise differentiation (or integration).

Dividing (2) by (3) term by term, we get

$$\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+\ldots\quad \left(R=\frac{\pi}{2}\right) \tag{4}$$

The law of the formation of coefficients cannot be expressed by an elementary formula and for this reason it is difficult to seek the radius of convergence via the theorem of Sec. 394. But it is clear that R does not exceed $\frac{\pi}{2}$; the series (4) already diverges when $x = \pm\frac{\pi}{2}$ since $\tan\left(\pm\frac{\pi}{2}\right) = \infty$.

Note. The radius of convergence of series (4) proves less than each of the radii of convergence of the series (2), (3), which produce (4) by termwise division. Cf. Sec. 397, Example 4.

The function $\cot x$ cannot be expanded in powers of x (because $\cot 0 = \infty$).

Hyperbolic functions. [1]

$$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \quad (R = \infty) \tag{2a}$$

(*hyperbolic sine;* symbol: $\sinh x$),

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \quad (R = \infty) \tag{3a}$$

(*hyperbolic cosine;* symbol: $\cosh x$),

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = x - \frac{1}{3}x^3 + \frac{2}{1}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 - \dots \left(R = \frac{\pi}{2}\right) \tag{4a}$$

(*hyperbolic tangent;* symbol: $\tanh x$).

The expansions (2a) and (3a) are obtained by subtracting and adding (1) and (1a); the expansion (4a) is obtained by termwise division of (2a) by (3a). Cf. note on formula (4).

The expansions of the hyperbolic functions differ from the expansions of the similar trigonometric functions in signs alone.

Logarithmic functions.

$$\ln(1 + x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (R = 1), \tag{5}$$

$$\ln(1 - x) = -\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \quad (R = 1) \tag{6}$$

Firmulas (5) and (6) are obtained by termise integration of the expansions $\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \dots$. Via termwise subtraction we get

$$\ln\frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right] \quad (R = 1) \tag{7}$$

---

[1] Hyperbolic functions are discussed in Sec. 403.

Series (7) is convenient for computing the logarithms of whole numbers. For example, when $x=\frac{1}{3}$ we get a rapidly convergent series for ln 2.

Binomial series.

$$(1+x)^m=1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\ldots \quad (R=1) \tag{8}$$

For positive integral m, the series terminates with a term of degree m (the subsequent coefficients are zeros). The resulting formula is called the binomial expansion.[1] The expansion (8) holds for any real m.

For $-\frac{1}{2}<x<1$ we can use the Lagrange remainder term in the proof.

Using other devices[2] it is possible to prove the validity of formula (8) for the entire interval $(-1, 1)$.

The following expansions are particular cases of (8):

$$(1+x)^{-1}=\frac{1}{1+x}=1-x+x^2-x^3+\ldots, \tag{9}$$

$$(1+x)^{-2}=\frac{1}{(1+x)^2}=1-2x+3x^2-4x^3+5x^4-\ldots, \tag{10}$$

$$(1+x)^{-3}=\frac{1}{(1+x)^3}=1-\frac{2\cdot 3}{2}x+\frac{3\cdot 4}{2}x^2-\frac{4\cdot 5}{2}x^3+\ldots, \tag{11}$$

$$(1+x^2)^{-1}=\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-\ldots, \tag{12}$$

$$(1+x)^{1/2}=\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{2\cdot 4}x^2+\frac{1\cdot 3}{2\cdot 4\cdot 6}x^3-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4+\ldots, \tag{13}$$

$$(1+x)^{-1/2}=\frac{1}{\sqrt{1+x}}=1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-$$

---

[1] The name is more aptly applied to the general formula (8) (cf. Sec. 270, Item 1).

[2] For the function $f(x)=(1+x)^m$ we have, identically, $mf(x)=(1+x)f'(x)$. Direct verification (termwise differentiation and multiplication) shows that the same relation $[mS(x)=(1+x)S'(x)]$ holds for the sum $S(x)$ of series (8). Hence

$$\frac{S(x)}{S'(x)}=\frac{f(x)}{f'(x)}, \text{ that is } [\ln f(x)]'=[\ln S(x)]'$$

Since the functions $\ln S(x)$ and $\ln f(x)$ have the same derivatives and the same values for $x=0$, they are equal. Consequently, $S(x)=f(x)$, as we had to prove.

$$-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4-\ldots, \tag{14}$$

$$(1-x^2)^{-1/2}=\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+$$
$$+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8+\ldots \tag{15}$$

Inverse trigonometric functions.

$$\arcsin x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\ldots\ (R=1), \tag{16}$$

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\ldots\ (R=1) \tag{17}$$

Expansions (16) and (17) are obtained, respectively, from (15) and (12) by termwise integration from zero to x.

The expansions

$\arccos x=\frac{\pi}{2}-\arcsin x$ and $\operatorname{arccot} x=\frac{\pi}{2}-\arctan x$ are obtained from (16) and (17).

Inverse hyperbolic functions. [1]

$$\ln(x+\sqrt{x^2+1})=x-\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\ldots\ (R=1) \tag{16a}$$

(*inverse hyperbolic sine*; notation: $\sinh^{-1}x$, or $\operatorname{arcsinh} x$).

$$\frac{1}{2}\ln\frac{1+x}{1-x}=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\ (R=1) \tag{17a}$$

(*inverse hyperbolic tangent*; notation: $\tanh^{-1}x$, or $\operatorname{arctanh} x$).

The functions

$$\ln(x+\sqrt{x^2-1})=\operatorname{arccosh} x$$

(*inverse hyperbolic cosine*) and

$$\frac{1}{2}\ln\frac{x+1}{x-1}=\operatorname{arccoth} x$$

(*inverse hyperbolic cotangent*) cannot be expanded in a series of powers of x [they are not defined at any point of the interval $(-1, 1)$, in particular at the point $x=0$].

The expansions (16a) and (17a) differ from the expansions (16) and (17) in signs alone.

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[1] Inverse hyperbolic functions are discussed in Sec. 404.

402. The Use of Series in Computing Integrals 1)

There are many integrals that cannot be expressed in terms of elementary functions in closed form and are represented by rapidly convergent infinite series. There is also sense in expanding in series such integrals as may be represented by finite expressions (if these expressions are complicated). The point is that errors arise also when using "exact" expressions because the values of these expressions are, as a rule, found with the aid of tables.

Example 1. The integral $\int\limits_0^x e^{-x^2}\,dx$ cannot be expressed in closed form in terms of elementary functions. Let us take advantage of the series

$$e^{-x^2}=1-\frac{x^2}{1!}+\frac{x^4}{2!}-\ldots+(-1)^n\frac{x^{2n}}{n!}+\ldots \tag{1}$$

convergent in the interval $(-\infty, +\infty)$. It yields

$$\int\limits_0^x e^{-x^2}\,dx=x-\frac{1}{1!}\,\frac{x^3}{3}+\frac{1}{2!}\,\frac{x^5}{5}-\ldots+(-1)^n\frac{1}{n!}\,\frac{x^{2n+1}}{2n+1}+\ldots \tag{2}$$

The interval of convergence is also $(-\infty, +\infty)$ (Sec. 398).

Example 2. Evaluate $\int\limits_0^1 e^{-x^2}\,dx$ to an accuracy of $0.5\cdot 10^{-4}$.

Solution. Substituting into (2) the value $x=1$, we get

$$\int\limits_0^1 e^{-x^2}\,dx=1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}+\frac{1}{216}-\frac{1}{1320}+\frac{1}{9360}-\frac{1}{75600}+\ldots \tag{3}$$

The term $-\frac{1}{75600}$ and subsequent terms are rejected since the error committed is much less than $0.5\cdot 10^{-4}$ [series (3) is an alternating series with decreasing terms, Sec. 376]. Calculating to five or six decimal places, we get

$$\int\limits_0^1 e^{-x^2}\,dx=0.7468$$

---

1) Infinite series appeared historically in connection with the problem of integration (cf. Sec. 270).

Example 3. Evaluate the integral $\int\limits_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$ to an accuracy of $0.5\ 10^{-3}$.

Solution. The indefinite integral $\int \frac{\sin x}{x}\,dx$ is not expressible in closed form. Expanding $\sin x$ in a series and dividing termwise by x, we get the series

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \tag{4}$$

which is convergent for any value of x (by the theorem of Sec. 394). Integrating, we obtain

$$\int\limits_0^x \frac{\sin x}{x}\,dx = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \dots,$$

$$\int\limits_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx = \frac{\pi}{2} - \frac{1}{18}\left(\frac{\pi}{2}\right)^3 + \frac{1}{600}\left(\frac{\pi}{2}\right)^5 - \frac{1}{35280}\left(\frac{\pi}{2}\right)^7 + \dots \tag{5}$$

The first rejected term $\frac{1}{9\cdot 9!}\left(\frac{\pi}{2}\right)^9$ is (by rough computation) much less than $0.5\cdot 10^{-3}$. We find

$$\begin{array}{rr@{\qquad}rr}
 & \frac{\pi}{2} = 1.5708 & + & \frac{1}{18}\left(\frac{\pi}{2}\right)^3 = 0.2153 \\
+ & \frac{1}{600}\cdot\left(\frac{\pi}{2}\right)^5 = 0.0159 & & \frac{1}{35280}\cdot\left(\frac{\pi}{2}\right)^7 = 0.0007 \\
\hline
 & 1.5867 & & 0.2160
\end{array}$$

$$\int\limits_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx = 1.5867 - 0.2160 \approx 1.371$$

403. Hyperbolic Functions

The power series

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \quad (R=\infty), \tag{1}$$

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots \quad (R=\infty) \tag{2}$$

similar to the expansions of $\sin x$, $\cos x$, have sums that are respectively equal to $\frac{e^x-e^{-x}}{2}$, $\frac{e^x+e^{-x}}{2}$. These functions are called, respectively, the *hyperbolic sine* (sinh) and the *hyperbolic cosine* (cosh):

$$\sinh x=\frac{e^x-e^{-x}}{2}, \tag{3}$$

$$\cosh x=\frac{e^x+e^{-x}}{2} \tag{4}$$

The *hyperbolic tangent* (tanh) and the *hyperbolic cotangent* (coth) are the functions

$$\tanh x=\frac{\sinh x}{\cosh x}=\frac{e^x-e^{-x}}{e^x+e^{-x}}, \tag{5}$$

$$\coth x=\frac{\cosh x}{\sinh x}=\frac{e^x+e^{-x}}{e^x-e^{-x}} \tag{6}$$

The functions $\sinh x$, $\cosh x$, $\tanh x$, $\coth x$ are called *hyperbolic functions*.[1] The graphs are given in Figs. 409 to 412.

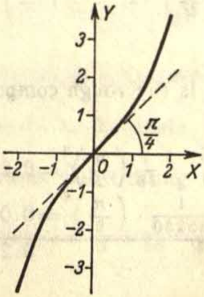


Fig. 409. $y=\sinh x$

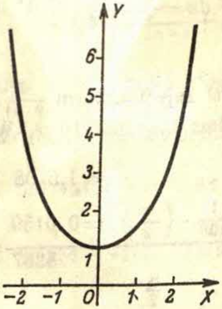


Fig. 410. $y=\cosh x$

The hyperbolic functions have definite values for all values of x (except $\coth x$ for $x=0$, where this function becomes infinite).

The function $\sinh x$ takes on all possible values, $\cosh x$ takes on those not less than unity ($\cosh 0=1$), the values of the function $\tanh x$ lie between -1 and $+1$, the values of $\coth x$ exceed 1 for $x > 0$ and are less than -1 for $x < 0$. The straight lines $y = +1$ and $y = -1$ serve as asymptotes for both lines $y = \tanh x$, $y = \coth x$.

---

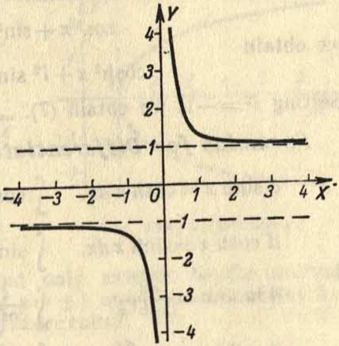
[1] The connection with the hyperbola is discussed in Sec. 405.

The hyperbolic functions are connected by the relations

$$\cosh^2 x - \sinh^2 x = 1, \quad (7)$$

$$\tanh x \cdot \coth x = 1, \quad (8)$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x} \quad (9)$$



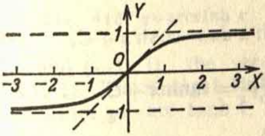


Fig. 411. $y = \tanh x$

Fig. 412. $y = \coth x$

and others, similar to trigonometric functions. Thus,

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y, \quad (10)$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y, \quad (11)$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \quad (12)$$

They all follow from the formulas (3) to (6).

Generally, every trigonometric formula that does not contain constant quantities under the signs of the trigonometric functions[1] is associated with an analogous relation between the hyperbolic functions. The latter relation is obtained if we replace $\cos\alpha$ everywhere by $\cosh\alpha$ and $\sin\alpha$ by $i\sinh\alpha$ (i is the imaginary unit); the imaginaries cancel out by themselves.

Example 1. From the trigonometric formula

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

---

[1] This proviso is essential. Thus, formula $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$ cannot be transformed by the rule given here.

we get, with the aid of the indicated substitution,

$$i \sinh (x+y) = i \sinh x \cosh y + \cosh x \, i \sinh y$$

Dividing both sides of the equation by i, we get (10).

Example 2. From the formula

$$\cos^2 x + \sin^2 x = 1$$

we obtain

$$\cosh^2 x + i^2 \sinh^2 x = 1$$

Setting $i^2 = -1$, we obtain (7).

Formulas for Differentiation and Integration

$$d \sinh x = \cosh x \, dx, \qquad \int \cosh x \, dx = \sinh x + C, \tag{13}$$

$$d \cosh x = \sinh x \, dx, \qquad \int \sinh x \, dx = \cosh x + C, \tag{14}$$

$$d \tanh x = \frac{dx}{\cosh^2 x}, \qquad \int \frac{dx}{\cosh^2 x} = \tanh x + C, \tag{15}$$

$$d \coth x = -\frac{dx}{\sinh^2 x}, \qquad \int \frac{dx}{\sinh^2 x} = -\coth x + C \tag{16}$$

These formulas are obtained from the corresponding trigonometric formulas if we make the above-indicated substitution and, besides, write $i\,dx$ in place of dx.

404. Inverse Hyperbolic Functions

The hyperbolic functions $\sinh x$, $\cosh x$, $\tanh x$, $\coth x$ have the following inverse functions:

arcsinh x (*inverse hyperbolic sine*, Fig. 413),
arccosh x (*inverse hyperbolic cosine*, Fig. 414),
arctanh x (*inverse hyperbolic tangent*, Fig. 415),
arccoth x (*inverse hyperbolic cotangent*, Fig. 416).

Cf. the graphs of the hyperbolic functions in Figs. 409 to 412.

The function $\sinh^{-1}$, or arcsinh x, is uniquelly defined over the whole real line. The function $\cosh^{-1}$, or arc cosh x, is defined only on the interval $(1, \infty)$ and is here double-valued (its values are equal in absolute value and differ in sign). Ordinarily, only the positive values are considered; the corresponding branch of the graph (*principal branch*) is depicted in Fig. 414 by a heavy line. With this proviso, the function arccosh x becomes single-valued.

The functions $\tanh^{-1}$, or arctanh x, and $\coth^{-1}$, or arccoth x, are single-valued, the first is defined only in the (open) interval $(-1, 1)$, the second only exterior to the interval $(-1, 1)$. The straight lines $x = \pm 1$ serve as asymptotes for both lines $y = \text{arc}\tanh x$, $y = \text{arc}\coth x$.

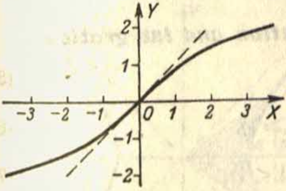


Fig. 413. $y = \text{arcsinh}\, x$

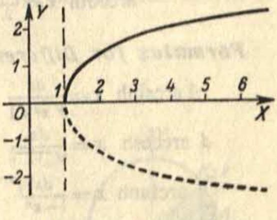


Fig. 414. $y = \text{arccosh}\, x$

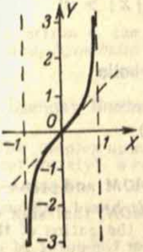


Fig. 415 $y = \text{arctanh}\, x$

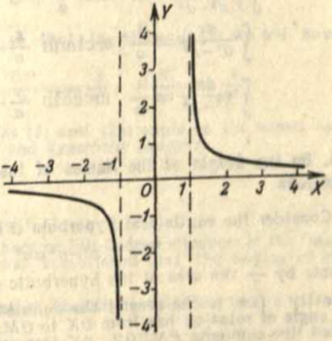


Fig. 416. $y = \text{arccoth}\, x$

The inverse hyperbolic functions are expressed in terms of the basic elementary functions in the following manner:

$$\text{arcsinh}\, x = \ln\left(x + \sqrt{x^2+1}\right), \tag{1}$$

$$\text{arccosh}\, x = \ln\left(x \pm \sqrt{x^2-1}\right) = \pm \ln\left(x + \sqrt{x^2-1}\right) \quad (x \geqslant 1) \tag{2}$$

the upper signs in formula (2) correspond to the principal value of arc cosh x;

$$\operatorname{arctanh} x=\frac{1}{2}\ln\frac{1+x}{1-x}\quad(|x|<1),\tag{3}$$

$$\operatorname{arccoth} x=\frac{1}{2}\ln\frac{x+1}{x-1}\quad(|x|>1)\tag{4}$$

Formulas for Differentiation and Integration [1]

$$d\operatorname{arcsinh} x=\frac{dx}{\sqrt{x^2+1}},\tag{5}$$

$$d\operatorname{arccosh} x=\frac{dx}{\sqrt{x^2-1}}\quad(x\geqslant 1),\tag{6}$$

$$d\operatorname{arctanh} x=\frac{dx}{1-x^2}\quad(|x|<1),\tag{7}$$

$$d\operatorname{arccoth} x=\frac{dx}{1-x^2}\quad(|x|>1),\tag{8}$$

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\operatorname{arcsinh}\frac{x}{a}+C,\tag{5a}$$

$$\int\frac{dx}{\sqrt{x^2-a^2}}=\operatorname{arccosh}\frac{x}{a}+C\qquad(x\geqslant a),\tag{6a}$$

$$\int\frac{dx}{a^2-x^2}=\frac{1}{a}\operatorname{arctanh}\frac{x}{a}+C\quad(|x|<a),\tag{7a}$$

$$\int\frac{dx}{a^2-x^2}=\frac{1}{a}\operatorname{arccoth}\frac{x}{a}+C\quad(|x|>a)\tag{8a}$$

405. On the Origin of the Names of the Hyperbolic Functions

Consider the equilateral hyperbola (Fig. 417)

$$x^2-y^2=a^2\tag{1}$$

Denote by $\frac{s}{2}$ the area of the hyperbolic sector AOM and prefix to the quantity s (i.e. to the area of the double sector MON) that sign which the angle of rotation has from OX to OM. Then the ratios of the directed line-segments PM, OP, AK (constructed for the point M of the hyperbola in similar fashion to the lines of the sine, cosine, and tangent; cf. Fig. 418) to the semiaxis a are expressed in terms of s as follows: [2]

$$\frac{PM}{a}=\sinh\frac{s}{a^2},\quad\frac{OP}{a}=\cosh\frac{s}{a^2},\quad\frac{AK}{a}=\tanh\frac{s}{a^2}\tag{2}$$

---

[1] arccosh x stands for the positive value of the function.

[2] We have (Sec. 333, Example 4)

$$s=2\text{ area }AOM=a^2\ln\frac{x+y}{a}$$ (cont'd on p. 601)

Now in place of the hyperbola (1) let us take a circle (Fig. 418):

$$x^2+y^2=a^2$$

Retaining the original notations, we see that the quantity $\frac{s}{a^2}$ taken with the proper sign (s is the area of the circular sector $MONA$) will

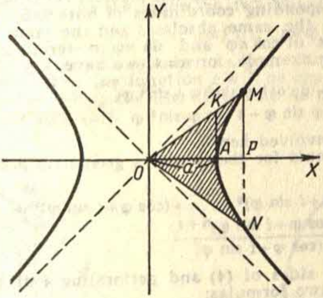


Fig. 417

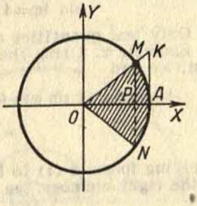


Fig. 418

yield the angle $\alpha=\angle AOM$ so that in place of (2) we will have to write

$$\frac{PM}{a}=\sin\frac{s}{a^2},\quad \frac{OP}{a}=\cos\frac{s}{a^2},\quad \frac{AK}{a}=\tan\frac{s}{a^2} \tag{2a}$$

A comparison of the formulas (2) and (2a) explains the names *hyperbolic sine, hyperbolic cosine*, and *hyperbolic tangent*.

406. Complex Numbers

The complex numbers ¹) became full-fledged members of the "mathematical society" when it was established that the finding of many

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Solving Eq. 1 and this equation simultaneously, we find

$$\frac{x}{a}=\frac{e^{\frac{s}{a^2}}+e^{-\frac{s}{a^2}}}{2}=\cosh\frac{s}{a^2},\quad \frac{y}{a}=\frac{e^{\frac{s}{a^2}}-e^{-\frac{s}{a^2}}}{2}=\sinh\frac{s}{a^2}$$

From the similarity of the triangles OAK, OPM we have

$$\frac{AK}{a}=\frac{PM}{OP}=\frac{y}{x}=\tanh\frac{s}{a^2}$$

¹) Operations on complex numbers and the geometrical interpretation of such operations may be found in handbooks of elementary mathematics.

relationships between real quantities is greatly facilitated with their aid.

Example 1. Multiplying successively the complex number $\cos\varphi+i\sin\varphi$ by itself, we get *de Moivre's theorem (formula)*

$$(\cos\varphi+i\sin\varphi)^n=(\cos n\varphi+i\sin n\varphi) \tag{1}$$

for positive integral n. ¹⁾ Let us apply to the left member the binomial formula and equate the corresponding coordinates of both sides (two equal complex numbers have the same abscissas and the same ordinates). We obtain expressions of $\cos n\varphi$ and $\sin n\varphi$ in terms of the powers of $\cos\varphi$ and $\sin\varphi$. For example, for $n=4$, we have

$$\cos 4\varphi=\cos^4\varphi-6\cos^2\varphi\sin^2\varphi+\sin^4\varphi, \tag{2}$$

$$\sin 4\varphi=4\cos^3\varphi\sin\varphi-4\cos\varphi\sin^3\varphi \tag{3}$$

Only real quantities are involved here.

Example 2. Using the formula for the sum of a geometric progression, we find

$$1+(\cos\varphi+i\sin\varphi)+(\cos\varphi+i\sin\varphi)^2+\ldots+(\cos\varphi+i\sin\varphi)^n=$$
$$=\frac{1-(\cos\varphi+i\sin\varphi)^{n+1}}{1-(\cos\varphi+i\sin\varphi)} \tag{4}$$

Applying formula (1) to both sides of (4) and performing a division in the right member, we get two formulas:

$$\cos\varphi+\cos 2\varphi+\cos 3\varphi+\ldots+\cos n\varphi=\frac{\sin\frac{(2n+1)\varphi}{2}-\sin\frac{\varphi}{2}}{2\sin\frac{\varphi}{2}}, \tag{5}$$

$$\sin\varphi+\sin 2\varphi+\sin 3\varphi+\ldots+\sin n\varphi=\frac{\cos\frac{\varphi}{2}-\cos\frac{(2n+1)\varphi}{2}}{2\sin\frac{\varphi}{2}} \tag{6}$$

Introducing *complex variables* and defining for them the concepts of function, limit, derivative, etc., we find many new relationships between real variables.

Sections 407 to 410 represent a departure from the general plan of this book: we consider the complex functions of a *real argument* (we do not discuss the functions of a complex argument at all).

407. A Complex Function of a Real Argument

The complex quantity

$$z=x+iy \tag{1}$$

(x, y are real numbers) is called a *function of the real argument t* if to each of the values of t (in the domain under consideration) there

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¹⁾ A negative power of a complex number may be defined in the same way as for real numbers; then formula (1) can be extended to negative exponents as well. Formula (1) may be taken as a definition for fractional and irrational exponents. The result is then multiple-valued (because the angle φ is multiple-valued: $\varphi=\varphi_0+2k\pi$, where k is any integer). The rules for operations on powers are the same as for a real base.

corresponds a definite value of z (i.e. a definite value of x and a definite value of y).

Here, each of the coordinates x, y is a (real) function of the argument t.

Notation:

$$z=f(t)+i\varphi(t) \tag{2}$$

is equivalent to the following two equations:

$$x=f(t), \quad y=\varphi(t) \tag{3}$$

If the complex number $x+iy$ is denoted by a point (x, y) in the xy-plane, then the function z will be depicted as a set of points either isolated or filling the line [this line is parametrically represented by the equations (3)].

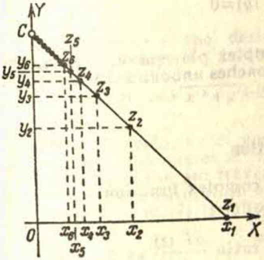

Fig. 419

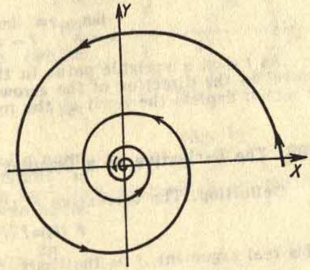

Fig. 420

The concepts of limit and infinitesimal quantity for complex functions are defined in the same way as for real functions (the absolute value of the complex number $x+iy$ is its modulus $|x+iy|=\sqrt{x^2+y^2}$). Points depicting the values of the function approach without bound the point depicting the limit when the argument t tends to the given value (or to infinity). To find the limit c of a complex function z, it is sufficient to find the limits a and b of its coordinates x and y. Then $c=a+bi$.

Example 1. The sequence

$$z_1=1, \quad z_2=\frac{1}{2}+\frac{1}{2}i, \quad z_3=\frac{1}{3}+\frac{2}{3}i, \ \ldots, \ z_n=\frac{1}{n}+\frac{n-1}{n}i, \ \ldots \tag{4}$$

is depicted (Fig. 419) as a set of isolated points (x_n, y_n):

$$x_n=\frac{1}{n}, \quad y_n=\frac{n-1}{n} \tag{5}$$

They lie on the straight line $x+y=1$. We have

$$\lim_{n\to\infty} x_n=0, \quad \lim_{n\to\infty} y_n=1, \tag{6}$$

$$\lim_{n\to\infty} z_n=\lim_{n\to\infty}(x_n+iy_n)=0+1i=i \tag{7}$$

The relation (7) means that the modulus $|z_n - i|$ of the difference $z_n - i$ decreases without bound as $n \to \infty$.

Geometrically, the points z_n approach without bound the point $C(0, 1)$.

Example 2. The complex function

$$z = e^{-0.1 t} (\cos t + i \sin t) \tag{8}$$

of the argument t is shown in Fig. 420 by the line

$$x = e^{-0.1t} \cos t, \quad y = e^{-0.1t} \sin t \tag{9}$$

(logarithmic spiral). We have

$$\lim_{t \to \infty} x = 0, \qquad \lim_{t \to \infty} y = 0;$$

$$\lim_{t \to \infty} z = \lim_{t \to \infty} (x + iy) = 0$$

As $t \to \infty$, a variable point in the complex plane moving along the spiral in the direction of the arrow approaches unboundedly the point O which depicts the limit of the function.

408. The Derivative of a Complex Function

Definition. The derivative $F'(t)$ of a complex function

$$F(t) = f(t) + i\varphi(t) \tag{1}$$

of a real argument t is the limit of the ratio $\dfrac{\Delta F(t)}{\Delta t}$ as $\Delta t \to 0$.

The coordinates of the derivative are the derivatives of the coordinates $f(t)$, $\varphi(t)$ of the given function:

$$F'(t) = f'(t) + i\varphi'(t) \tag{2}$$

The vector depicting $F'(t)$ is the vector of the tangent at the corresponding point of the graph

$$x = f(t), \quad y = \varphi(t) \tag{3}$$

If t is the time, then the modulus of the derivative is equal to the absolute value of the velocity of the point along the graph (3).

The differential of a complex function is defined in the same way as that for a real function and has the same properties.

If a complex function $F(t)$ is represented by a polynomial

$$F(t) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \tag{4}$$

where z is a complex function of a real argument t, then

$$F'(t) = (a_1 + 2a_2 z + \ldots + n a_n z^{n-1}) z'(t) \tag{5}$$

The formulas for the derivative of a product and of a quotient are the same as for real functions.

Example 1. The derivative of the function

$$F(t) = a\left(\cos 2\pi \frac{t}{T} + i \sin 2\pi \frac{t}{T}\right) \tag{6}$$

is equal to

$$F'(t)=\frac{2\pi a}{T}\left(-\sin 2\pi\frac{t}{T}+i\cos 2\pi\frac{t}{T}\right) \tag{7}$$

The function (6) is depicted by the set of points of a circle (Fig. 421) of radius a:

$$x=a\cos 2\pi\frac{t}{T}, \quad y=a\sin 2\pi\frac{t}{T} \tag{8}$$

The derivative (7) is portrayed as the vector of the tangent MK with coordinates

$$x'=-\frac{2\pi a}{T}\sin 2\pi\frac{t}{T}, \quad y'=\frac{2\pi a}{T}\cos 2\pi\frac{t}{T} \tag{9}$$

The modulus of the derivative (which expresses velocity if t is time) is

$$|F'(t)|=\sqrt{x'^2+y'^2}=\frac{2\pi a}{T} \tag{10}$$

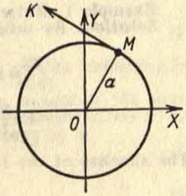

Fig. 421

Thus, the velocity of motion of a point along the circumference is constant so that $|F'(t)|$ is the arc traversed in unit time. Hence T is the period of one complete revolution about the circle.

From (6) and (7) it follows that

$$F'(t)=F(t)\frac{2\pi}{T}i \tag{11}$$

Geometrically, the vector MK is obtained from the vector OM by stretching (compressing) by a factor of $\frac{2\pi}{T}$ and by rotation through 90° (multiplication by i is equivalent to a rotation through 90°).

Example 2. The derivative of the function

$$F(t)=(x+iy)^2 \tag{12}$$

where x and y are functions of t, is

$$F'(t)=2(x+iy)(x'+iy')=2(xx'-yy')+2i(xy'+yx') \tag{13}$$

We get the same result if we first represent (12) in the form

$$F(t)=(x^2-y^2)+2xyi \tag{14}$$

409. Raising a Positive Number to a Complex Power

For all *real* values of u, the series

$$1+\frac{u}{1!}+\frac{u^2}{2!}+\frac{u^3}{3!}+\ldots+\frac{u^n}{n!}+\ldots \tag{1}$$

converges everywhere and has the sum e^u.

The series (1) also converges for any *complex* value of u, i.e. its partial sums s_n (which are now complex numbers) tend to a finite limit (also a complex number).

This is the basis for the following definition of a new operation: the *raising of a positive number to a complex power*.[1]

Definition. To raise the number e (the base of natural logarithms) to a complex power $u=x+iy$ means taking the sum of the series (1). For the complex power u of any other *positive* number a we take the quantity $e^{u \ln a}$ (for real values of u it is identical with a^u).

Note. All rules involving operations with powers may be extended to complex powers of positive numbers. But they have to be proved separately.

Example 1. Raise e to the power πi.

Solution. By definition

$$e^{\pi i}=1+\frac{\pi i}{1!}+\frac{\pi^2 i^2}{2!}+\frac{\pi^3 i^3}{3!}+\frac{\pi^4 i^4}{4!}+\frac{\pi^5 i^5}{5!}+\ldots=$$
$$=1+\frac{\pi i}{1!}-\frac{\pi^2}{2!}-\frac{\pi^3 i}{3!}+\frac{\pi^4}{4!}+\frac{\pi^5 i}{5!}-\frac{\pi^6}{6!}-\frac{\pi^7 i}{7!}+\ldots$$

The abscissa of the sum is equal to

$$1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\ldots=\cos \pi=-1$$

(cf. Sec. 272). The ordinate of the sum is

$$\frac{\pi}{1!}-\frac{\pi^3}{3!}+\frac{\pi^5}{5!}-\frac{\pi^7}{7!}+\ldots=\sin \pi=0$$

Hence

$$e^{\pi i}=-1$$

In this case we obtained a real number.

Example 2. Compute 10^i.

Solution. By definition

$$10^i=e^{i \ln 10}=e^{\frac{1}{M} i}$$

where $\frac{1}{M} \approx 2.3026$ (see Sec. 242).

$$10^i=1+\frac{1}{1!M} i-\frac{1}{2!M^2}-\frac{1}{3!M^3} i+\frac{1}{4!M^4}+\frac{1}{5!M^5} i-\frac{1}{6!M^6}-$$
$$-\frac{1}{7!M^7} i+\ldots=\left(1-\frac{1}{2!M^2}+\frac{1}{4!M^4}-\frac{1}{6!M^6}+\ldots\right)+$$
$$+i\left(\frac{1}{1!M}-\frac{1}{3!M^3}+\frac{1}{5!M^5}-\frac{1}{7!M^7}+\ldots\right)=\cos\frac{1}{M}+i \sin \frac{1}{M} \approx$$
$$\approx \cos 2.3026+i \sin 2.3026 \approx \cos 131^\circ 56'+i \sin 131^\circ 56'=$$
$$=-0.6680+i \cdot 0.7440$$

---

[1] See footnote on p. 602 for the raising of a complex number to a real power. It is also possible to define the raising of a complex number to a complex power, but that is more complicated.

Example 3. Compute 10^{2+i}.
Solution. We have (cf. Example 2)

$$10^{2+i}=10^{2}\cdot 10^{i}\approx -66.80+74.40\,i$$

410. Euler's Formula

The relation

$$e^{i\varphi}=\cos\varphi+i\sin\varphi \tag{1}$$

is called *Euler's formula*. It is a consequence of the definition of Sec. 409 (and is derived as in Example 1 of Sec. 409).

If one defines $\cos\varphi$, $\sin\varphi$ for complex φ by means of the same series whose sums, by what has been proved, yield $\cos\varphi$, $\sin\varphi$ for real φ, then formula (1) will hold true for any complex φ.

From formula (1) we get

$$e^{-i\varphi}=\cos\varphi-i\sin\varphi \tag{2}$$

and from (1) and (2) we find

$$\cos\varphi=\frac{e^{i\varphi}+e^{-i\varphi}}{2},\qquad \sin\varphi=\frac{e^{i\varphi}-e^{-i\varphi}}{2i}. \tag{3}$$

These formulas are very much like the expressions of the hyperbolic functions

$$\cosh\varphi=\frac{e^{\varphi}+e^{-\varphi}}{2},\qquad \sinh\varphi=\frac{e^{\varphi}-e^{-\varphi}}{2}$$

From (1) there also follows the formula

$$e^{x+iy}=e^{x}(\cos y+i\sin y) \tag{4}$$

(cf. Sec. 409, Note).

If x and y in formula (4) are functions of the argument t, then (4) may be differentiated in the same way as if i were a real constant:

$$e^{x+iy}(x'+iy')=x'e^{x}(\cos y+i\sin y)+y'e^{x}(-\sin y+i\cos y) \tag{5}$$

Verification of the validity of formula (5) is straightforward.

Example. Find the derivative of the function

$$F(t)=e^{0.1t}(\cos 2t+i\sin 2t)$$

Solution. Represent $F(t)$ in the form

$$F(t)=e^{(0.1+2i)t}$$

We then get

$$F'(t)=(0.1+2i)\,e^{(0.1+2i)t}=(0.1+2i)\,e^{0.1t}(\cos 2t+i\sin 2t)=$$
$$=e^{0.1t}\left[(0.1\cos 2t-2\sin 2t)+i\,(0.1\sin 2t+2\cos 2t)\right]$$

411. Trigonometric Series

A *trigonometric series* is a series of the form

$$\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+\ldots+a_n\cos nx+$$
$$+b_n\sin nx+\ldots \qquad (1)$$

where $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are constants called the *coefficients* of the series.

Note 1. The constant term $\left(\text{which may be written as } \frac{a_0}{2}\cos 0x\right)$ is denoted by $\frac{a_0}{2}$ (and not by a_0) so that the formulas for the coefficients (cf. Sec. 414) are uniform.

Note 2. All the terms of (1) are *periodic functions* with period 2π. This means that when the argument x increases by a multiple of 2π, all the terms retain their values.

Note 3. The term trigonometric series is also applied to the more general expression

$$\frac{a_0}{2}+a_1\cos\frac{\pi x}{l}+b_1\sin\frac{\pi x}{l}+a_2\cos 2\frac{\pi x}{l}+b_2\sin 2\frac{\pi x}{l}+\ldots$$
$$\ldots+a_n\cos n\frac{\pi x}{l}+b_n\sin n\frac{\pi x}{l}+\ldots \qquad (2)$$

where l is a positive constant called the *half-period* [all terms of (2) are periodic functions with period $2l$; cf. Note 2]. Series (1) is a particular case of series (2) when the half-period $l=\pi$.

412. Trigonometric Series (Historical Background)

Trigonometric series were introduced by D. Bernoulli[1] in 1753 in connection with studies of the vibrations of a string. The problem which arose of the possibility of expanding the given function in a trigonometric series gave rise to hot debates among the outstanding mathematicians of the day (Euler, d'Alembert, Lagrange). The differences were due to the fact that the concept of a function had not yet been clearly established. These debates contributed much to a clarification of the concept of a function.

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[1] Daniel Bernoulli (1700-1782), noted Swiss mathematician and mechanician, one of the founders of hydrodynamics.

Formulas expressing the coefficients of series (1) in terms of a given function (Sec. 414) were given by Clairaut[1] in 1757, but did not attract any attention. Euler obtained these formulas again in 1777 (in a paper published posthumously in 1793). Their rigorous derivation was outlined by Fourier in 1823. Developing the ideas of Fourier, Dirichlet[2] established (in 1829) and rigorously proved a sufficient criterion for expansibility of a function in a trigonometric series (Sec. 418).

Other sifficient conditions were subsequently established, and functions not satisfying these conditions were investigated. The following Russian and Soviet mathematicians have made significant contributions to the theory of trigonometric functions and their practical applications: N. Lobachevsky, A. Krylov (1863-1945), S. Bernstein (1880-), N. Luzin (1883-1950), D. Menshov (1892-), N. Bari (1901-1961), A. Kolmogorov (1903-) and others.

413. The Orthogonality of the System of Functions cos nx, sin nx

Definition 1. Two functions $\varphi(x)$, $\psi(x)$ are called *orthogonal in an interval* (a, b) if the integral of the product $\varphi(x)\psi(x)$ taken between the limits a and b is zero.

Example 1. The functions

$$\varphi(x) = \sin 5x$$

and

$$\psi(x) = \cos 2x$$

are orthogonal in the interval $(-\pi, \pi)$ because

$$\int_{-\pi}^{\pi} \sin 5x \cos 2x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\sin 7x + \sin 3x)\, dx =$$

$$= -\frac{1}{14}\cos 7x - \frac{1}{6}\cos 3x \Big|_{-\pi}^{\pi} = 0$$

Example 2. The functions

$$\varphi(x) = \sin 4x$$

and

$$\psi(x) = \sin 2x$$

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[1] Alexis Claude Clairaut (1713-1765), outstanding French mathematician, astronomer and geophysicist. Elected member of the Paris Academy of Sciences at the age of 16.

[2] Peter Gustav Lejeune-Dirichlet (1805-1859), celebrated German mathematician.

are orthogonal in the interval $(-\pi, \pi)$ because

$$\int_{-\pi}^{\pi} \sin 4x \sin 2x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 2x - \cos 6x)\,dx = 0$$

Theorem. Any two distinct functions taken from the system of functions

$$1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots \quad (1)$$

are orthogonal in the interval $(-\pi, \pi)$, that is,

$$\int_{-\pi}^{\pi} 1\cdot \cos mx\,dx = 0\ (m \neq 0), \quad \int_{-\pi}^{\pi} 1\cdot \sin mx\,dx = 0, \quad (2)$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0, \quad \int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0 \quad (3)$$

$$(\text{for } m \neq n),$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0 \quad (4)$$

(m and n any natural numbers).

Proof: follow the pattern of Examples 1 and 2.

Note 1. If in place of two distinct functions of system (1) we take two identical functions, then the integral between $-\pi$ and π is equal to π for all functions (1), except the first, for which it is double:

$$\int_{-\pi}^{\pi} 1\cdot 1\,dx = 2\pi, \quad (5)$$

$$\int_{-\pi}^{\pi} \cos^2 nx\,ax = \pi, \quad \int_{-\pi}^{\pi} \sin^2 nx\,dx = \pi\ (n = 1, 2, 3, \ldots) \quad (6)$$

Formulas (6) are obtained with the aid of the transformations

$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx), \qquad \sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

Note 2. Formulas (2) to (6) remain valid for any interval of length 2π. For example,

$$\int_{-\frac{\pi}{4}}^{1\frac{3}{4}\pi} \sin 4x \sin 2x \, dx = \int_{0}^{2\pi} \sin 4x \sin 2x \, dx = 0,$$

$$\int_{-2\pi}^{0} \cos^2 3x \, dx = \int_{\frac{\pi}{2}}^{\frac{5}{2}\pi} \cos^2 3x \, dx = \pi$$

Definition 2. If in some system of functions every two functions are orthogonal, then the system itself is termed *orthogonal*. By virtue of the theorem of this section, the system (1) is orthogonal in the interval $(-\pi, \pi)$ (and also in any interval of length 2π).

414. Euler-Fourier Formulas

Theorem. Let the trigonometric series

$$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$$
$$\ldots + a_n \cos nx + b_n \sin nx + \ldots \qquad (1)$$

converge for all values of x to some function $f(x)$ (this function is periodic with period 2π). If for this function (which may also be discontinuous) there exists a (proper or improper) integral $\int_{-\pi}^{\pi} |f(x)| \, dx$, then for the coefficients of the series (1) the following *Euler-Fourier formulas* hold true (see Sec. 411):

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx,$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x \, dx, \quad b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x \, dx,$$

$$a_2=\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos 2x\,dx,\quad b_2=\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin 2x\,dx,$$

$$a_3=\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos 3x\,dx,\quad b_3=\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin 3x\,dx,$$

. . . .

and, generally,

$$a_n=\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos nx\,dx,\quad b_n=\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (2)$$

Note. The expression for a_0 is obtained from the general formula for a_n if we put $n=0$ in the latter. This uniformity is upset if by a_0 we denote the constant term of series (1) and not its doubled magnitude. Cf. Sec. 411, Note 1.

Explanation. We have

$$f(x)=\frac{a_0}{2}+a_1\cos x+b_1\sin x+\ldots+a_n\cos nx+b_n\sin nx+\ldots \qquad (3)$$

Integrating this equation from $-\pi$ to π and assuming that the given series admits termwise integration,[1] we get

$$\int\limits_{-\pi}^{\pi} f(x)\,dx=\int\limits_{-\pi}^{\pi}\frac{a_0}{2}\,dx+a_1\int\limits_{-\pi}^{\pi}\cos x\,dx+b_1\int\limits_{-\pi}^{\pi}\sin x\,dx+\ldots \qquad (4)$$

All integrals on the right, except the first, are equal to zero by (2), Sec. 413, and we obtain

$$\int\limits_{-\pi}^{\pi} f(x)\,dx=\pi a_0,\quad \text{i. e. } a_0=\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\,dx$$

We obtained the first of formulas (2) for the case $n=0$, the remaining formulas are obtained by the same method if we first multiply (3) by $\cos nx$ or $\sin nx$.

---

[1] If the integral $\int\limits_{-\pi}^{\pi}|f(x)|\,dx$ is convergent, the trigonometric series (1), which converges to the function $f(x)$, admits term-by-term integration.

Thus, multiplying (3) by $\cos 2x$ and integrating term by term, we get

$$\int_{-\pi}^{\pi} f(x)\cos 2x\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos 2x\,dx + a_1\int_{-\pi}^{\pi}\cos x\cos 2x\,dx + $$

$$+ b_1\int_{-\pi}^{\pi}\sin x\cos 2x\,dx + a_2\int_{-\pi}^{\pi}\cos^2 2x\,dx + $$

$$+ b_2\int_{-\pi}^{\pi}\sin 2x\cos 2x\,dx + \qquad (5)$$

On the right, all the integrals are equal to zero except the fourth, by virtue of (2), (3), and (4), Sec. 413. The fourth one is equal to π by (6), Sec. 413. Hence

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx$$

Trigonometric series with arbitrary period. Let the trigonometric series, with period $2l$,

$$\frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l} + a_2\cos 2\frac{\pi x}{l} + b_2\sin 2\frac{\pi x}{l} + \ldots$$

$$\ldots + a_n\cos n\frac{\pi x}{l} + b_n\sin n\frac{\pi x}{l} + \ldots \qquad (6)$$

converge for all values of x to some function $f(x)$ (this function also has the period $2l$). If there exists a (proper or improper) integral $\int_{-l}^{l}|f(x)|\,dx$, then for the coefficients of the series (6) the following Euler-Fourier formulas hold:

$$\left.\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos n\frac{\pi x}{l}\,dx \quad (n=0,\ 1,\ 2,\ 3,\ldots), \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin n\frac{\pi x}{l}\,dx \quad (n=1,\ 2,\ 3,\ldots) \end{aligned}\right\} \qquad (7)$$

Formulas (2) are obtained from (7) for $l=\pi$.

415. Fourier Series

In Sec. 414 we considered the sum $f(x)$ of a *given* convergent trigonometric series Of practical importance is the following converse problem: given a function $f(x)$ with period

2π;[1] required to find the trigonometric series, convergent everywhere,

$$\frac{a_0}{2}+a_1\cos x+b_1\sin x+\ldots+a_n\cos nx+b_n\sin nx+\ldots \quad (1)$$

having the sum $f(x)$.

If this problem has a solution, then it is unique, and the coefficients of the required series (1) are found from the Euler-Fourier formulas (Sec. 414):

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad (2)$$

The series obtained is called the *Fourier series of the function $f(x)$.*

It may happen that the problem posed here *does not have* a solution: the Fourier series [even if the function $f(x)$ is continuous] may prove to be divergent at an infinity of points on the interval $(-\pi, \pi)$. Therefore, the relationship between the function $f(x)$ and its Fourier series is denoted as follows:

$$f(x)\sim\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+\ldots \quad (3)$$

avoiding the equals sign.

However, *for all continuous functions of practical importance* the problem has a solution, that is, the Fourier series of a continuous periodic function $f(x)$ actually turns out to be everywhere convergent and its sum is equal to the given function and not to any other one. This is evident from Sec. 416, where a sufficient condition is given for the expansibility of a continuous function in a Fourier series.

What is more, discontinuous periodic functions which are of practical importance can also be expanded in a Fourier series, but with one proviso: at the points of discontinuity of the function $f(x)$, its Fourier series can have a sum different from the corresponding value of the function itself (see Sec. 418).

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[1] It is assumed that for this function there exists a (proper or improper) integral $\int_{-\pi}^{\pi} |f(x)|\,dx$.

Note. Nonperiodic functions defined in the interval $(-\pi, \pi)$ can also be expanded in a Fourier series, but with the following proviso: exterior to the interval $(-\pi, \pi)$ and at its end-points the Fourier series of the function $f(x)$ will have a sum that, as a rule, will differ from the corresponding value of the function itself [this is natural, since the sum of a trigonometric series is a periodic function (see Sec. 417, Example 2)]. This is inessential, however, since we are interested in the values of the function only interior to the interval $(-\pi, \pi)$.

416. The Fourier Series of a Continuous Function

Theorem. Let a function $f(x)$ be continuous in a closed interval $(-\pi, \pi)$ and either have no extrema there or have a finite number of them.[1] Then the Fourier series of this function is everywhere convergent. Its sum is equal to $f(x)$ for any value of x interior to the interval $(-\pi, \pi)$. At both extremities the sum is equal to

$$\frac{1}{2}[f(-\pi)+f(\pi)]$$

i. e. the arithmetic mean between $f(-\pi)$ and $f(+\pi)$.

Example. Let us consider the function $f(x)=x$; it is continuous in the closed interval $(-\pi, \pi)$ and does not have any extrema. The coefficients $a_0, a_1, a_2, \ldots$ of its Fourier series are zeros. Indeed,

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx+$$

$$+\frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx \tag{1}$$

After the substitution $x=-x'$ the first term becomes $\frac{1}{\pi}\int_{\pi}^{0} x'\cos nx'\,dx'$ and, combined with the second, yields zero:

$$a_n=0, \quad (n=0,\ 1,\ 2,\ 3,\ \ldots) \tag{2}$$

---

[1] An instance of a continuous function having an infinite number of maxima and minima on a finite interval is $f(x)=x\sin\frac{1}{x}$ considered in any interval about the point $x=0$ (at this point the function is assigned the value 0; cf. Sec. 231).

The coefficients b_n are found by integration by parts:

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx=-\frac{1}{\pi n}x\cos nx\Big|_{-\pi}^{\pi}+\frac{1}{\pi n}\int_{-\pi}^{\pi}\cos nx\,dx=$$

$$=-\frac{2\pi\cos\pi n}{\pi n}=2(-1)^{n+1}\cdot\frac{1}{n} \qquad (3)$$

The Fourier series of the function x is of the form

$$2\left[\frac{1}{1}\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\ldots+\right.$$
$$\left.+\frac{(-1)^{n+1}}{n}\sin nx+\ldots\right] \qquad (4)$$

By the theorem, the series (4) converges everywhere; for $-\pi<x<\pi$ its sum is

$$2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots+(-1)^{n+1}\frac{\sin nx}{n}+\ldots\right]=x$$
$$(-\pi<x<\pi) \qquad (5)$$

For $x=\pm\pi$ the sum is

$$\frac{1}{2}[-\pi+\pi]=0 \qquad (6)$$

This is obvious because all the terms of the series vanish. For $x=\frac{\pi}{2}$ formula (5) yields the Leibniz series (Sec. 398)

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4} \qquad (7)$$

Fig. 422, which depicts the graph of the 5th partial sum of the Fourier series of the function $f(x)=x$

$$s_5=2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\frac{\sin 5x}{5}\right) \qquad (8)$$

gives some idea of how close the partial sum s_n of the series (4) inside the interval $(-\pi, \pi)$ is to the function $f(x)$ itself. The graph of $y=s_5(x)$ oscillates about the straight line $y=x$; for some values of x we get values that are too small, for others, values that are too large.

The curve $y=s_5(x)$ passes through the points $(-\pi, 0)$, $(\pi, 0)$ and therefore departs sharply from the straight line $y=x$ near these points.

The pattern is the same for subsequent partial sums s_n as well. But the size of the interval of sharp departure decreases unboundedly with increasing n. At the end-points of the interval $(-\pi, \pi)$ all partial sums are equal to zero and, hence, do not approach the values of the function $f(x)=x$ at the points $x=\pm\pi$. However, in any interior interval whose end-points do not coincide with the points

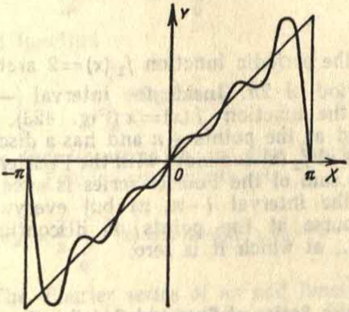

Fig. 422

$x=\pm\pi$, the series (4) converges (and converges uniformly) to the function $f(x)=x$. But the convergence is bad; thus, taking the value $x=\frac{\pi}{2}$, we get the series (7), which (by the Leibniz test, Sec. 376) converges very slowly.

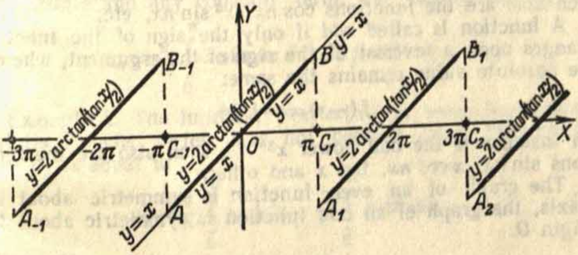

Fig. 423

Note 1. The function $f(x)=x$ is defined outside the interval $(-\pi, \pi)$ as well, but since it is not periodic, then for $x \geqslant \pi$ and for $x \leqslant -\pi$ the sum of the series (4) is not equal to x (cf. Sec. 415, Note). The graph of the sum of (4) consists (Fig. 423) of many segments obtained by horizontal displacement of the segment AB by the amount $\pm 2k\pi$ ($k=0$, 1, 2, 3, ...). All the segments $A_{-1}B_{-1}$, AB, A_1B_1, ... are devoid of end-points, which are replaced by the points C_{-1}, C_1, C_2, ... which bisect the segments $B_{-1}A$, BA_1, B_1A_2 and so forth.

Note 2.

Consider the periodic function $f_1(x)=2\arctan\left(\tan\frac{x}{2}\right)$; it has a period of 2π. Inside the interval $(-\pi, \pi)$ it coincides with the function $f(x)=x$ (Fig. 423). The function is not defined at the points $\pm\pi$ and has a discontinuity. The Fourier series of $f_1(x)$ coincides with the Fourier series of $f(x)$, and now the sum of the Fourier series is equal to $f_1(x)$ not only inside the interval $(-\pi, \pi)$ but everywhere as well, except of course at the points of discontinuity $x=\pm\pi$, $x=\pm 3\pi$, etc., at which it is zero.

417. The Fourier Series of Even and Odd Functions

Definition. Let the function $f(x)$ be defined in the interval $(-a, a)$. It is called *even* if the value of the function does not change upon a reversal of the sign of the argument:

$$f(-x)=f(x) \qquad (1)$$

Such is the even power x^{2m} (whence the term "even function"), such also are the functions $\cos nx$, $x^3 \sin nx$, etc.

A function is called *odd* if only the sign of the function changes upon a reversal of the sign of the argument, whereas the absolute value remains the same:

$$f(-x)=-f(x) \qquad (2)$$

An instance is the odd power x^{2m-1}, such also are the functions $\sin nx$, $x\cos nx$, $\tan x$ and others.

The graph of an even function is symmetric about the y-axis, the graph of an odd function is symmetric about the origin O.

Note 1. For an even function, the integrals $\int_{-a}^{0} f(x)\,dx$ and $\int_{0}^{a} f(x)\,dx$ are equal, for an odd function, they differ in sign. Therefore, for an even function we have

$$\int_{-a}^{a} f(x)\,dx=2\int_{0}^{a} f(x)\,dx \tag{3}$$

and for an odd function

$$\int_{-a}^{a} f(x)\,dx=0 \tag{4}$$

Note 2. The Fourier series of an even function does not contain sines; the Fourier coefficients are

$$a_n=\frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx, \quad b_n=0 \tag{5}$$

(cf. Note 1). The Fourier series of an odd function does not contain cosines and a constant term; the Fourier coefficients are

$$a_n=0, \quad b_n=\frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx \tag{6}$$

Example 1. The function $f(x)=x$ considered in the example of Sec. 416 is odd. Its Fourier series does not contain cosines and any constant term. The coefficients b_n are

$$b_n=\frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx=2(-1)^{n+1}\cdot\frac{1}{n}$$

Example 2. The function $f(x)=|x|$ is even; that means that its Fourier series does not contain sines. The coefficient a_0 is equal to

$$a_0=\frac{2}{\pi}\int_{0}^{\pi}|x|\,dx=\frac{2}{\pi}\int_{0}^{\pi}x\,dx=\pi \tag{7}$$

For $n\neq 0$ we get

$$a_n=\frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx=\frac{2}{\pi}\,x\,\frac{\sin nx}{n}\Big|_{0}^{\pi}-\frac{2}{n\pi}\int_{0}^{\pi}\sin nx\,dx=$$

$$=2\,\frac{\cos nx-1}{n^2\pi} \tag{8}$$

that is,

$$a_{2k}=0,\quad a_{2k-1}=-\frac{4}{(2k-1)^2\pi}\quad (k=1,\ 2,\ 3,\ \ldots)\tag{9}$$

Hence, the Fourier series of the function $f(x)=|x|$ will be

$$\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\ \ldots+\frac{\cos(2n-1)x}{(2n-1)^2}+\ldots\right)\tag{10}$$

The function $f(x)=|x|$ satisfies the hypothesis of the theorem of Sec. 416. Hence, the series (10) is everywhere convergent. Its sum is equal to $|x|$ for any value of x inside the interval $(-\pi,\ \pi)$. What is more, since the function $f(x)=|x|$ is even, the sum of its Fourier series is $f(x)$ at the extremities of the interval $(-\pi,\ \pi)$ as well. Indeed, for an even function we have $f(-\pi)=f(\pi)$ so that the arithmetic mean between the values $f(-\pi)$ and $f(\pi)$ coincides with each one of these values. Thus, we have

$$|x|=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\ldots\right)\quad(-\pi\leqslant x\leqslant\pi)\tag{10a}$$

In particular, substituting into (10a) one of the values $x=\pm\pi$ or $x=0$, we find that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}\tag{11}$$

The series (11) [and, generally, the series (10a)] converges poorly, though better than (4), Sec. 416 (cf. the graphs in Figs. 422 and 424).

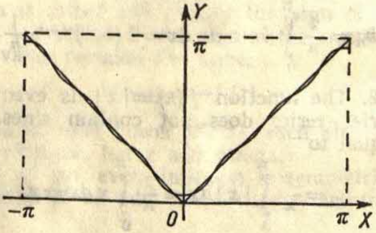

Fig. 424

Fig. 424 shows the graph of the partial sum s_4 of the series (10),

$$s_4=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}\right)$$

in the interval $(-\pi, \pi)$. The broken line about which the curve $y=s_4(x)$ oscillates is the graph of the sum $f_1(x)$ of the series (10). Fig. 425 depicts the graph of the sum $f_1(x)$ in the interval $(-3\pi, 3\pi)$. Also shown (as two rays emanating from the point O) is the graph of the function $f(x)=|x|$.

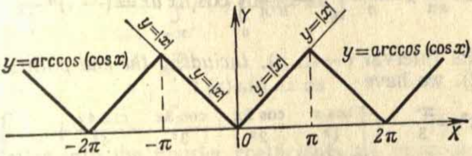

Fig. 425

The functions $f(x)$ and $f_1(x)$ coincide in the closed interval $(-\pi, \pi)$.

Note 3. The function $f_1(x)$ may be represented by the formula

$$f_1(x)=\arccos(\cos x)$$

Example 3. Expand the function $f(x)=x^2$ in a Fourier series (Fig. 426).

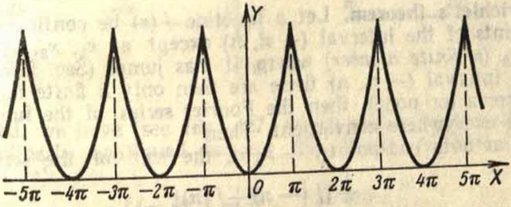

Fig. 426

Solution. The function is even, and so we have

$$a_0=\frac{2}{\pi}\int_0^\pi x^2\,dx=\frac{2\pi^2}{3}$$

To compute a_n for $n \neq 0$ we integrate by parts twice:

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx = \frac{2}{\pi} x^2 \frac{\sin nx}{n}\Big|_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x \sin nx\, dx =$$

$$= \frac{4}{n\pi} x \frac{\cos nx}{n}\Big|_0^{\pi} - \frac{4}{n^2\pi}\int_0^{\pi} \cos nx\, dx = (-1)^n \frac{4}{n^2} \qquad (12)$$

In the interval $(-\pi, \pi)$, *including the end-points* (cf. Example 2), we have

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right] \qquad (13)$$

For $x = \pi$ and $x = 0$, we get, respectively,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} + \ldots = \frac{\pi^2}{6}, \qquad (14)$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots + \frac{(-1)^{n-1}}{n^2} + \ldots = \frac{\pi^2}{12} \qquad (15)$$

Adding (14) and (15) term by term, we again get (11).

418. The Fourier Series of a Discontinuous Function

The theorem of Sec. 416 admits the following generalization.

Dirichlet's theorem. Let a function $f(x)$ be continuous at all points of the interval $(-\pi, \pi)$ except at $x_1, x_2, x_3, \ldots, x_k$ (*a finite number*) where it has jumps (Sec. 219a). If in the interval $(-\pi, \pi)$ there are then only a finite number of extrema (or none), then the Fourier series of the function $f(x)$ is everywhere convergent. Then[1]

(1) at both end-points, $-\pi$, π, the sum of the series is

$$\frac{1}{2}[f(-\pi) + f(\pi)] \qquad (1)$$

(2) at every point of discontinuity $x = x_i$ the sum of the series is

$$\frac{1}{2}[f(x_i - 0) + f(x_i + 0)] \qquad (2)$$

where the symbol $f(x_i - 0)$ denotes the limit to which $f(x)$

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[1] What follows may be stated more succinctly (see Note 2)

tends when x approaches x_i from the left and $f(x_i+0)$ denotes the limit of $f(x)$ as $x \to x_i$ from the right;

(3) at the remaining points of the interval $(-\pi, \pi)$ the sum of the series is equal to $f(x)$.

Note 1. The integrals

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx,$$

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

which enter into the Fourier coefficients are improper in the case at hand (Sec. 328).

Example. Consider the function $f(x)$ defined in the interval $(-\pi, \pi)$ as follows:

$$\left.\begin{aligned} f(x) &= -\frac{\pi}{4} \quad \text{for } -\pi \leqslant x < 0, \\ f(x) &= \frac{\pi}{4} \quad \text{for } \quad 0 \leqslant x \leqslant \pi \end{aligned}\right\} \tag{3}$$

This function is discontinuous at $x=0$, where it has a jump.

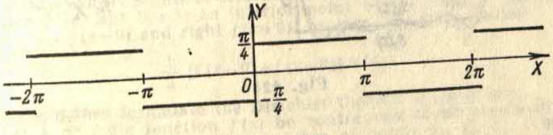

Fig. 427

Indeed, we have [see Fig. 427, which shows the function $f(x)$ periodically continued beyond the limits of the interval $(-\pi, \pi)$]:

$$f(-0) = -\frac{\pi}{4}, \quad f(+0) = \frac{\pi}{4} \tag{4}$$

We find the Fourier coefficients [the function $f(x)$ is odd]:

$$\left.\begin{aligned} a_n &= 0, \\ b_n &= \frac{2}{\pi} \int_0^{\pi} \frac{\pi}{4} \sin nx \, dx = \frac{1}{2n}[1-(-1)^n] \end{aligned}\right\} \tag{5}$$

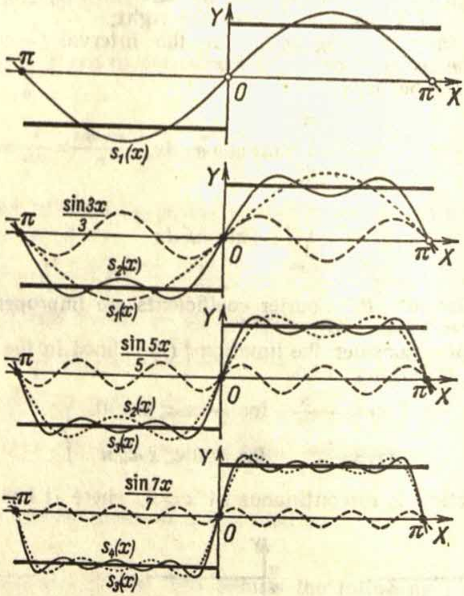

Fig. 428

Hence

$$\left.\begin{aligned} b_{2k-1} &= \frac{1}{2k-1}, \\ b_{2k} &= 0 \end{aligned}\right\} \quad (k = 1, 2, 3, \ldots) \tag{6}$$

At all interior points of the interval $(-\pi, \pi)$, except at the discontinuity $x = 0$, the sum of the Fourier series is equal to $f(x)$; that is, for $-\pi < x < 0$ we have

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin (2n-1)x}{2n-1} + \ldots = -\frac{\pi}{4} \tag{7}$$

and for $0 < x < \pi$ we have

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin (2n-1)x}{2n-1} + \ldots = \frac{\pi}{4} \tag{8}$$

At the discontinuity $x=0$, the sum of the Fourier series is

$$\frac{1}{2}\left(-\frac{\pi}{4}+\frac{\pi}{4}\right)=0$$

(all terms of the series are zeros). At the end-points of the interval $(-\pi, \pi)$ the sum is also

$$\frac{1}{2}\left(-\frac{\pi}{4}+\frac{\pi}{4}\right)=0$$

Fig. 428 shows how the partial sums $s_1(x)$, $s_2(x)$, $s_3(x)$, $s_4(x)$ gradually approach $f(x)$. The top band shows the graph of $s_1(x)$; the next from the top depicts the graph of $s_2(x)$ as a solid line:

$$s_2(x)=s_1(x)+\frac{\sin 3x}{3}$$

Here also (dashed line) is the graph of $\frac{\sin 3x}{3}$ and likewise (dotted line) the graph of $s_1(x)$. This is followed (below) by the graph of $s_3(x)$, the dotted and dashed lines depicting $s_2(x)$ and $\frac{\sin 5x}{5}$. The bottom graph is similarly constructed.

Note 2. Items 1 and 3 of the Dirichlet theorem are actually particular cases of Item 2. Indeed, if $f(-\pi)\neq f(\pi)$, then the end-points of the interval are points of discontinuity of the periodically continued function $f(x)$. But if x is an interior point of continuity, then both limits, left $f(x-0)$ and right $f(x+0)$, are equal to $f(x)$, so that

$$\frac{1}{2}[f(x-0)+f(x+0)]=f(x) \tag{9}$$

We can thus formulate the Dirichlet theorem more briefly:

Let a periodic function $f(x)$ be continuous at all points of the interval $(-\pi, \pi)$ except a finite number of points $x_1, x_2, x_3, \ldots, x_k$, where it has jumps. If in the interval $(-\pi, \pi)$ there are then only a finite number of extrema (or none at all), then the Fourier series of the function $f(x)$ is everywhere convergent and its sum is everywhere equal to

$$\frac{1}{2}[f(x-0)+f(x+0)]$$

DIFFERENTIATION AND INTEGRATION OF FUNCTIONS OF SEVERAL VARIABLES

419. A Function of Two Arguments

Definition. A quantity z is called a *function of two variable quantities* x and y if every pair of numbers that may (by the conditions of the problem) be the values of the variables x and y is associated with one or several definite values of z. The variables x and y are called *arguments* (cf. Sec. 196, Definition 1).

Single-valued and multiple-valued functions are distinguished as in Definition 2, Sec. 196.

Example 1. The height h (above sea-level) of a point on the earth's surface is a function of the geographic coordinates of latitude φ and longitude ψ. The latitude can vary between $-90°$ and $+90°$, the longitude, between $-180°$ and $+180°$.

Example 2. The product of two factors x and y is a function of the two arguments x and y. The values of the arguments x and y may be arbitrary.

Number plane. For pictorialness, the pair of values x, y can be depicted geometrically by a point $M(x, y)$ referred to a rectangular coordinate system XOY. The plane embodying this system is called the *number plane*.

The expression "point $M(x, y)$" is equivalent to the expression "the pair of values of the arguments x and y". For example, the expression "the point $M(1, -3)$" is the same as "the pair of values $x=1$, $y=-3$". Accordingly, a function of two variables is called a *point function* (see Sec. 457). It often happens that the value of a function is physically determined by the choice of a point in a plane or on a curved surface (cf. Example 1).

Domain of definition of a function. All pairs of those numbers which (by the conditions of the problem) can be values of the arguments x, y of the function $f(x, y)$ constitute the *domain of definition* of the function.

Geometrically, the domain of definition is depicted as some collection of points in the xy-plane.

In Example 1, the domain of definition of the function h of the arguments φ and ψ is the set of points of the number plane lying within and on the boundary of some rectangle which has 360 scale units in length and 180 in width, whose sides are parallel to the coordinate axes and whose centre coincides with the origin. In Example 2, the domain of definition of the function is the entire number plane.

Notations. The notation

$$z=f(x, y)$$

(read: "z equals f of x, y") means that z is a function of the two variables x and y. The notation $f(3, 5)$ means that we consider the value of the function $f(x, y)$ at the point $M(3, 5)$; it is that value of the function which corresponds to the values of the arguments $x=3$, $y=5$ (see Sec. 202). Other letters may be used in place of f.

Sometimes the same letter is used for the function symbol as is used to denote the function itself, that is, we write $z=z(x, y)$, $w=w(u, v)$, and so on.

Note. It may happen that the value of the function $f(x, y)$ varies with x but remains the same when the argument y varies. Then the function of the two arguments may be regarded as a function of one argument (x). If the value of $f(x, y)$ remains the same for any values of the two arguments, then the function of the two arguments is a constant quantity.

Example 3. The daily amount of precipitation (h millimetres) within Moscow Region is a function of the latitude φ and the longitude ψ of the point of observation. However it may happen that the daily amount of precipitation remains constant from south to north and only varies from east to west. Then h may be regarded as a function of the one argument ψ.

If there has been no precipitation throughout the region during one 24-hour period, then h is a constant (equal to zero).

420. A Function of Three and More Arguments

The concepts of a function of three, four, etc., arguments and the domain of its definition are introduced in the same way as for the case of two arguments (Sec. 419).

The domain of definition of a function of three arguments may be depicted as a certain set of points in space. Accordingly, a function of three variables (and, by analogy, of a greater number of variables) is called a *point function.*

Notaton:

$$u=f(x, y, z)$$

signifies that u is a function of three arguments: x, y, z

Note. It may happen that the value of the function $f(x, y, z)$ varies with x and y but remains the same when z varies. Then the function of three variables $f(x, y, z)$ is, at the same time, a function of two variables: x, y. A function $f(x, y, z)$ can also be a function of a single variable; it can even be a constant (cf. Sec. 419, Note).

Generally, a function of n variables may prove to be a function of a smaller number of variables.

421. Modes of Representing Functions of Several Arguments

1. A function of two or more arguments may be specified by a *formula* (or several formulas). A function given by a formula may be *explicit* or *implicit* (cf. Sec. 197, Item c).

Example 1. The formula

$$pv = A(273.2 + t) \tag{1}$$

where $A = 0.02927$, expresses a relationship between the volume v of one kilogram of air (in cubic metres), its pressure p $\left(\text{in } \frac{\text{tons}}{\text{m}^2}\right)$ and its temperature t (in degrees Celsius). Each of the variables p, v, t is an implicit function of the other two.

The formula

$$v = \frac{A(273.2 + t)}{p} \tag{2}$$

specifies v as an explicit function of the two arguments p and t. The domain of definition of this function is the collection of physically possible values of the pressure and temperature (t can take on only those values which exceed $-273°$, p, only positive values).

Note. A function of several arguments is frequently represented by a formula without any indication of the physical meaning of the quantities involved. If there are no indications about the domain of definition of the function, then it is assumed that the domain embraces all those points for which the formula is meaningful.

Example 2. Let a function of the two arguments x, y be given by the formula

$$z = \sqrt{R^2 - (x^2 + y^2)} \tag{3}$$

without any indication of the domain of the function. Formula (3) is meaningful only when $x^2 + y^2 \leqslant R^2$. Hence, the domain is the collection of all points lying inside and on the boundary of a circle of radius R with centre at the coordinate origin.

Example 3. The formula $u=\sqrt{a^2-(x^2+y^2+z^2)}$ specifies a function of three variables. The formula is meaningful provided that $x^2+y^2+z^2 \leqslant a^2$; the domain of definition is the collection of points lying inside and on the surface of a sphere of radius a with centre at the origin.

2. A function of two or more arguments may be represented by a *table*. In the case of two arguments, the table is conveniently arranged in the form of a rectangle. The values of one of the arguments are specified in the top row, the values of the other in the left column. The value of the function is read at the intersection of the appropriate row and column *(table of double entry)*.

Example 4. The following table gives the volume of 1 kg of air as a function of pressure and temperature (see Example 1):

t° $p\ \frac{\text{tons}}{\text{m}^2}$	−20	−10	0	10	20
10.0	0.7411	0.7704	0.7997	0.8289	0.8582
10.1	0.7338	0.7628	0.7918	0.8207	0.8497
10.2	0.7266	0.7553	0.7840	0.8126	0.8414
10.3	0.7195	0.7480	0.7764	0.8048	0.8332
10.4	0.7126	0.7408	0.7689	0.7970	0.8252
10.5	0.7058	0.7337	0.7616	0.7894	0.8173

3. A function of two arguments may be represented by a *spatial model (bar diagram)*. A spatial model of a function $f(x, y)$ is some surface S referred to a rectangular coordinate system $OXYZ$; the projection of a point M of the surface S on the xy-plane serves to represent pairs of values of the arguments x, y, the z-coordinate of M depicts the corresponding value of the function $f(x, y)$.

This method is not applicable to a function of three and more arguments.

Example 5. A function specified by the formula

$$z=\sqrt{a^2-x^2-y^2}$$

is represented by a hemisphere (Fig. 429; cf. Example 2).

4. A function of two variables may be represented in the plane by means of *level curves* (*level-curve representation*). A pair of values of x and y is depicted by a point $M(x, y)$ and the value of z by a numerical label. (This method is employed in cartography to indicate altitude.) Points of the same value of z are connected by a line (a level curve — contour line — with the elevation indicated). When a point (x, y) lies on one of the level curves, the value of the function is read off in straightforward fashion; if it does not lie on one, we take the two closest level curves between which the point (x, y) lies and interpolate by eye.

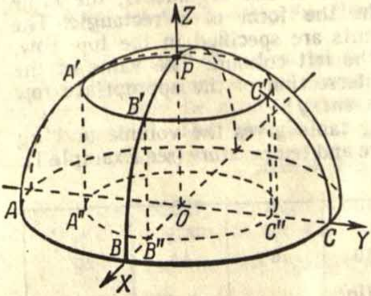

Fig. 429

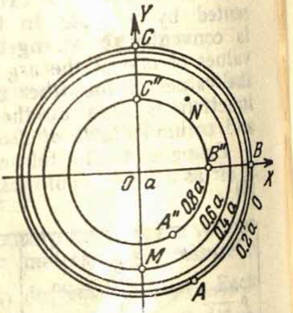

Fig. 430

Example 6. Fig. 430 gives the level curves of the function $z=\sqrt{a^2-x^2-y^2}$ which correspond to function increases of $0.2a$ $(OB=a)$. The value of z at the point $M(0, -0.8a)$ is read from the label $0.6a$. To find the value of z at the point $N\left(\frac{1}{2}a, \frac{1}{2}a\right)$, take the labels $0.6a$ and $0.8a$ for the nearest level curves. Since N is roughly midway between the curves, $z \approx 0.7a$.

Note. If we cut the surface $z=f(x, y)$ with a plane $z=k$ and project the section on the xy-plane, we get a level curve with the label k. Thus, if we cut a hemisphere $z=\sqrt{a^2-x^2-y^2}$ with a plane $z=0.8a$, we get a section $A'B'C'$ (Fig. 429). Its projection $A''B''C''$ (Figs. 429 and 430) on the xy-plane is a level curve with label $0.8a$.

Similarly, a function of three variables $u=f(x, y, z)$ may be represented by the level-curve method in space. The role of level curves is played here by *level surfaces*.

422. The Limit of a Function of Several Arguments

The concept of a limit of a function of several arguments is established in the same way as for a function of one argument. For definiteness, consider the case of a function of two arguments.

The number l is called the limit of the function $z=f(x, y)$ at the point $M_0(a, b)$ if z approaches l without bound every time that the point $M(x, y)$ approaches M_0 without bound (cf. Sec. 204).

Notation:

$$\lim_{M \to M_0} f(x, y)=l$$

or

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y)=l$$

Note 1. It is assumed that the function $f(x, y)$ is defined at all points not coincident with M_0 inside a circular neighbourhood of M_0; at the point M_0 itself the function $f(x, y)$ is either defined or is not defined (cf. Sec. 204, Note 1).

Note 2. The mathematical meaning of the expression "approaches without bound" becomes clear from the following exact definition.

Definition. A number l is called the *limit* of a function $f(x, y)$ at a point $M_0(a, b)$ if the absolute value of the difference $f(x, y)-l$ remains less than any preassigned positive number ε whenever the distance $M_0M=\sqrt{(x-a)^2+(y-b)^2}$ from point $M_0(a, b)$ to point $M(x, y)$ (distinct from M_0) is less than some positive number δ (dependent on ε).

Geometrical meaning. The z-coordinate of a surface $z=f(x, y)$ differs from l by less than ε whenever the projection of a point lying on the surface falls within a circle of radius δ with centre at the point $M_0(a, b)$.

Note 3. For the case of a function of three arguments, $f(x, y, z)$, the distance M_0M is represented by the expression $\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$. For the case of four arguments, when a geometrical interpretation of the expression $\sqrt{(x-a)^2+(y-b)^2+(z-c)^2+(u-d)^2}$ becomes impossible, it is still called (by analogy) the *distance* between the points $M(x, y, z, u)$ and $M_0(a, b, c, d)$.

The concept of an infinitely small and infinitely large quantity is established in the same way as for a function of one argument (Secs. 207, 208). The order of smallness is discussed in Sec. 423. The concept of a limit is extended as in Sec. 211.

423. On the Order of Smallness of a Function of Several Arguments

In Sec. 217, when comparing two infinitesimal functions α and β of a single argument, we distinguished the following cases:

(1) the ratio $\frac{\alpha}{\beta}$ has a nonzero finite limit; then the infinitesimals α and β are of the same order;

(2) $\lim \frac{\alpha}{\beta}=0$; then α is of higher order than β;

(3) $\lim \frac{\alpha}{\beta}=\infty$; then α is of lower order than β;

(4) the ratio $\frac{\alpha}{\beta}$ has no limit; then α and β cannot be compared.

Case 4 is an exceptional case in the study of elementary functions of a single argument. For functions of two and more arguments, *Case 1 is exceptional*. Cases 2, 3, and 4 are of practical significance.

Thus, the ratio of two infinitesimal functions of several arguments typically has no limit (see Example 1). In other cases, one of the two infinitesimals (say α) is of higher order than the other (see Examples 2 and 3). Then the second is of lower order than the first.

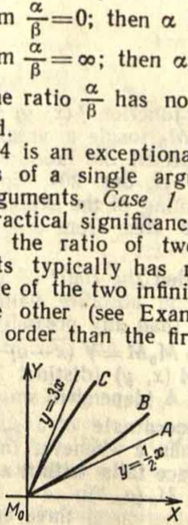

Fig. 431

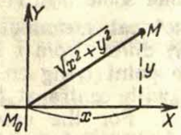

Fig. 432

Example 1. The quantities $2x^2+y^2$ and x^2+y^2 are infinitesimal as $x \longrightarrow 0$, $y \longrightarrow 0$, but their ratio has no limit.

Indeed, the point $M(x, y)$ can tend to $M_0(0, 0)$ along a line tangent, at the point M_0, to the straight line $y=\frac{1}{2}x$ (the line BM_0 in Fig. 431), or to the straight line $y=3x$, or to the straight line $y=x$, etc. In the first case, the ratio $\frac{y}{x}$ tends to $\frac{1}{2}$, in the second, to 3, in the third, to 1. Hence, the ratio

$$(2x^2+y^2):(x^2+y^2)=\left[2+\left(\frac{y}{x}\right)^2\right]:\left[1+\left(\frac{y}{x}\right)^2\right]$$

tends to $\frac{9}{5}$ in the first case, to $\frac{11}{10}$ in the second, to $\frac{3}{2}$ in the third, etc.

Note. The infinitesimal x^2+y^2 is the square of the distance M_0M between the point M_0 and the point M which is approaching $M_0(0, 0)$. Generally, the case when one of the infinitesimals being compared is some power of the distance between M and its limit M_0 is particularly important (cf. Secs. 430, 444).

Example 2. The function $2x^2-y^2$ is of higher order than the distance

$$MM_0=\sqrt{x^2+y^2}$$

as $M \longrightarrow M_0(0, 0)$. Indeed, the ratio $(2x^2-y^2):\sqrt{x^2+y^2}$ is transformed as follows:

$$\frac{2x^2-y^2}{\sqrt{x^2+y^2}}=2x\frac{x}{\sqrt{x^2+y^2}}-y\frac{y}{\sqrt{x^2+y^2}} \tag{1}$$

Neither of the quantites $\frac{x}{\sqrt{x^2+y^2}}$, $\frac{y}{\sqrt{x^2+y^2}}$ exceeds (in absolute value) unity (see Fig. 432), but each of the quantities $2x$, y tends to zero. Consequently, both terms on the right of (1) tend to zero. Hence, $(2x^2-y^2):\sqrt{x^2+y^2}$ tends to zero as well.

Example 3. The function $f(x, y)=(x-x_0)^2(y-y_0)^2$ is of higher order than the square of the distance MM_0, that is, than $(x-x_0)^2+(y-y_0)^2$. Indeed,

$$\frac{f(x, y)}{MM_0^2}=(x-x_0)\frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

The first factor tends to zero and neither one or the other of the two exceeds unity (cf. Example 2).

424. Continuity of a Function of Several Arguments

Definition 1. A function $f(x, y)$ is called *continuous at a point* $M_0(x_0, y_0)$ if the following two conditions are fulfilled:

(1) the function has a definite value l at M_0,

(2) the function has a limit, also equal to l, at M_0.

If even one of these conditions is violated, the function is called *discontinuous at the point* M_0.

The same holds for the case of three and more arguments.

Definition 2. A function $f(x, y)$ is called *continuous in some region* if it is continuous at every point of the region.

Example 1. A function $f(x, y)$ specified by the formulas

$$f(0, 0)=0,$$
$$f(x, y)=\frac{2x^2-y^2}{\sqrt{x^2+y^2}} \quad (x^2+y^2 \neq 0)$$

is continuous at the point $M_0(0, 0)$. Indeed, at M_0 it has the value zero; besides, it has a limit here, which is also equal to zero (cf. Example 2, Sec. 423). At all the remaining points of the number plane the function $f(x, y)$ is also continuous. It is therefore continuous in any region.

Example 2. A function $\varphi(x, y)$ given by the formulas

$$\varphi(0, 0)=0,$$
$$\varphi(x, y)=\frac{2x^2+y^2}{x^2+y^2} \quad (x^2+y^2 \neq 0)$$

is discontinuous at the point $M_0(0, 0)$. The first condition of Definition 1 is fulfilled. But the second one is not: the function $\varphi(x, y)$ has no limit as $M \longrightarrow M_0$ (see Example 1, Sec. 423).

425. Partial Derivatives

Definition. A *partial derivative* of a function $u=f(x, y, z)$ with respect to the argument x is the limit of the ratio

$$\frac{f(x+\Delta x, y, z)-f(x, y, z)}{\Delta x} \quad \text{as } \Delta x \longrightarrow 0$$

Notations:

$$u'_x, \quad f'_x(x, y, z), \quad \frac{\partial u}{\partial x}, \frac{\partial f(x, y, z)}{\partial x}. \tag{1}$$

For the meanings of the symbols ∂u, ∂x see Sec. 429.

Note 1. In the process of finding the limit, the arguments x, y, z are held constant; the resulting partial derivative is a function of x, y, z (cf. Sec. 224).

The partial derivatives with respect to the arguments y and z are defined and denoted similarly, for example,

$$u'_y=\frac{\partial u}{\partial y}=f'_y(x, y, z)=\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z)-f(x, y, z)}{\Delta y} \tag{2}$$

Note 2. To find the partial derivative u_x, it is sufficient to find the ordinary derivative of the variable u, considering

u as a function of the argument x alone. If it is necessary to find all three partial derivatives, it is more practical to use the method given in Sec. 438.

Example. Find the values of the partial derivatives of the function

$$u=f(x, y, z)=2x^2+y^2-3z^2-3xy-2xz \tag{3}$$

at the point $M_0(0, 0, 1)$.

Solution. Considering u as a function of the argument x alone, we find that its derivative $\frac{\partial u}{\partial x}$ is equal to $4x-3y-2z$. At the point (0, 0, 1) the value of this derivative is -2.

Notation:

$$f'_x(0, 0, 1)=4x-3y-2z\,|_{x=0,\ y=0,\ z=1}=-2,$$

$$f'_y(0, 0, 1)=2y-3x\,|_{x=0,\ y=0,\ z=1}=0,$$

$$f'_z(0, 0, 1)=-6$$

426. A Geometrical Interpretation of Partial Derivatives for the Case of Two Arguments

Let point $M_0(x_0, y_0)$ (Fig. 433) be associated with point N_0 of the surface $z=f(x, y)$ (Sec. 421). Draw through N_0 a plane N_0M_0U parallel to the xz-plane. At the intersection, we get the line L_1N_0 along which y remains constant ($y=y_0$). The z-coordinate of the line L_1N_0 is a function of the argument x alone. The partial derivative $f'_x(x_0, y_0)$ is numerically equal to the slope of the tangent line UN_0, that is, to the tangent of the angle M_0UN_0 formed by the tangent line US with the coordinate xy-plane XOY.

Drawing the plane N_0M_0V parallel to YOZ, we get the section L_2N_0. The partial derivative $f'_y(x_0, y_0)$ is equal to the tangent of the angle M_0VN_0 formed by the tangent line VT and the xy-plane.

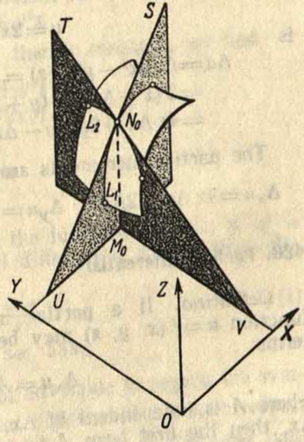

Fig. 433

427. Total and Partial Increments

Let us take some values x_0, y_0, z_0 of the arguments x, y, z and increment them arbitrarily: Δx, Δy, Δz. The function $u=f(x, y, z)$ will then receive the *total increment*

$$\Delta u=\Delta f(x, y, z)=f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)-f(x_0, y_0, z_0)$$

It may happen that the increments Δy, Δz are equal to zero, that is, y and z remain unchanged; then the function $f(x, y, z)$ receives a *partial increment*

$$\Delta_x u=\Delta_x f(x, y, z)=f(x_0+\Delta x, y_0, z_0)-f(x_0, y_0, z_0)$$

Similarly we obtain the partial increments

$$\Delta_y u=\Delta_y f(x, y, z)=f(x_0, y_0+\Delta y, z_0)-f(x_0, y_0, z_0),$$
$$\Delta_z u=\Delta_z f(x, y, z)=f(x_0, y_0, z_0+\Delta z)-f(x_0, y_0, z_0)$$

Note. For the case of two arguments, the total increment of a function is geometrically portrayed as the increment of the z-coordinate M_0N_0 (Fig. 433) for an arbitrary displacement of a point N_0 along the surface $z=f(x, y)$. The partial increment $\Delta_x f(x, y)$ is obtained in a displacement of N_0 along the section L_1N_0, the partial increment $\Delta_y f(x, y)$, in a displacement along L_2N_0.

Example. The total increment of the function

$$u=2x^2-y^2-z$$

is

$$\begin{aligned}\Delta u&=\Delta(2x^2-y^2-z)=\\&=2(x+\Delta x)^2-(y+\Delta y)^2-(z+\Delta z)-2x^2+y^2+z=\\&=4x\,\Delta x-2y\,\Delta y-\Delta z+2\Delta x^2-\Delta y^2\end{aligned}$$

The partial increments are

$$\Delta_x u=4x\,\Delta x+2\Delta x^2,\quad \Delta_y u=-2y\,\Delta y-\Delta y^2,\quad \Delta_z u=-\Delta z$$

428. Partial Differential

Definition. If a partial increment $\Delta_x u$ (Sec. 427) of a function $u=f(x, y, z)$ may be partitioned into a sum of two terms:

$$\Delta_x u=A\,\Delta x+\alpha \tag{1}$$

where A is independent of Δx, and α is of higher order than Δx, then the first term $A\,\Delta x$ is called the *partial differential* of the function $f(x, y, z)$ with respect to the argument x and is denoted $d_x f(x, y, z)$ or $d_x u$:

$$d_x u = d_x f(x, y, z) = A\,\Delta x \tag{2}$$

In other words, a partial differential is the differential (Sec. 228) of a function $f(x, y, z)$ taken on the assumption that the quantities y and z do not vary ($\Delta y = \Delta z = 0$). On this assumption, x is the sole argument, and for this reason in place of Δx we can write dx (cf. Sec. 234) so that

$$d_x u = d_x f(x, y, z) = A\,dx$$

The partial differentials $d_y f(x, y, z)$, $d_z f(x, y, z)$ with respect to the arguments y and z are defined similarly.

The coefficient A is equal to the partial derivative u'_x, i.e. the partial differential of a function is equal to the product of the corresponding partial derivative by the increment of the argument (Sec. 228, Theorem 1)

$$d_x u = u'_x dx \tag{3}$$

Similarly,

$$d_y u = u'_y dy, \tag{4}$$

$$d_z u = u'_z dz \tag{5}$$

Example. Find the partial differentials of the function

$$u = x^2 y + y^2 x$$

Solution. Holding first y and then x constant, we find

$$d_x u = (2xy + y^2)\,dx,$$
$$d_y u = (x^2 + 2xy)\,dy$$

429. Expressing a Partial Derivative in Terms of a Differential

The partial derivative u_x of the function $u = f(x, y, z)$ is equal to the ratio of the partial differential $d_x u$ to the differential dx:

$$u'_x = \frac{d_x u}{dx} \tag{1}$$

This follows from Sec. 428 (cf. Sec. 235).

In the notation $\frac{\partial u}{\partial x}$, it is not advisable to regard the symbol ∂u in the sense of the *partial differential* $d_x u$ with respect to the argument x because in the notation $\frac{\partial u}{\partial y}$ *the same sym-*

bol ∂u would have to be understood as the partial differential $d_y u$, and in the notation $\frac{\partial u}{\partial z}$, as $d_z u$.

For this reason, the expression $\frac{\partial u}{\partial x}$ should be regarded as *inseparable symbol* of a partial derivative (and not as a ratio of differentials).

Example. Let $u=xy$; then $x=\frac{u}{y}$ and $y=\frac{u}{x}$. We have

$$\frac{\partial u}{\partial x}=y, \quad \frac{\partial x}{\partial y}=-\frac{u}{y^2}, \quad \frac{\partial y}{\partial u}=\frac{1}{x}$$

Whence we find

$$\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial u}=y\cdot\left(-\frac{u}{y^2}\right)\cdot\frac{1}{x}=-\frac{u}{xy}=-1$$

If we regarded the symbols ∂u, ∂x, ∂y as independent quantities, we would have the erroneous result $+1$ in place of -1.

430. Total Differential

Definition. Suppose the total increment $\Delta f(x, y, z)$ (Sec. 427) of the function $f(x, y, z)$ can be partitioned into a sum of two terms:

$$\Delta f(x, y, z)=(A\,\Delta x+B\,\Delta y+C\,\Delta z)+\varepsilon \tag{1}$$

where none of the coefficients A, B, C is dependent either on Δx, or Δy, or Δz, and the quantity ε (regarded as a function of Δx, Δy, Δz) is of higher order (Sec. 423) than the distance $\rho=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$.

Then the first term

$$A\,\Delta x+B\,\Delta y+C\,\Delta z \tag{2}$$

is called the *total differential* of the function $f(x, y, z)$ (or, simply, the *differential*) and is denoted by $df(x, y, z)$ (cf. Secs. 228, 428).

Example 1. Let us take the function

$$f(x, y, z)=2x^2-y^2-z \tag{3}$$

We have (Sec. 427, Example)

$$\Delta f(x, y, z)=(4x\,\Delta x-2y\,\Delta y-\Delta z)+(2\Delta x^2-\Delta y^2)$$

The coefficients $A=4x$, $B=-2y$, $C=-1$ are not dependent either on Δx, or Δy, or Δz, the quantity $\varepsilon=2\Delta x^2-\Delta y^2$ is of higher order than $\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$ (cf. Sec. 423, Example 2). Hence, the expression $4x\,\Delta x-2y\,\Delta y-\Delta z$ is the total differential of the function $2x^2-y^2-z$:

$$d(2x^2-y^2-z)=4x\,\Delta x-2y\,\Delta y-\Delta z \tag{4}$$

Theorem. The coefficients A, B, C are equal, respectively, to the partial derivatives of the function $f(x, y, z)$:

$$A=f'_x(x, y, z), \quad B=f'_y(x, y, z), \quad C=f'_z(x, y, z) \tag{5}$$

In other words, *the total differential is equal to the sum of the partial differentials* (Sec. 428):

$$df(x, y, z)=d_xf(x, y, z)+d_yf(x, y, z)+d_zf(x, y, z) \tag{6}$$

or

$$df(x, y, z)=f'_x(x, y, z)\,\Delta x+f'_y(x, y, z)\,\Delta y+f'_z(x, y, z)\,\Delta z \tag{7}$$

Example 2. In formula (4) the coefficients $A=4x$, $B=-2y$, $C=-1$ are partial derivatives of the function $2x^2-y^2-z$ with respect to the arguments x, y, z

$$\left.\begin{aligned} 4x&=\frac{\partial}{\partial x}(2x^2-y^2-z),\\ -2y&=\frac{\partial}{\partial y}(2x^2-y^2-z),\\ -1&=\frac{\partial}{\partial z}(2x^2-y^2-z) \end{aligned}\right\} \tag{8}$$

Note 1. By virtue of formula (7), the total differentials dx, dy, dz of the arguments x, y, z are, respectively, equal to Δx, Δy, Δz. We therefore have

$$df(x, y, z)=f'_x(x, y, z)\,dx+f'_y(x, y, z)\,dy+f'_z(x, y, z)\,dz \tag{9}$$

For instance (cf. Example 1),

$$d(2x^2-y^2-z)=4x\,dx-2y\,dy-dz \tag{10}$$

Formula (9) is invariant (see Sec. 432) and therefore is to be preferred to (7).

Note 2. If u is a function of one argument, the total differential turns into an ordinary differential and the sole partial derivative into an ordinary derivative.

431. Geometrical Interpretation of the Total Differential (for the Case of Two Arguments)

Let a plane P be tangent (Sec. 435) at a point $M(x, y, z)$ of a surface S depicting the function $z=f(x, y)$ (Item 3, Sec. 421). Displace the projection $M_0(x, y, 0)$ of point M to the position $M_1(x+\Delta x, y+\Delta y, 0)$. Then the z-coordinate of the *tangent plane* will receive an increment equal to the total differential:

$$dz=f'_x(x, y)\Delta x+f'_y(x, y)\Delta y \tag{1}$$

The corresponding increment in the z-coordinate of the *surface* S is equal to the total increment Δz of the function $z=f(x, y)$.

Hence (Sec. 430, Definition), the distance between the surface S and the tangent plane P (reckoned in the direction of the z-coordinate) is of higher order than the distance

$$\rho=M_0M_1=\sqrt{\Delta x^2+\Delta y^2}$$

(cf. Sec. 230).

432. Invariance of the Expression $f'_x dx+f'_y dy+f'_z dz$ of the Total Differential

The expression $f'_x\Delta x+f'_y\Delta y+f'_z\Delta z$ (Sec. 430) is the total differential of the function $u=f(x, y, z)$ if x, y, z are regarded as arguments.[1] But if the variables x, y, z are themselves functions of one, two, or more arguments, then this expression is not, as a rule, a differential. On the contrary, the expression

$$f'_x\,dx+f'_y\,dy+f'_z\,dz$$

is always[1] the total differential of the function $f(x, y, z)$ (cf. Sec. 234).

Example 1. Let us consider the function $u=xy$ of the arguments x, y. We have

$$du=u'_x\,dx+u'_y\,dy=y\,dx+x\,dy \tag{1}$$

This formula also holds true when x, y are functions of the arguments t, s given by the formulas

$$x=t^2+s^2, \quad y=t^2-s^2 \tag{2}$$

---

[1] It is assumed that the total differential exists. See Sec. 434 on functions having partial derivatives but with no total differential.

Indeed, in this case we have

$$u=t^4-s^4, \tag{3}$$

$$du=u'_t\,dt+u'_s\,ds=4t^3\,dt-4s^3\,ds \tag{4}$$

We obtain the same result using formula (1) if in place of x, y we substitute the expressions (2) and in place of dx, dy, the expressions

$$dx=2t\,dt+2s\,ds, \quad dy=2t\,dt-2s\,ds \tag{5}$$

which are found with the aid of formulas (2). But if in place of (1) we take the formula

$$u=y\,\Delta x+x\,\Delta y \tag{6}$$

it will be incorrect for the arguments t, s.

Example 2. Formula (1) also holds when x and y are functions of one argument.

Example 3. The formula $d\arctan x=\frac{dx}{1+x^2}$ holds true if we put $x=rst$:

$$d\arctan rst=\frac{d(rst)}{1+r^2s^2t^2}$$

433. The Technique of Differentiation

In most cases, when seeking partial derivatives, it is convenient first to find the total differential, which is computed by the same rules as the differential of a function of one argument (cf. Sec. 432 and Sec. 430, Note 2).

Example 1. Find the partial derivatives of the function

$$u=\arctan\frac{y}{x}$$

Solution. We compute the total differential by the rules of Secs. 247 and 240. This yields

$$du=\frac{d\frac{y}{x}}{1+\frac{y^2}{x^2}}=\frac{x\,dy-y\,dx}{x^2+y^2} \tag{1}$$

The coefficients of dx, dy are the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$. Therefore

$$\frac{\partial u}{\partial x}=-\frac{y}{x^2+y^2}, \quad \frac{\partial u}{\partial y}=\frac{x}{x^2+y^2} \tag{2}$$

Direct computation of the derivatives would have required a great deal of work and attention.

Example 2. Find the partial derivatives of the function $u=\ln\sqrt{x^2+y^2}$.

Solution.

$$d\ln\sqrt{x^2+y^2}=\frac{1}{2}\,d\ln(x^2+y^2)=\frac{x\,dx+y\,dy}{x^2+y^2},\tag{3}$$

$$\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2},\qquad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}\tag{4}$$

When differentiating a function of one argument, it is sometimes convenient to take advantage of the total differential of a function of two, three, etc. arguments.

Example 3. Find the differential of the function $u=x^x$.

Solution. We seek dy^z (y and z are independent variables); to do this we first find the partial derivatives and then set $y=x$, $z=x$,

$$dy^z=\frac{\partial u}{\partial y}\,dy+\frac{\partial u}{\partial z}\,dz=zy^{z-1}\,dy+y^z\ln y\,dz,\tag{5}$$

$$dx^x=xx^{x-1}\,dx+x^x\ln x\,dx=x^x(1+\ln x)\,dx\tag{6}$$

With a little skill, the writing is confined to formula (6), the rest being done mentally.

434. Differentiable Functions

A function $u=f(x,y,z)$ with a total differential at the point M_0 is called a *differentiable function* at that point.

A differentiable function always possesses finite partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ and partial differentials

$$d_xu=\frac{\partial u}{\partial x}\Delta x,\qquad d_yu=\frac{\partial u}{\partial y}\Delta y,\qquad d_zu=\frac{\partial u}{\partial z}\Delta z$$

the sum of which yields the total differential (Sec. 430).

But the existence of partial differentials (or finite partial derivatives) does not ensure the existence of a total differential.

Example. Consider the function $f(x,y)$ defined at the point $M_0(0,0)$ by the formula

$$f(0,0)=4\tag{1}$$

and at the other points by the formula

$$f(x,y)=4+2x+y+\frac{x^2y}{x^2+y^2}\tag{2}$$

This function is continuous at the point $M_0(0, 0)$ and has partial derivatives here:

$$f'_x(0, 0)=\lim_{\Delta x\to 0}\frac{f(\Delta x, 0)-4}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\Delta x}{\Delta x}=2,$$

$$f'_y(0, 0)=\lim_{\Delta y\to 0}\frac{f(0\ \ \Delta y)-4}{\Delta y}=1$$

But the expression $f'_x(0, 0)\,\Delta x+f'_y(0, 0)\,\Delta y=2\Delta x+\Delta y$ is not a total differential. Indeed, the total increment is of the form

$$\Delta f(0, 0)=f(\Delta x, \Delta y)-4=(2\Delta x+\Delta y)+\frac{\Delta x^2\Delta y}{\Delta x^2+\Delta y^2}$$

The first term is not a total differential since the second term $\varepsilon=\frac{\Delta x^2\Delta y}{\Delta x^2+\Delta y^2}$ is not of higher order than

$$\rho=\sqrt{\Delta x^2+\Delta y^2}$$

That is, the ratio $\varepsilon:\rho$ does not tend to zero as $M(\Delta x, \Delta y)\to 0$. Thus, if M tends to M_0 along the ray $y=3t$, $x=4t$, then $\varepsilon:\rho$ maintains the value $\frac{36}{125}$.

Another example of a nondifferentiable function is considered in Sec. 442 (Example 2)

Note 1. If all partial derivatives are continuous at the point under consideration, then the function is differentiable at that point. In the preceding example, both partial derivatives were discontinuous at $M_0(0, 0)$.

Note 2. As a rule, the elementary functions are differentiable. Differentiability is violated only at certain points or along separate lines.

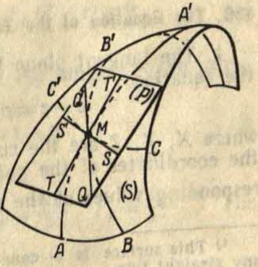

Fig. 434

435. The Tangent Plane and the Normal to a Surface

Definition 1. Through point M of the surface S (Fig. 434) draw (on the surface) lines AA', BB', CC', ..., which at M have tangents TT', QQ' SS', ... The plane P in which all possible tangent lines lie is called the *tangent plane to the surface S at the point M (point of tangency)*.

Example 1. Let the straight line MT be tangent to some spherical curve. Then MT is perpendicular to the radius, i.e. it lies in the P plane passing through point M perpendicular to the radius. Hence, P is the tangent plane to the sphere.

Example 2. A conical surface does not have a tangent plane at the vertex K. Indeed, if we draw all possible lines through K, their tangents at the point K will not lie in one plane.

Note. The surface $z=f(x, y)$ does not have a tangent plane at the point M if and only if the function $f(x, y)$ is not differentiable at that point. Physically realizable surfaces can lose the tangent plane only at separate points (*conical points*) or along certain lines (*edges*) (cf. Sec. 434, Note 2).

Example 3. The function $f(x, y)=\frac{x^2y}{x^2+y^2}+2x+y+4$, redefined by the condition $f(0, 0)=4$, is not differentiable at the point $x=0$, $y=0$ (Sec. 434, Example). Accordingly, the surface

$$z=\frac{x^2y}{x^2+y^2}+2x+y+4 \tag{1}$$

has no tangent plane at the point $A(0, 0, 4)$.[1]

Definition 2. The *normal to surface S at the point M* is the normal to the tangent plane drawn through M.

Example 4. The normal to a spherical surface at every point passes through the centre of the sphere.

436. The Equation of the Tangent Plane

1. The tangent plane to a surface $z=f(x, y)$ is given by the equation

$$Z-z=p(X-x)+q(Y-y) \tag{1}$$

where X, Y, Z are the current coordinates, and x, y, z are the coordinates of the point of tangency; p, q are the corresponding values of the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

---

[1] This surface is a cone (not circular) with vertex A. Indeed, any straight line

$$y=ax, \quad z=\left(\frac{a}{1+a^2}+a+2\right)x+4 \tag{2}$$

(a is a constant) passes through A and lies on surface (1), of which we convince ourselves by substituting (2) into (1). The set of straight lines (2) forms a conical surface.

Explanation. The plane (1) passes through the straight line

$$Z-z=p(X-x), \quad Y-y=0 \tag{A}$$

which is evident from substitution into Eq. (1). The straight line (A) is a tangent to the section drawn through the point (x, y, z) parallel to the xz-plane (Sec. 426). In the same way we see that the plane (1) passes through the tangent line to the section parallel to the zy-plane. Hence (Sec. 435), plane (1) coincides with the tangent plane (if the latter exists; cf. Sec. 435, Note).

Example 1. Find the equation of the tangent plane to the hyperbolic paraboloid $z=\frac{x^2-y^2}{2a}$ at the point $\left(2a, a, \frac{3}{2}a\right)$.

Solution. We have $\frac{\partial z}{\partial x}=\frac{x}{a}=2$, $\frac{\partial z}{\partial y}=-\frac{y}{a}=-1$. The equation of the desired tangent plane is

$$Z-\frac{3}{2}a=2(X-2a)-(Y-a)$$

or $Z=2X-Y-\frac{3}{2}a$.

2. If a surface is given by an equation of the form $F(x, y, z)=0$, then the tangent plane will be represented by the equation

$$F'_x(X-x)+F'_y(Y-y)+F'_z(Z-z)=0 \tag{2}$$

Eq. (1) is a special form of Eq. (2).

Example 2. Find the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \tag{3}$$

at the point $M(x, y, z)$.

Solution. We have $F'_x=\frac{2x}{a^2}$, $F'_y=\frac{2y}{b^2}$, $F'_z=\frac{2z}{c^2}$. The required equation is

$$\frac{2x}{a^2}(X-x)+\frac{2y}{b^2}(Y-y)+\frac{2z}{c^2}(Z-z)=0 \tag{4}$$

or, cancelling 2 and taking into account the equation of the ellipsoid,

$$\frac{xX}{a^2}+\frac{yY}{b^2}+\frac{zZ}{c^2}-1=0$$

Note. The equation of the tangent plane is most simply obtained from the equation of the given surface as follows: differentiate the given equation and in place of dx, dy, dz write $X-x$, $Y-y$, $Z-z$. Thus, differentiating Eq. (3), we get

$$\frac{2x\,dx}{a^2}+\frac{2y\,dy}{b^2}+\frac{2z\,dz}{c^2}=0$$

Replacing the differentials dx, dy, dz by the differences $X-x$, $Y-y$, $Z-z$, we get Eq. (4).

437. The Equation of the Normal

The normal to the surface $F(x, y, z)=0$ at the point $M(x, y, z)$ is given by the equations

$$\frac{X-x}{F'_x}=\frac{Y-y}{F'_y}=\frac{Z-z}{F'_z} \qquad (1)$$

(cf. Secs. 436 and 156). In particular, if the surface is given by the equation $z=f(x, y)$, then the equations of the normal, in the notations of Sec. 436, are of the form

$$\frac{X-x}{p}=\frac{Y-y}{q}=\frac{Z-z}{-1} \qquad (2)$$

Example. The equations of the normal to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (cf. Sec. 436, Example 2) are

$$\frac{a^2(X-x)}{x}=\frac{b^2(Y-y)}{y}=\frac{c^2(Z-z)}{z}$$

438. Differentiation of a Composite Function

A quantity w is a *composite function* if it is regarded as a function of (auxiliary) variables $x, y, \ldots$, which in turn depend on one or several arguments $u, v, \ldots$ (cf. Sec. 236).

Finding the total differential of a composite function does not require special rules (due to the invariance of the expression of a differential; Sec. 432). After the total differential has been found, the expressions for the partial derivatives are obtained automatically (Sec. 433). The general form of these expressions is given in Sec. 440.

Example. Find the total differential and the partial derivatives of the function

$$w=e^{uv}\sin(u+v) \qquad (1)$$

If we represent w in the form $e^x \sin y$, where $x=uv$ and $y=u+v$, then w will be a composite function of the arguments u, v. The total differential is found in the same way, as if x and y were independent variables:

$$dw=e^x \sin y\,dx+e^x \cos y\,dy=e^x(\sin y\,dx+\cos y\,dy)$$

Substituting $x=uv$, $y=u+v$, we get

$$dw=e^{uv}[\sin(u+v)(v\,du+u\,dv)+\cos(u+v)(du+dv)] \quad (2)$$

This is the total differential of the given function; its partial derivatives are the coefficients of du, dv. Namely,

$$\frac{\partial w}{\partial u}=e^{uv}[v\sin(u+v)+\cos(u+v)], \quad (3)$$

$$\frac{\partial w}{\partial v}=e^{uv}[u\sin(u+v)+\cos(u+v)] \quad (4)$$

Note. In practical cases, no special designations are introduced for auxiliary variables. In Example 1, it is usual to do as follows:

$$dw=d[e^{uv}\sin(u+v)]=\sin(u+v)\,de^{uv}+e^{uv}\,d\sin(u+v)=$$
$$=\sin(u+v)\,e^{uv}\,d(uv)+e^{uv}\cos(u+v)\,d(u+v)$$

Expanding expressions $d(uv)$, $d(u+v)$, we get (2).

439. Changing from Rectangular to Polar Coordinates

Let $z=f(x, y)$ be a function of the rectangular coordinates x, y and suppose we know the values of the partial derivatives f'_x, f'_y at the point M. Then the partial derivatives $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \varphi}$ in polar coordinates are found from the formulas

$$\frac{\partial z}{\partial r}=f'_x\cos\varphi+f'_y\sin\varphi, \qquad \frac{\partial z}{\partial \varphi}=r(f'_y\cos\varphi-f'_x\sin\varphi) \quad (1)$$

Explanation. Since $x=r\cos\varphi$, $y=r\sin\varphi$ (Sec. 73), then z is a composite function of r, φ. By the method of Sec. 438, we find

$$dz=f'_x\,dx+f'_y dy=f'_x d(r\cos\varphi)+f'_y d(r\sin\varphi)=$$
$$=f'_x(\cos\varphi\,dr-r\sin\varphi\,d\varphi)+f'_y(\sin\varphi\,dr+r\cos\varphi\,d\varphi)$$

The derivatives $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \varphi}$ are the coefficients of dr, $d\varphi$.

Example. From the given values

$$f'_x(3,\ 4)=7,\quad f'_y(3,\ 4)=2$$

find the values of $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial \varphi}$ at the point (3, 4).

Solution. At the given point we have: $r=\sqrt{3^2+4^2}=5$, $\cos\varphi=\frac{3}{5}$, $\sin\varphi=\frac{4}{5}$. From formulas (1) we obtain

$$\frac{\partial z}{\partial r}=7\cdot\frac{3}{5}+2\cdot\frac{4}{5}=5.8;\quad \frac{\partial z}{\partial \varphi}=5\left(2\cdot\frac{3}{5}-7\cdot\frac{4}{5}\right)=-22$$

440. Formulas for Derivatives of a Composite Function

Let w be a composite function of any number of arguments $u, v, \ldots, t$ (Sec. 438) which is specified in terms of the auxiliary variables $x, y, \ldots, z$ (any number whatsoever). Then

$$\left.\begin{aligned}\frac{\partial w}{\partial u}&=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\ldots+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u},\\ \frac{\partial w}{\partial v}&=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\ldots+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \frac{\partial w}{\partial t}&=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}+\ldots+\frac{\partial w}{\partial z}\frac{\partial z}{\partial t}\end{aligned}\right\}\tag{1}$$

That is, the partial derivative with respect to some argument is equal to the sum of the products of the partial derivatives with respect to all auxiliary variables by the derivatives of these variables with respect to the corresponding argument.

Explanation. Formulas (1) are obtained from the expression for the total differential:

$$dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\ldots+\frac{\partial w}{\partial z}dz\tag{2}$$

if we substitute

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\ldots+\frac{\partial x}{\partial t}dt\tag{3}$$

and similar expressions for $dy, \ldots, dz$ (cf. Sec. 438).

441. Total Derivative

Let w be regarded as a function of the variables $x, y, \ldots, z$:

$$w=f(x, y, \ldots, z) \tag{1}$$

where x is the argument and the other variables depend on x.[1] The derivative of w, with respect to the argument x, taken with allowance made for this relationship, is called the *total derivative* and is denoted by $\frac{dw}{dx}$ in contrast to the partial derivative $\frac{\partial w}{\partial x}$ (Sec. 425). The total derivative is expressed by the formula

$$\frac{dw}{dx}=\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{dy}{dx}+\ldots+\frac{\partial w}{\partial z}\frac{dz}{dx} \tag{2}$$

It is obtained from the expression of the total differential dw (by dividing by dx).

Example 1. Find the total derivative of the function $w=x^3e^{y^2}$ where y is some function of x

Solution.

$$dw=e^{y^2}d(x^3)+x^3de^{y^2}=3e^{y^2}x^2dx+x^3e^{y^2}d(y^2)=$$
$$=3e^{y^2}x^2dx+2x^3e^{y^2}ydy,$$
$$\frac{dw}{dx}=3e^{y^2}x^2+2x^3ye^{y^2}\frac{dy}{dx}$$

Example 2. Find the total derivative of the function $w=xy'$.

Solution. The role of the variable y is played here by the derivative $y'=\frac{dy}{dx}$. By formula (2) we find

$$\frac{dw}{dx}=\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y'}\frac{dy'}{dx}=y'+x\frac{d^2y}{dx^2}$$

We get the same expression by dividing the equation

$$dw=y'\,dx+x\,dy'=y'\,dx+xy''\,dx$$

term by term by dx.

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[1] This is a special form of a composite function (Sec. 438) of one argument u (the variables $y, \ldots, z$ are dependent on u in arbitrary fashion, and the variable x is connected with u by the equality $x=u$).

442. Differentiation of an Implicit Function of Several Variables

Rule 1. The equation

$$F(x, y, z)=0 \tag{1}$$

gives the variable z as an implicit function of the arguments x, y for certain conditions.[1] In order to find the total differential of this function it is necessary to differentiate Eq. (1), that is, to equate to zero the total differential of the left-hand side. The equation obtained has to be solved for dz; then we find the total differential of the function z. The coefficients of dx, dy yield corresponding partial derivatives.

We proceed in exactly the same fashion for any number of arguments.

Example 1. Find the total differential and the partial derivatives of the implicit function z (of the arguments x, y) given by the equation

$$x^2+y^2+z^2=9 \tag{2}$$

at the point $x=1$, $y=-2$, $z=-2$.

Solution. Differentiating, we find

$$2x\,dx+2y\,dy+2z\,dz=0$$

Solving this equation for dz, we obtain the total differential of the function z (at an arbitrary point):

$$dz=-\frac{x}{z}\,dx-\frac{y}{z}\,dy \tag{3}$$

At the given point $(1, -2, -2)$ we have

$$dz=\frac{1}{2}\,dx-dy \tag{4}$$

The coefficients of dx, dy yield the values of the partial derivatives at the given point:

$$\frac{\partial z}{\partial x}=\frac{1}{2}, \quad \frac{\partial z}{\partial y}=-1 \tag{5}$$

Check. Solving Eq. (2) for z, we get

$$z=-\sqrt{9-x^2-y^2} \tag{6}$$

(we take the minus sign before the radical because for $x=1$,

---

[1] See Note 1 below.

$y=-2$ we have to have $z=-2$). From (6) we find

$$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{9-x^2-y^2}}, \quad \frac{\partial z}{\partial y}=\frac{y}{\sqrt{9-x^2-y^2}}$$

Substituting here the values $x=1$, $y=-2$, we again find (5).

Note 1. It is assumed in Rule 1 that the function $F(x, y, z)$ is differentiable at some point $M_0(x_0, y_0, z_0)$, which satisfies Eq. (1), and in a sufficiently small neighbourhood of it (that is, at all points of some sphere centred at M_0). Besides, it is assumed that the equation obtained by differentiation is uniquely solvable for dz (i.e. that the coefficient of dz is different from zero). Under these conditions it may be asserted that:

(1) Eq. (1) does indeed specify z as an implicit function of the arguments x, y; it is defined in some circle centred at (x_0, y_0) and takes the value z_0 for $x=x_0$, $y=y_0$;

(2) the function z is differentiable in the indicated circle and, in particular, at the point (x_0, y_0).

The foregoing conditions are fulfilled in Example 1. In the next example we consider one of the great diversity of cases in which they are violated.

Example 2. The equation

$$x^3+8y^3-z^3=0 \tag{7}$$

represents z as an implicit function of the arguments x, y. Its explicit expression is

$$z=\sqrt[3]{x^3+8y^3} \tag{8}$$

In an attempt to apply Rule 1 to finding the total differential of the function z at the point $x=0$, $y=0$, $z=0$, we would get, from (7), the equation

$$3x^2\,dx+24y^2\,dy-3z^2\,dz=0 \tag{9}$$

At the point $x=0$, $y=0$, $z=0$ this equation does not admit a unique solution for dz because it becomes an identity, $0=0$. Thus, Rule 1 does not make it possible for us to find either the total differential or the partial derivatives of the function z at the point under consideration. A supplementary investigation shows that the function z is not differentiable at this point (Sec. 434) but has partial derivatives

$$\frac{\partial z}{\partial x}=1, \quad \frac{\partial z}{\partial y}=2^{1)}$$

---

1) Indeed, putting $y=0$, we get $z=\sqrt[3]{x^3}=x$ so that $\left(\frac{\partial z}{\partial x}\right)_{x=0}=1$; putting $x=0$, we get $z=\sqrt[3]{8y^3}=2y$ so that $\left(\frac{\partial z}{\partial y}\right)_{\substack{x=0\\y=0}}=2$ But the expression $\Delta x+2\Delta y$ is not the total differential because the difference

$$\Delta z-(\Delta x+2\Delta y)=\varepsilon$$

is not of higher order than $\rho=\sqrt{\Delta x^2+\Delta y^2}$. Thus, if the point (x, y) tends to the point $(0, 0)$, say along the bisector of the first quadrant,

Rule 2. The system of two equations

$$F_1(x, y, z, u, v)=0, \quad F_2(x, y, z, u, v)=0 \tag{10}$$

represents, under certain conditions, 1) the two variables u, v as implicit functions of the arguments x, y, z. In order to find the total differentials of these functions we have to differentiate Eqs. (10). The system of equations has to be solved for du, dv, and we then find the total differentials of the functions u, v. The coefficients of dx, dy, dz will yield the appropriate partial derivatives.

We proceed in the same fashion when the number of equations in the system is greater than two (for any number of arguments).

Example 3. Find the total differentials and partial derivatives of the implicit functions u, v given by the system of equations

$$x+y+u+v=a, \quad x^2+y^2+u^2+v^2=b^2 \tag{11}$$

Solution. Differentiating, we get

$$\begin{gathered} dx+dy+du+dv=0, \\ x\,dx+y\,dy+u\,du+v\,dv=0 \end{gathered} \tag{12}$$

Solving the system (12) for du, dv, we get the total differentials of the functions u, v:

$$du=\frac{(v-x)\,dx+(v-y)\,dy}{u-v}, \quad dv=\frac{(u-x)\,dx+(u-y)\,dy}{v-u} \tag{13}$$

The coefficients of dx, dy give the partial derivatives

$$\frac{\partial u}{\partial x}=\frac{v-x}{u-v}, \quad \frac{\partial u}{\partial y}=\frac{v-y}{u-v}, \quad \frac{\partial v}{\partial x}=\frac{u-x}{v-u}, \quad \frac{\partial v}{\partial y}=\frac{u-y}{v-u} \tag{14}$$

Note 2. It is assumed in Rule 2 that the functions $F_1(x, y, z, u, v)=0$, $F_2(x, y, z, u, v)=0$ are differentiable at some point $M_0(x_0, y_0, z_0, u_0, v_0)$ and in a sufficiently close neighbourhood of it. Also, it is assumed that the system of equations obtained by differentiation is uniquely solvable for du, dv (that is, that a determinant made up of the coefficients of du, dv is different from zero). Under these conditions, we can assert that:

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then the ratio $\frac{\varepsilon}{\rho}$ has the value

$$\frac{\varepsilon}{\rho}=\frac{\sqrt[3]{\Delta x^3+8\,\Delta y^3}-(\Delta x+2\,\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}}=\frac{\sqrt[3]{9}-3}{\sqrt{2}}$$

i.e. it does not tend to zero.

1) See Note 2 below.

(1) system (10) does indeed specify u, v as implicit functions of the arguments x, y, z; these functions are defined in some sphere centred at (x_0, y_0, z_0) and take on values u_0, v_0 for $x=x_0$, $y=y_0$, $z=z_0$;

(2) the functions u, v are differentiable in the indicated sphere and, in particular, at the point (x_0, y_0, z_0).

443. Higher-Order Partial Derivatives

Definition 1. The partial derivatives of the functions

$$\frac{\partial z}{\partial x}=f'_x(x, y), \quad \frac{\partial z}{\partial y}=f'_y(x, y) \tag{1}$$

are called *partial derivatives of the second order* (or *second partial derivatives*) of the function $z=f(x, y)$.

The total number of second partial derivatives is four. The partial derivative of $\frac{\partial z}{\partial x}$ with respect to the argument x is denoted by $\frac{\partial^2 z}{\partial x^2}$, or $\frac{\partial^2 f(x, y)}{\partial x^2}$, or $f''_{xx}(x, y)$. The others are denoted in similar fashion, so we have

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 f(x, y)}{\partial x^2}= {}''_{xx}(x, y), \tag{2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\, \partial y}=\frac{\partial^2 f(x, y)}{\partial x\, \partial y}=f''_{xy}(x, y), \tag{3}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\, \partial x}=\frac{\partial^2 f(x, y)}{\partial y\, \partial x}=f''_{yx}(x, y), \tag{4}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 f(x, y)}{\partial y^2}=f''_{yy}(x, y) \tag{5}$$

The second derivatives (2) and (5) are called *pure*, the second derivatives (3) and (4) are called *mixed*.

Theorem 1. Mixed derivatives of the second order (they differ as to the order of differentiation) are equal, provided they are continuous at the point under consideration.

Example 1. Find the second partial derivatives of the function $z=x^3y^2+2x^2y-6$. We have

$$\frac{\partial z}{\partial x}=3x^2y^2+4xy, \qquad \frac{\partial z}{\partial y}=2x^3y+2x^2,$$

$$\frac{\partial^2 z}{\partial x^2}=6xy^2+4y, \qquad \frac{\partial^2 z}{\partial y\, \partial x}=6x^2y+4x,$$

$$\frac{\partial^2 z}{\partial x\, \partial y}=6x^2y+4x, \qquad \frac{\partial^2 z}{\partial y^2}=2x^3$$

The mixed derivatives $\frac{\partial^2 z}{\partial y\, \partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$ are equal.

Note 1. By virtue of Theorem 1, the four partial derivatives of the second order reduce to three: $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$.

Definition 2. Partial derivatives of partial derivatives of the second order are called *partial derivatives of the third order* (or *third partial derivatives*) and are denoted by f'''_{xxx}, f'''_{yyy} (pure derivatives), f'''_{xxy}, f'''_{xyx}, f'''_{xyy}, etc. (mixed derivatives) or $\frac{\partial^3 z}{\partial x^3}$, $\frac{\partial^3 z}{\partial y^3}$, $\frac{\partial^3 z}{\partial x^2\,\partial y}$, $\frac{\partial^3 z}{\partial x\,\partial y\,\partial x}$, etc.

Theorem 2. Mixed derivatives of the third order which differ only in the order of differentiation are equal (provided that they are continuous at the point under consideration)

For example, $\frac{\partial^3 z}{\partial x^2\,\partial y}=\frac{\partial^3 z}{\partial x\,\partial y\,\partial x}$.

Example 2. The partial third-order derivatives of the function $z=x^3y^2+2x^2y-6$ (cf. Example 1) are

$$\frac{\partial^3 z}{\partial x^3}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right)=6y^2,\quad \frac{\partial^3 z}{\partial y^3}=\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right)=0,$$

$$\frac{\partial^3 z}{\partial x^2\,\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right)=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)=12xy+4,$$

$$\frac{\partial^3 z}{\partial x\,\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right)=6x^2$$

Note 2. By virtue of Theorem 2, eight partial derivatives of the third order are reduced to four:

$$\frac{\partial^3 z}{\partial x^3},\quad \frac{\partial^3 z}{\partial x^2\,\partial y},\quad \frac{\partial^3 z}{\partial x\,\partial y^2},\quad \frac{\partial^3 z}{\partial y^3}$$

Note 3. We similarly define and denote the partial derivatives of the fourth and higher orders of a function $f(x, y)$ and also of functions of three and more arguments. In all cases, theorems similar to 1 and 2 hold true.

444. Total Differentials of Higher Orders

Let us form the total increment (Sec. 427) Δz of the function $z=f(x, y)$; then, *holding the same values* Δx, Δy, let us form the total increment $\Delta(\Delta z)$ of the quantity Δz (regarding it as a function of x, y). We get the *second difference* $\Delta^2 z$ of the function z.

Suppose $\Delta^2 z$ decomposes into a sum of two terms:

$$\Delta^2 z=(r\Delta x^2+2s\,\Delta x\,\Delta y+t\,\Delta y^2)+\alpha \tag{1}$$

where r, s, t do not depend either on Δx or Δy, and α is of higher order than $\rho^2=\Delta x^2+\Delta y^2$. Then the first term is called the *second* (total) *differential* of the function z and is denoted d^2z.

Example 1. Consider the function $z=x^3y^2$. We find

$$\Delta z=(x+\Delta x)^3(y+\Delta y)^2-x^3y^2,$$

$$\Delta^2 z=(x+2\Delta x)^3(y+2\Delta y)^2-2(x+\Delta x)^3(y+\Delta y)^2+ \\ +x^3y^2=(6xy^2\,\Delta x^2+12x^2y\,\Delta x\,\Delta y+2x^3\,\Delta y^2)+\alpha \qquad (2)$$

where α is of higher order than ρ^2. Now the first term of the sum (2) is of the form $r\Delta x^2+2s\,\Delta x\,\Delta y+t\,\Delta y^2$ and the quantities $r=6xy^2$, $s=6x^2y$, $t=2x^3$ do not depend either on Δx or on Δy. Hence, the first term is the second differential of the function $z=x^3y^2$:

$$d^2z=6xy^2\,\Delta x^2+12x^2y\,\Delta x\,\Delta y+2x^3\,\Delta y^2 \qquad (3)$$

Theorem 1. The quantities r, s, t in formula (1) are equal to the corresponding second partial derivatives of the function z:

$$r=\frac{\partial^2 z}{\partial x^2},\quad s=\frac{\partial^2 z}{\partial x\,\partial y},\quad t=\frac{\partial^2 z}{\partial y^2}$$

Example 2. In the preceding example we had

$$r=6xy^2=\frac{\partial^2 z}{\partial x^2},\qquad s=6x^2y=\frac{\partial^2 z}{\partial x\,\partial y},\qquad t=2x^3=\frac{\partial^2 z}{\partial y^2}$$

Expression for the second differential. By Theorem 1 we have

$$d^2z=\frac{\partial^2 z}{\partial x^2}\,\Delta x^2+2\,\frac{\partial^2 z}{\partial x\,\partial y}\,\Delta x\,\Delta y+\frac{\partial^2 z}{\partial y^2}\,\Delta y^2 \qquad (4)$$

Note. Since

$$\Delta x=dx,\qquad \Delta y=dy$$

(Sec. 430, Note 1), then in place of (4) we can write

$$d^2z=\frac{\partial^2 z}{\partial x^2}\,dx^2+2\,\frac{\partial^2 z}{\partial x\,\partial y}\,dx\,dy+\frac{\partial^2 z}{\partial y^2}\,dy^2 \qquad (5)$$

In contrast to the corresponding expression for the first differential (cf. Sec. 432), formula (5) does not as a rule hold true if x and y are not arguments (cf. footnote, Sec. 258).

Theorem 2. If we consider the differentials dx, dy as not depending either on x or y, then the second differential d^2z is equal to the differential of the first differential dz (cf. Sec. 258, Theorem 2):

$$d\,[df\,(x,\,y)]=d^2f\,(x,\,y) \qquad (6)$$

Example 3. Let $z=x^3y^2$. We have

$$dz=3x^2y^2\,dx+2x^3y\,dy$$

Differentiate once again, holding dx, dy fixed. We obtain

$$d\,(dz)=d\,(3x^2y^2)\,dx+d\,(2x^3y)\,dy=6xy^2\,dx^2+ \\ +12x^2y\,dx\,dy+2x^3\,dy^2$$

This is the second total differential of the function x^3y^2 (see Example 1).

Total differentials of the third, fourth, etc. orders (d^3z, d^4z, etc.) are defined similarly and are expressed by the following formulas:

$$d^3z=\frac{\partial^3 z}{\partial x^3}\,dx^3+3\frac{\partial^3 z}{\partial x^2\,\partial y}\,dx^2\,dy+3\frac{\partial^3 z}{\partial x\,\partial y^2}\,dx\,dy^2+\frac{\partial^3 z}{\partial y^3}\,dy^3, \qquad (7)$$

$$d^4z=\frac{\partial^4 z}{\partial x^4}\,dx^4+4\frac{\partial^4 z}{\partial x^3\,\partial y}\,dx^3\,dy+6\frac{\partial^4 z}{\partial x^2\,\partial y^2}\,dx^2\,dy^2+ \\ +4\frac{\partial^4 z}{\partial x\,\partial y^3}\,dx\,dy^3+\frac{\partial^4 z}{\partial y^4}\,dy^4 \qquad (8)$$

The numerical factors are equal to the corresponding binomial coefficients.

Formulas (7), (8), etc. do not as a rule hold if x and y are not arguments.

The foregoing can be extended completely to functions of three and more variables.

445. The Technique of Repeated Differentiation

To find partial derivatives of higher order it is convenient first to find the total differential of the corresponding order.

Example. Find the partial derivatives of the function $z=x^3y^2$ up to third order inclusive.

Solution. First find the first differential dz:

$$dz=3x^3y^2\,dx+2x^3y\,dy \qquad (1)$$

then the second; to do this, differentiate (1) holding dx, dy constant:

$$d^2z=6xy^2\,dx^2+12x^2y\,dx\,dy+2x^3\,dy^2 \qquad (2)$$

(cf. Sec. 444, Example 3). Differentiating (2), we again hold dx, dy constant and obtain

$$d^3z=(6y^2\,dx^3+12xy\,dx^2\,dy)+(24xy\,dx^2\,dy+12x^2\,dx\,dy^2)+ \\ +6x^2\,dx\,dy^2$$

or

$$d^3z = 6y^2\,dx^3 + 3\cdot 12xy\,dx^2\,dy + 3\cdot 6x^2\,dx\,dy^2 \tag{3}$$

From the coefficients of the expressions (1), (2), and (3), and taking into account formulas (5) and (7), Sec. 444, we get

$$\frac{\partial z}{\partial x} = 3x^2y^2, \quad \frac{\partial z}{\partial y} = 2x^3y;$$

$$\frac{\partial^2 z}{\partial x^2} = 6xy^2, \quad \frac{\partial^2 z}{\partial x\,\partial y} = 6x^2y, \quad \frac{\partial^2 z}{\partial y^2} = 2x^3;$$

$$\frac{\partial^3 z}{\partial x^3} = 6y^2, \quad \frac{\partial^3 z}{\partial x^2\,\partial y} = 12xy, \quad \frac{\partial^3 z}{\partial x\,\partial y^2} = 6x^2, \quad \frac{\partial^3 z}{\partial y^3} = 0$$

446. Symbolism of Differentials

As the order increases, the expressions for differentials become more complicated. To simplify matters, we introduce the following conventional symbolism for a *k*th order differential of a function $z = f(x, y)$:

$$d^kz = \left(\frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy\right)^k z \tag{1}$$

This is to be understood as: first raise the binomial $\frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy$ to the *k*th power as if the symbols ∂x, ∂y, ∂ denoted independent algebraic quantities. Then remove the brackets and affix to each symbol ∂^k the factor z. Only then invest all the symbols with their true meaning.

Example. The notation $d^3z = \left(\frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy\right)^3 z$ is deciphered as follows: raising to the third power, we get

$$\left(\frac{\partial^3}{\partial x^3}\,dx^3 + 3\,\frac{\partial^3}{\partial x^2\,\partial y}\,dx^2\,dy + 3\,\frac{\partial^3}{\partial x\,\partial y^2}\,dx\,dy^2 + \frac{\partial^3}{\partial y^3}\,dy^3\right) z$$

Removing brackets, we find

$$d^3z = \frac{\partial^3 z}{\partial x^3}\,dx^3 + 3\,\frac{\partial^3 z}{\partial x^2\,\partial y}\,dx^2\,dy + 3\,\frac{\partial^3 z}{\partial x\,\partial y^2}\,dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}\,dy^3$$

[cf. (7), Sec. 444].

Note. For three, four and so forth arguments, the conventional symbols are the same: for example the notation

$$d^2u = \left(\frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy + \frac{\partial}{\partial z}\,dz\right)^2 u$$

signifies that

$$d^2u=\frac{\partial^2 u}{\partial x^2}dx^2+\frac{\partial^2 u}{\partial y^2}dy^2+\frac{\partial^2 u}{\partial z^2}dz^2+2\frac{\partial^2 u}{\partial x\,\partial y}dx\,dy+$$
$$+2\frac{\partial^2 u}{\partial y\,\partial z}dy\,dz+2\frac{\partial^2 u}{\partial z\,\partial x}dz\,dx$$

447. Taylor's Formula for a Function of Several Arguments

For a function of one argument, Taylor's formula (Sec. 271) may be written in the form

$$f(x+\Delta x)=f(x)+\frac{1}{1!}f'(x)\Delta x+\frac{1}{2!}f''(x)\Delta x^2+\ldots+$$
$$+\frac{1}{n!}f^{(n)}(x)\Delta x^n+\frac{1}{(n+1)!}f^{(n+1)}(x+\Theta\Delta x)\Delta x^{n+1} \qquad (1)$$

where Θ is some positive number less than unity:[1]

$$0<\Theta<1 \qquad (2)$$

Here the expressions $f'(x)\Delta x$, $f''(x)\Delta x^2$, ... are differentials of order one, two, etc.

Taylor's formula for a function of several variables[2] is similarly constructed, only the differentials taken are *total*. Thus for two arguments for $n=2$ we have

$$f(x+\Delta x,\ y+\Delta y)=$$
$$=f(x,\ y)+\frac{1}{1}[f'_x(x,\ y)\Delta x+f'_y(x,\ y)\Delta y]+$$
$$+\frac{1}{2!}[f''_{xx}(x,\ y)\Delta x^2+2f''_{xy}(x,\ y)\Delta x\,\Delta y+f''_{yy}(x,\ y)\Delta y^2]+$$
$$+\frac{1}{3!}[f'''_{xxx}(x+\Theta\Delta x,\ y+\Theta\Delta y)\Delta x^3+$$
$$+3f'''_{xxy}(x+\Theta\Delta x,\ y+\Theta\Delta y)\Delta x^2\,\Delta y+$$
$$+3f'''_{xyy}(x+\Theta\Delta x,\ y+\Theta\Delta y)\Delta x\,\Delta y^2+$$
$$+f'''_{yyy}(x+\Theta\Delta x,\ y+\Theta\Delta y)\Delta y^3] \qquad (3)$$

where Θ satisfies inequality (2).

---

[1] The number ξ in (1), Sec. 271, lies between x and $x+\Delta x$; therefore the difference $\xi-x$ has the same sign as Δx, but is less in absolute value than Δx. Hence, the quotient $(\xi-x):\Delta x$ is some positive number Θ less than unity. From the equation $(\xi-x):\Delta x=\Theta$ we find $\xi=x+\Theta\Delta x$.

[2] The condition under which the formula holds is given in the note below.

The expressions in square brackets are (Sec. 444) total differentials. In the last term, the partial derivatives are taken for intermediate values of the arguments.1)

Taylor's formula for any number of terms is surveyable (even for two arguments) only if we use the conventional symbolism of Sec. 446. Then it has the form

$$\begin{aligned}\Delta f(x,\ y) = {} & \frac{1}{1!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right) f(x,\ y) + \\ & +\frac{1}{2!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^2 f(x,\ y)+\dots+ \\ & +\frac{1}{n!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^n f(x,\ y)+ \\ & +\frac{1}{(n+1)!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{n+1} f(x+\Theta\Delta x,\ y+\Theta\Delta y)\end{aligned} \tag{4}$$

or

$$\begin{aligned}\Delta f(x,\ y) = {} & \frac{1}{1!}\,df(x,\ y)+\frac{1}{2!}\,d^2f(x,\ y)+\dots+\frac{1}{n!}\,d^nf(x,\ y)+ \\ & +\frac{1}{(n+1)!}\,d^{n+1}f(x+\Theta\Delta x,\ y+\Theta\Delta y)\end{aligned} \tag{5}$$

and similarly for a greater number of arguments.

Note. Taylor's formula holds true provided the function $f(x,\ y)$ possesses a total differential of the $(u+1)$th order at all points of a segment connecting points $M(x,\ y)$ and $M_1(x+\Delta x,\ y+\Delta y)$.

Example. Let us verify formula (3) using the function

$$f(x,\ y)=xy^2$$

for $x=y=1,\ \Delta x=0.1,\ \Delta y=0.2$. We get

$$\begin{aligned}(x+\Delta x)(y+\Delta y)^2 = {} & xy^2+[y^2\,\Delta x+2xy\,\Delta y]+ \\ & +\frac{1}{2}[4(y+\Theta\Delta y)\,\Delta x\,\Delta y+2(x+\Theta\,\Delta x)\,\Delta y^2]\end{aligned}$$

Substituting the given values, we obtain the equation $0.004=0.012\Theta$, whence $\Theta=\frac{1}{3}$; thus, Θ indeed lies between zero and unity.

---

1) The point $\overline{M}(x+\Theta\Delta x,\ y+\Theta\Delta y)$ lies on a segment connecting points $M(x,\ y)$ and $M_1(x+\Delta x,\ y+\Delta y)$. The number Θ yields the ratio

$$\overline{MM}:MM_1$$

448. The Extremum (Maximum or Minimum) of a Function of Several Arguments

Definition. A function $f(x, y)$ has a *maximum (minimum)* at a point $P_0(a, b)$ if at all points sufficiently close to P_0 the value of $f(x, y)$ is less than (more than) the value $f(a, b)$ (cf. Sec. 275).

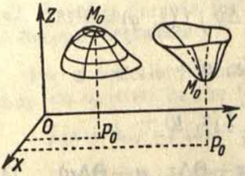

Fig. 435

Geometrically, above the point P_0 (Fig. 435) the surface $z=f(x, y)$ has a point M_0 which lies above (below) all neighbouring points.

A necessary condition for an extremum. If the function $f(x, y)$ has an extremum at a point $P_0(a, b)$, then the total differential at this point is either identically zero or does not exist.

Note 1. The condition $df(x, y)=0$ is equivalent to the system of two equations:

$$f'_x(x, y)=0, \quad f'_y(x, y)=0$$

The equation $f'_x(x, y)=0$ taken separately is a necessary condition for an extremum when y is fixed (Sec. 276). Geometrically it means that the section of a surface parallel to the xz-plane has at the point M_0 a tangent line parallel to the x-axis (cf. Sec. 426). The equation $f'_y(x, y)=0$ has a similar meaning.

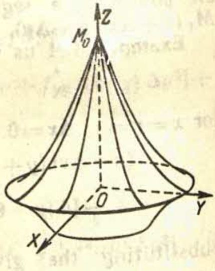

Fig. 436

Geometrically, at the point M_0 lying above (below) all neighbouring ones, the surface $z=f(x, y)$ either has a horizontal tangent plane (as in Fig. 435), or does not have any tangent plane at all (as in Fig. 436).

Note 2. The definition of an extremum and the necessary condition remain the same for any number of arguments.

449. Rule for Finding an Extremum

Let the function $f(x, y)$ be differentiable in some region of its domain. In order to find all its extrema in that region, we have to

(1) Solve the system of equations

$$f'_x(x, y)=0 \quad f'_y(x, y)=0 \tag{1}$$

The solution will yield *critical points*.

(2) Investigate, for every critical point $P_0(a, b)$ whether the sign of the difference

$$f(x, y)-f(a, b) \tag{2}$$

remains unchanged or not for all points (x, y) sufficiently close to P_0. If the difference (2) preserves the positive sign, then at the point P_0 we have a minimum, if the negative sign, then a maximum. If the difference (2) does not preserve sign, then there is no extremum at the point P_0.

In similar fashion we find the extrema of a function of a larger number of arguments.

Note. For two arguments, the investigation is sometimes simplified by the use of the sufficient condition of Sec. 450. For a larger number of arguments, this condition becomes complicated. Therefore, in practical applications one attempts to employ the special properties of the given function.

Example. Find the extrema of the function

$$f(x, y)=x^3+y^3-3xy+1$$

Solution. (1) Equating to zero the partial derivatives $f'_x=3x^2-3y$, $f'_y=3y^2-3x$, we get the system of equations

$$x^2-y=0, \quad y^2-x=0 \tag{3}$$

It has two solutions:

$$x_1=y_1=0, \quad x_2=y_2=1 \tag{4}$$

Let us investigate the sign of the difference (2) for each of the two critical points $P_1(0, 0)$, $P_2(1, 1)$.

(2a) For the point $P_1(0, 0)$ we have

$$f(x, y)-f(0, 0)=x^3+y^3-3xy \tag{5}$$

The difference (5) *does not preserve sign*, i. e. *in any close neighbourhood* of P_1 there are points of two types: for some the difference (5) is positive, for others it is negative. Thus, if the point $P(x, y)$ is taken on the straight line $y=x$, then (5) is equal to $2x^3-3x^2=x^2(2x-3)$. Near P_1 $\left(\text{for } x<\frac{3}{2}\right)$ this difference is negative. But if point $P(x, y)$ is taken on the straight line $y=-x$, then the difference (5) is equal to $3x^2$, and this quantity is always positive.

Since the difference (5) does not preserve sign, there is no extremum at the point $P_1(0, 0)$. The surface

$$z = x^3 + y^3 - 3xy + 1$$

at the point (0, 0, 1) is saddle-shaped (like a hyperbolic paraboloid).

(2b) For the point $P_2(1, 1)$ we have

$$f(x, y) - f(1, 1) = x^3 + y^3 - 3xy + 1 \tag{6}$$

We shall prove that this difference preserves the plus sign in a sufficiently close neighbourhood of the point (1, 1). Put

$$x = 1 + \alpha, \quad y = 1 + \beta \tag{7}$$

The difference (6) is transformed to

$$3(\alpha^2 - \alpha\beta + \beta^2) + (\alpha^3 + \beta^3) \tag{8}$$

The first term for all nonzero values of α, β is positive and more than[1] $\frac{3}{2}(\alpha^2 + \beta^2)$. The second term may also be negative, but if $|\alpha|$ and $|\beta|$ are sufficiently small, then it is less than $\alpha^2 + \beta^2$ in absolute value.[2] Hence, the difference (8) is positive.

Thus, the given function has a minimum at the point (1, 1).

450. Sufficient Conditions for an Extremum (for the Case of Two Arguments)

Theorem 1. Let

$$A\,dx^2 + 2B\,dx\,dy + C\,dy^2 \tag{1}$$

be the second differential of the function $f(x, y)$ at its critical point (Sec. 449) $P_0(a, b)$ (so that the numbers A, B, C give the values of the second derivatives f''_{xx}, f''_{xy}, f''_{yy} at the point P_0). Then, if the inequality

$$AC - B^2 > 0 \tag{2}$$

holds, the function $f(x, y)$ has an extremum at P_0: a maximum when A (or C) is negative, a minimum when A (or C) is positive.

---

[1] We have the identity $3(\alpha^2 - \alpha\beta + \beta^2) = \frac{3}{2}(\alpha^2 + \beta^2) + \frac{3}{2}(\alpha - \beta)^2$. The quantity $(\alpha - \beta)^2$ is positive or zero.

[2] For $|\alpha| < 1$, $|\beta| < 1$ we have $|\alpha^3| < a^2$, $|\beta^3| < \beta^2$.

Note 1. The numbers A and C always have the same signs, provided (2) holds.

Theorem 1 gives a *sufficient condition for the existence of an extremum.*

Example 1. The function $f(x, y)=x^3+y^3-3xy+1$ (cf. Example, Sec. 449) has an extremum at the point (1, 1) because the first derivatives are zero at this point and the second derivatives $\frac{\partial^2 f}{\partial x^2}=6x$, $\frac{\partial^2 f}{\partial x\,\partial y}=-3$, $\frac{\partial^2 f}{\partial y^2}=6y$ have the values $A=6$, $B=-3$, $C=6$ so that the inequality (2) is satisfied. The extremum is a minimum because A and C are positive.

Theorem 2. If at the critical point $P_0(a, b)$ the inequality (in the notation of Theorem 1)

$$AC-B^2<0 \tag{3}$$

holds, then the function $f(x, y)$ does not have an extremum at P_0.

Theorem 2 yields a *sufficient condition for the absence of an extremum.*

Example 2. The function $f(x, y)=x^3+y^3-3xy+1$ (cf. Example, Sec. 449) does not have an extremum at the point (0, 0): although the first derivatives vanish, we now have

$$A=0, \quad B=-3, \quad C=0$$

so that

$$AC-B^2=-9<0$$

Note 2. If the equation

$$AC-B^2=0 \tag{4}$$

holds at a critical point, then the function can have an extremum (maximum or minimum) there, but it may not have. This case requires more investigation.

451. Double Integral[1]

Let a function $f(x, y)$ be continuous inside some domain D (Fig. 437) and on its boundary. We partition the domain D into n subdomains $D_1, D_2, \ldots, D_n$; we denote their areas

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[1] The concept of a double integral is an extension of the concept of a definite integral to the case of two arguments. It is therefore advisable to reread Sec. 314.

S_n, that is, the integral $\iint\limits_D f(x, y)\,d\sigma$ yields the exact value of the volume V.

453. Properties of a Double Integral

Property 1. If a domain D is partitioned into two parts D_1 and D_2, then

$$\iint\limits_D f(x, y)\,d\sigma = \iint\limits_{D_1} f(x, y)\,d\sigma + \iint\limits_{D_2} f(x, y)\,d\sigma$$

(cf. Sec. 315, Item 2). Similarly for a subdivision of D into three, four and more parts.

Property 2. The double integral of the algebraic sum of a fixed number of functions is equal to the algebraic sum of the double integrals taken for each term (cf. Sec. 315, Item 3); thus, for three terms:

$$\iint\limits_D [f(x, y) + \varphi(x, y) - \psi(x, y)]\,d\sigma =$$

$$= \iint\limits_D f(x, y)\,d\sigma + \iint\limits_D \varphi(x, y)\,d\sigma - \iint\limits_D \psi(x, y)\,d\sigma$$

Property 3. A constant factor may be taken outside the integral sign (cf. Sec. 315, Item 4):

$$\iint\limits_D mf(x, y)\,d\sigma = m\iint\limits_D f(x, y)\,d\sigma \quad (m \text{ a constant}).$$

454. Estimating a Double Integral

Let m be the smallest and M the greatest value of a function $f(x, y)$ in a domain D and let S be the area of D. Then

$$mS \leqslant \iint\limits_D f(x, y)\,d\sigma \leqslant MS$$

Geometrically this means that the volume of a cylindrical solid is contained between the volumes of two cylinders having the same base; the altitude of the first is the smallest z-coordinate, the altitude of the second is the greatest z-coordinate (cf Sec. 318, Theorem 1).

455. Computing a Double Integral (Simplest Case)

Let a domain D be given by the inequalities

$$a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d \tag{1}$$

It is depicted by the rectangle $KLMN$ (Fig. 441). Then the double integral is computed from one of the formulas

$$\iint\limits_D f(x, y)\,dx\,dy = \int\limits_c^d dy \int\limits_a^b f(x, y)\,dx, \tag{2}$$

$$\iint\limits_D f(x, y)\,dx\,dy = \int\limits_a^b dx \int\limits_c^d f(x, y)\,dy \tag{3}$$

The expressions in the right-hand members are called *iterated integrals.*

Note. In formula (2) the definite integral $\int\limits_a^b f(x, y)\,dx$ is calculated first. *During this integration,* y is regarded as a constant. But the *result of the integration* is viewed as a function of y, and the second integration (from c to d) is performed with respect to the argument y. In formula (3) the order of operations is reversed.

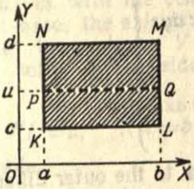

Fig. 441

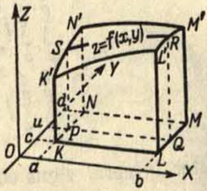

Fig. 442

Explanation. The double integral $\iint\limits_{(KLMN)} f(x, y)\,dx\,dy$ expresses the volume V of the prismatic solid KM' (Fig. 442) having the base $KLMN$:

$$V=\iint_D f(x, y)\,dx\,dy \tag{4}$$

The same volume is obtained from the variable area F of the longitudinal section $PQRS$ (it depends on the ordinate $y=Ou$) by means of the formula (Sec. 336)

$$V=\int_c^d F(y)\,dy \tag{5}$$

The area of $PQRS$ is given by

$$F(y)=\int_a^b z\,dx=\int_a^b f(x, y)\,dx \tag{6}$$

Comparing (4), (5) and (6), we get (2). We obtain (3) in similar fashion.

Notations. The double integral $\iint_D f(x, y)\,dx\,dy$ taken over a rectangle whose sides are parallel to the x- and y-axes, is denoted by

$$\left.\begin{array}{l} \displaystyle\int_c^d\int_a^b f(x, y)\,dx\,dy \\ \text{or} \\ \displaystyle\int_a^b\int_c^d f(x, y)\,dy\,dx \end{array}\right\} \tag{7}$$

(the outer integral signs correspond to the outer differentials).

Example 1. Compute the double integral $\displaystyle\int_1^2\int_3^4 \frac{dx\,dy}{(x+y)^2}$.

Solution. The domain of integration is defined by the inequalities

$$3\leqslant x\leqslant 4, \quad 1\leqslant y\leqslant 2$$

and is a rectangle with sides parallel to the x- and y-axes.

First compute the definite integral $\displaystyle\int_3^4 \frac{dx}{(x+y)^2}$ where y is held constant:

$$\int_3^4 \frac{dx}{(x+y)^2}=\frac{1}{y+3}-\frac{1}{y+4}$$

Then from formula (2) we obtain

$$\int_1^2\int_3^4 \frac{dx\,dy}{(x+y)^2}=\int_1^2\left(\frac{1}{y+3}-\frac{1}{y+4}\right)dy=\ln\frac{25}{24}\approx 0.0408$$

Example 2. Compute the double integral

$$I=\int_1^3\int_2^5 (5x^2y-2y^3)\,dx\,dy$$

Solution. From formula (3) we find

$$I=\int_1^3 dy\int_2^5 (5x^2y-2y^3)\,dx=\int_1^3 (195y-6y^3)\,dy=660$$

Example 3. A rectangular parallelepiped KM_1 (Fig. 443) is cut from above by a paraboloid of revolution with parameter p. The vertex of the paraboloid coincides with the centre C of the upper base, the axis is vertical. Determine the volume V of the resulting solid if the sides of its base are

$$KL=a, \quad KN=b$$

and altitude

$$OC=h$$

Solution. Choose a coordinate system $OXYZ$, as indicated in Fig. 443. The equation of the paraboloid will be

$$z=h-\frac{x^2+y^2}{2p} \qquad (8)$$

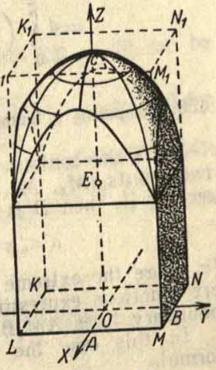

Fig. 443

The required volume is equal to the double integral $\iint\limits_{(KLMN)} z\,dx\,dy$ over the rectangular area

KLMN, or

$$V=\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left(h-\frac{x^2+y^2}{2p}\right)dy\,dx \tag{9}$$

In place of this integral we can take a quadrupled integral over the region *OAMB* (due to the symmetry of the solid about the *xz*- and *yz*-planes), i. e.

$$V=4\int_0^{\frac{a}{2}}\int_0^{\frac{b}{2}}\left(h-\frac{x^2+y^2}{2p}\right)dy\,dx$$

We successively find

$$V=4\int_0^{\frac{a}{2}}\left[hy-\frac{x^2}{2p}y-\frac{y^3}{6p}\right]_0^{\frac{b}{2}}dx=$$

$$=4\int_0^{\frac{a}{2}}\left(\frac{bh}{2}-\frac{bx^2}{4p}-\frac{b^3}{48p}\right)dx=abh-\frac{ab}{24p}(a^2+b^2)$$

456. Computing a Double Integral (General Case)

1. If the contour of the domain D meets, at no more than two points (M_1, M_2 in Fig. 444), every *vertical* line intersecting it, then D is given by the inequalities

$$a\leqslant x\leqslant b,\quad \varphi_1(x)\leqslant y\leqslant \varphi_2(x) \tag{1}$$

[a, b are the extreme abscissas of the domain, $\varphi_1(x)$ and $\varphi_2(x)$ are functions expressing the ordinates of the lower and upper boundary lines AM_1B_1, AM_2B_2].

In this case the double integral is computed from the formula

$$\iint\limits_D f(x,\ y)\,d\sigma=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy \tag{2}$$

2. If the contour of the domain meets, at no more than two points, every *horizontal* straight line intersecting it, we analogously have (in the notation of Fig. 445)

$$\iint\limits_D f(x, y)\, d\sigma = \int\limits_c^d dy \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx \qquad (3)$$

Note. If the contour does not fit either of the foregoing cases, then the domain D is partitioned into several parts (D_1, D_2, D_3 in Fig. 446) so that formula (2) or (3) can be applied to each part.

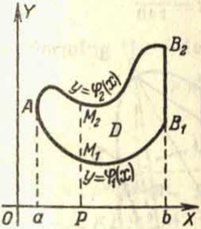

Fig. 444

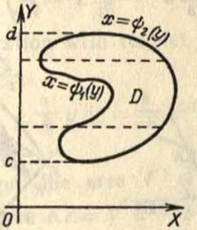

Fig. 445

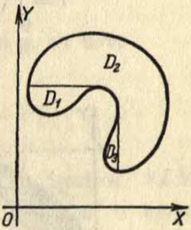

Fig. 446

Example 1. Find the integral $I=\iint\limits_D (y^2+x)\,dx\,dy$ if the domain D is bounded by the parabolas $y=x^2$, $y^2=x$ (Fig. 447; the contour fits both cases 1 and 2).

First solution. Apply (2); put $a=0$, $b=1$, $\varphi_1(x)=x^2$, $\varphi_2(x)=\sqrt{x}$. This yields

$$\iint\limits_D (y^2+x)\,dx\,dy = \int\limits_0^1 dx \int\limits_{x^2}^{\sqrt{x}} (y^2+x)\,dy$$

We compute the integral $\int\limits_{x^2}^{\sqrt{x}} (y^2+x)\,dy$ treating x as a constant:

$$\int_{x^2}^{\sqrt{x}} (y^2+x)\,dy=\left[\frac{y^3}{3}+xy\right]_{y=x^2}^{y=\sqrt{x}}=\left(\frac{1}{3}x^{\frac{3}{2}}+x^{\frac{3}{2}}\right)-$$
$$-\left(\frac{1}{3}x^6+x^3\right)$$

The resulting expression is then integrated with respect to x; this gives

$$I=\int_0^1\left(\frac{4}{3}x^{\frac{3}{2}}-\frac{1}{3}x^6-x^3\right)dx=\frac{33}{140}$$

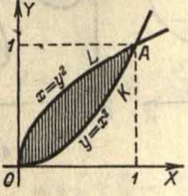

Fig. 447

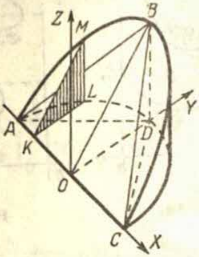

Fig. 448

Second solution. We apply formula (3), putting $c=0$, $d=1$, $\psi_1(y)=y^2$, $\psi_2(y)=\sqrt{y}$. We successively obtain

$$I=\int_0^1 dy\int_{y^2}^{\sqrt{y}}(y^2+x)\,dx=\int_0^1 dy\left[xy^2+\frac{x^2}{2}\right]_{x=y^2}^{x=\sqrt{y}}=$$
$$=\int_0^1\left(y^{\frac{5}{2}}+\frac{y}{2}-\frac{3}{2}y^4\right)dy=\frac{33}{140}$$

Example 2. Find the volume V of a "cylindrical hoof" (the solid $ACDB$ in Fig. 448) cut from a semicylinder by the plane ABC drawn through the diameter AC of the base. Given: radius of base $R=OA$ and altitude of hoof $DB=h$.

Solution. Choose a system of coordinates as indicated in Fig. 448 (then the contour fits both cases, 1 and 2). The equation of the plane ABC will be $z=\frac{h}{R}y$. We have $V = \iint\limits_{(ADC)} \frac{h}{R}\, y\, dx\, dy$.

First method. Put, in (2), Fig. 448,

$$a=-R,\quad b=R,\quad \varphi_1(x)=0,\quad \varphi_2(x)=\sqrt{R^2-x^2}\,(=KL)$$

We obtain

$$V=\int_{-R}^{+R} dx \int_{0}^{\sqrt{R^2-x^2}} \frac{h}{R}\, y\, dy$$

Performing the integration with respect to y, we have

$$\int_{0}^{\sqrt{R^2-x^2}} \frac{h}{R}\, y\, dy=\frac{h}{2R}(R^2-x^2)$$

This expression gives the area F of the section KLM $\left(F=\frac{1}{2}KL\times LM\right.$, where $KL=\sqrt{R^2-x^2}$ and LM is found from the similarity of the triangles $\left.KLM,\ ODB\right)$. We finally have

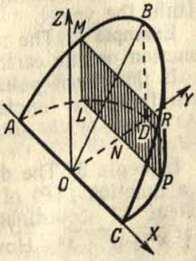

Fig. 449

$$V=\int_{-R}^{R} \frac{h}{2R}(R^2-x^2)\, dx=\frac{2}{3}R^2h$$

Thus, the cylindrical hoof is equal to twice the volume of the pyramid $BACD$.[1]

Second method. In formula (3), Fig. 449, put $c=0$, $d=R$, $\psi_1(y)=-\sqrt{R^2-y^2}\,(=NL)$, $\psi_2(y)=\sqrt{R^2-y^2}\,(=NP)$. We obtain

$$V=\int_{0}^{R} dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{h}{R}\, y\, dx$$

---

[1] This result was found by Archimedes.

The first integration yields

$$\int\limits_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{h}{R}\, y\, dx = 2\frac{h}{R}\, y\sqrt{R^2-y^2}$$

This expression is the area S of the section $PLMR$. We finally get

$$V=\int\limits_0^R 2\frac{h}{R}\sqrt{R^2-y^2}\, y\, dy=\frac{2}{3}R^2h$$

457. Point Function

Let there be given a set of points (say the set of points of a given segment, of a patch of surface, or of a solid). If each point P of this set is associated with a definite value of a quantity z (scalar or vector), then the quantity is called the *function of the point* P. The given point set is termed the *domain of the function*.

Notation: $z=f(P)$.

Example 1. The temperature of a gas in a vessel is a point function; the domain of the function is the set of points lying within the vessel.

Example 2. The annual amount of precipitation is a point function on the earth's surface.

If the given point set is referred to some coordinate system, then the point function becomes a function of the coordinates. *The aspect of the latter depends on the choice of the coordinate system.*

Example 3. The distance of point P from a fixed point O is a function $f(P)$ of the point P. If we take a rectangular system of coordinates with origin at O, then $f(P)=\sqrt{x^2+y^2+z^2}$. However, if the origin is chosen at some other point, then $f(P)=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$, where a, b, c are the coordinates of the point O.

Example 4. The integrand $f(x, y)$ of the double integral $\iint\limits_D f(x, y)\, d\sigma$ is a function of the point $P(x, y)$, and for this reason the integral $\iint\limits_D f(x, y)\, d\sigma$ is written in the form $\iint\limits_D f(P)\, d\sigma$.

458. Expressing a Double Integral in Polar Coordinates

The double integral $\iint\limits_D f(P)\,d\sigma$ is expressed in the polar coordinates of a point P by the formula

$$\iint\limits_D f(P)\,d\sigma = \iint\limits_D F(r, \varphi)\,r\,dr\,d\varphi \tag{1}$$

Here, $F(r, \varphi)$ is that function of the coordinates r, φ which represents the given function $f(P)$ of the point P. The expression $r\,dr\,d\varphi$ is called the *element of area in polar coordinates*. It is equivalent to the area of the quadrangle $ABCD$ (Fig. 450, where $AD \approx OA \cdot \Delta\varphi = r\,d\varphi$ and $AB = DC = dr$).

Integral (1) is expressed by an iterated integral (Sec. 455) as if r and φ were rectangular coordinates [for the integrand $F(r, \varphi)\,r$].

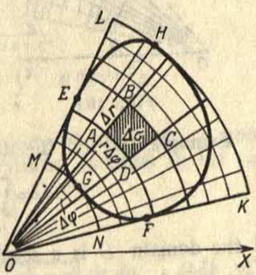

Fig. 450

If the pole is exterior to the contour and each polar ray intersecting the contour meets it at most twice (Fig. 450), then

$$\iint\limits_D F(r, \varphi)\,r\,dr\,d\varphi = \int\limits_{\varphi_1}^{\varphi_2} d\varphi \int\limits_{r_1}^{r_2} F(r, \varphi)\,r\,dr \tag{2}$$

Here, $\varphi_1 = \angle XOK$, $\varphi_2 = \angle XOL$ and r_1 and r_2 are functions of φ which represent the boundary arcs FGE, FHE. In particular, these functions (one or both) can be constant (Fig. 451).

If the pole is interior to the contour (Fig. 452) and each polar ray meets the contour once, then in formula (2) we have to put $r_1 = 0$, $\varphi_1 = 0$, $\varphi_2 = 2\pi$; but if the pole lies on the contour, then $r_1 = 0$, $\varphi_1 = \angle XOA$, $\varphi_2 = \angle XOB$ (Fig. 453).

If every circle with centre at the pole intersects the contour at most twice (Fig. 450), then

$$\iint\limits_D F(r, \varphi)\,r\,dr\,d\varphi = \int\limits_{r_1}^{r_2} r\,dr \int\limits_{\varphi}^{\varphi_2} F(r, \varphi)\,d\varphi \tag{3}$$

Here $r_1=OG$, $r_2=OH$ and φ_1, φ_2 are functions of r representing the boundary arcs GEH, GFH.

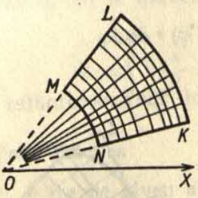

Fig. 451

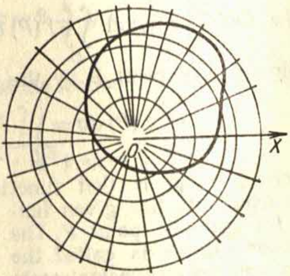

Fig. 452

Example 1. Find the double integral

$$I=\iint_D r\sin\varphi\,d\sigma \tag{4}$$

if the domain D is a semicircle of diameter a (depicted in Fig. 454).

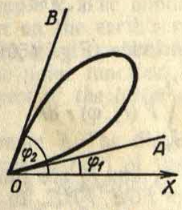

Fig. 453

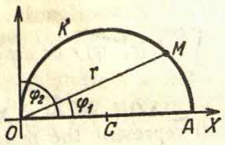

Fig. 454

Solution. For the points M of the semicircle AKO we have (Sec. 71, Example 2): $r=a\cos\varphi$. Apply formula (2), setting $r_1=0$, $r_2=a\cos\varphi$, $\varphi_1=0$, $\varphi_2=\frac{\pi}{2}$; we get

$$\iint_D r\sin\varphi\,d\sigma=\int_0^{\frac{\pi}{2}}d\varphi\int_0^{a\cos\varphi}r^2\sin\varphi\,dr=$$

$$=\int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \int_0^{a\cos\varphi} r^2\, dr=\int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi\, \frac{a^3\cos^3\varphi}{3}=\frac{a^3}{12}$$

Note 1. In order to express integral (4) in rectangular coordinates, we have to put

$$r\sin\varphi=y, \quad d\sigma=dx\,dy$$

Taking into account that the equation of the semicircle AKO is $y^2=ax-x^2$, we get

$$I=\iint_D y\,dx\,dy=\int_0^a dx \int_0^{\sqrt{ax-x^2}} y\,dy=\frac{a^3}{12}$$

Note 2. Integral (4) gives the volume of a cylindrical hoof (cf. Sec. 456, Example 2) whose altitude is equal to the radius of the base.

Example 2. Compute the integral

$$I=\int_{-a}^{+a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$$

Solution. Domain D is a circle of radius a with centre at point $(0, 0)$ (the integral I expresses the volume of a hemisphere of radius a). Computation in rectangular coordinates is cumbersome. Let us pass to polar coordinates. For the pole take the centre of the circle, i.e. the coordinate origin. The integrand will become $\sqrt{a^2-r^2}$, and we get

$$I=\iint_D \sqrt{a^2-r^2}\,d\sigma=\iint_D \sqrt{a^2-r^2}\,r\,dr\,d\varphi$$

Applying (2), we find

$$I=\int_0^{2\pi} d\varphi \int_0^a \sqrt{a^2-r^2}\,r\,dr=\int_0^{2\pi}\frac{a^3}{3}\,d\varphi=\frac{2}{3}\pi a^3$$

Example 3. Find the volume V of a solid cut out of a hemisphere of radius a (Fig. 455) by a cylindrical surface whose diameter is equal to the radius of the sphere, and one of the generatrices coincides with the axis of the hemisphere *(solid of Viviani)*.[1]

Solution. Place the axes as in Fig. 455. The sought-for volume is expressed by the integral

$$I=\iint\limits_D z\,d\sigma=\iint\limits_D \sqrt{a^2-x^2-y^2}\,dx\,dy$$

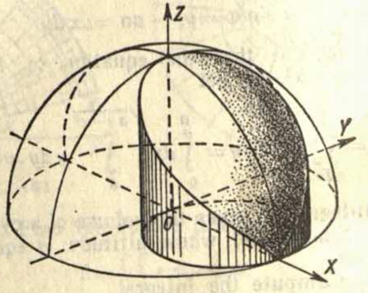

Fig. 455

Computation in rectangular coordinates is involved. Let us take polar coordinates with pole at the centre O of the hemisphere (cf. Examples 1, 2); we get

$$I=\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\varphi \int\limits_0^{a\cos\varphi}\sqrt{a^2-r^2}\,r\,dr=$$

$$=2\int\limits_0^{\frac{\pi}{2}}\frac{a^3(1-\sin^3\varphi)}{3}\,d\varphi=\frac{2}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

459. The Area of a Piece of Surface

Let there be some piece $K'L'M'$ (Fig. 456) of surface S which is projected on the domain D of the xy-plane (KLM in Fig. 456); *only one point* N' of the piece is projected on each point N of domain D.

---

[1] Vincenzo Viviani (1622-1703), mathematician and architect, pupil of Galileo: Viviani used the contour of the upper base for windows in a spherical dome.

Then the area F of the piece $K'L'M'$ is expressed[1] by the double integral

$$F=\iint\limits_D \sqrt{1+p^2+q^2}\,d\sigma \tag{1}$$

where $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$.

Explanation. Let γ be the angle between the tangent plane P at the point N' and the xy-plane. Then (Secs. 127, 436) $\cos\gamma=\frac{1}{\sqrt{1+p^2+q^2}}$. A cylindrical surface having as base the element $\Delta\sigma$ ($ABCD$ in Fig. 456) cuts off of plane P a piece $A'B'C'D'$ the area of which is $\frac{\Delta\sigma}{\cos\gamma}=\sqrt{1+p^2+q^2}\,\Delta\sigma$. The area of the element $abcd$ (of surface S) which is projected on the element $ABCD$ is approximately equal to the area of the piece $A'B'C'D'$, so that the sum of the areas of the pieces $A'B'C'D'$ yields[2] F in the limit:

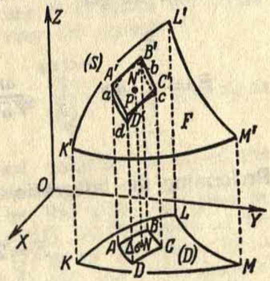

Fig. 456

$$F=\lim\left(\sqrt{1+p_1^2+q_1^2}\,\Delta\sigma_1+\ldots+\sqrt{1+p_n^2+q_n^2}\,\Delta\sigma_n\right) \tag{2}$$

Whence (Sec. 451) formula (1).

Example. Find the area of the upper base of a Viviani solid (Sec. 458, Example 3).

Solution. We have

$$z=\sqrt{a^2-x^2-y^2},\quad p=\frac{\partial z}{\partial x}=-\frac{x}{\sqrt{a^2-x^2-y^2}},$$

$$q=\frac{\partial z}{\partial y}=-\frac{y}{\sqrt{a^2-x^2-y^2}},\qquad \sqrt{1+p^2+q^2}=\frac{a}{\sqrt{a^2-x^2-y^2}}$$

---

[1] It is assumed that the surface has a tangent plane at every point of the patch under consideration, also that the tangent plane is continuously varying (i.e. the angle between the two tangent planes is infinitesimal together with the distance between the points of tangency).

[2] See Note 1 below.

The required area is

$$F=\iint\limits_D \sqrt{1+p^2+q^2}\,d\sigma=\iint\limits_D \frac{a\,d\sigma}{\sqrt{a^2-x^2-y^2}}$$

The domain D is bounded by the circle

$$x^2+y^2-ax=0$$

Expressing the double integral in terms of polar coordinates (Sec. 458), we get

$$F=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_{0}^{a\cos\varphi} \frac{ar\,dr}{\sqrt{a^2-r^2}}=2a\int\limits_{0}^{\frac{\pi}{2}} d\varphi \int\limits_{0}^{a\cos\varphi} \frac{r\,dr}{\sqrt{a^2-r^2}}$$

Performing the integration, we find

$$F=2a^2\left(\frac{\pi}{2}-1\right)$$

Note 1. We assumed that the sum of the areas $A'B'C'D'$ gives, in the limit, the area F. This property (which is in agreement with the graphical conceptions of practical experience) is frequently taken as a definition, which is stated as follows:

Definition. The piece of surface under consideration is partitioned into *abcd*; in each part we choose a point N'. Through the points N' draw tangent planes and project *abcd* on the corresponding tangent plane P by straight lines parallel to OZ. The area of the piece is the limit to which the sum of the areas of the projections tends as the partition is refined indefinitely.

The conditions indicated in Footnote 1 on page 679 ensure the existence of this limit.

Note 2. Given such a definition, it is necessary not only to establish the presence of a limit but also to prove its independence of the choice of coordinate system. This latter problem disappears if we change the definition; namely, if we project *abcd* on the P plane in a direction perpendicular to P. But then the derivation of formula (1) becomes involved.

460. Triple Integral

Definition. [1] Let the function $f(x, y, z)$ of the point $P(x, y, z)$ be continuous within a spatial domain D and on its boundary. Partition D into n parts; let $\Delta v_1, \Delta v_2, \ldots, \Delta v_n$ be their volumes. Take a point in each part and form the sum

$$S_n = f(x_1, y_1, z_1)\,\Delta v_1 + f(x_2, y_2, z_2)\,\Delta v_2 + \ldots + f(x_n, y_n, z_n)\,\Delta v_n \tag{1}$$

The limit to which S_n tends when the largest of the diameters of the subdomains tends to zero, [2] is called the *triple integral of the function* $f(x, y, z)$ *over the domain* D.

Notations:

$$\iiint\limits_D f(x, y, z)\,dv, \quad \text{or} \quad \iiint\limits_D f(P)\,dv, \quad \text{or} \quad \iiint\limits_D f(x, y, z)\,dx\,dy\,dz$$

The expression $dx\,dy\,dz$ in the last notation is called the *element of volume in rectangular coordinates*.

Physical interpretation. Let D be the space occupied by a physical solid, and $f(P)$ the density of the solid at the point $P(x, y, z)$. Then the sum (1) gives an approximate value of the mass M of the solid D and the triple integral $\iiint\limits_D f(P)\,dv$ yields its exact value.

The *properties of a triple integral* are the same as those of a double integral (Sec 453).

461. Computing a Triple Integral (Simplest Case)

Let a spatial domain D be specified by the inequalities

$$a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d, \quad e \leqslant z \leqslant f \tag{1}$$

i.e. let it be depicted by a parallelepiped, the edges of which are parallel to the coordinate axes. Then the triple integral is computed from the formula

$$\iiint\limits_D f(x, y, z)\,dx\,dy\,dz = \int_e^f dz \int_c^d dy \int_a^b f(x, y, z)\,dx \tag{2}$$

---

[1] It is similar to the definition of the double integral (Sec. 451).

[2] A theorem holds here that is similar to the theorem of Sec. 451.

or a similar one (the arguments x, y, z can be interchanged) (cf. Sec. 455).

The expression on the right side of (2) is called an *iterated integral*.

A triple integral taken over a parallelepiped whose edges are parallel to the coordinate axes is also denoted by

$$\int_e^f \int_c^d \int_a^b f(x,\ y,\ z)\,dx\,dy\,dz, \quad \int_a^b \int_c^d \int_e^f f(x,\ y,\ z)\,dz\,dy\,dx$$

and so forth (the outer integral sign corresponds to the outer differential, the inner sign to the inner differential).

Example. Find the integral

$$I=\int_0^1 \int_2^4 \int_0^3 (x+y+z)\,dx\,dy\,dz$$

Solution.

$$I=\int_0^1 dz \int_2^4 dy \int_0^3 (x+y+z)\,dx=$$

$$=\int_0^1 dz \int_2^4 dy \left[\frac{x^2}{2}+(y+z)\,x\right]_{x=0}^{x=3}=$$

$$=\int_0^1 dz \int_2^4 \left(\frac{9}{2}+3y+3z\right) dy$$

Further calculations are as in Sec. 455. We obtain $I=30$.

462. Computing a Triple Integral (General Case)

The given spatial region is partitioned, if necessary, into parts (cf. Sec. 456) so that the "horizontal" projection $\overline{D}$ (Fig. 457) of each part of D is a plane region of the simplest kind (Sec. 456, Items 1 and 2) and so that each "vertical" straight line meeting the boundary of the domain D has at most two common points with it (M_1, M_2 in Fig. 457).

The triple integral taken over every subdomain of D is reduced to a double integral by the formula

$$\iiint\limits_{D} f(x,\ y,\ z)\,dx\,dy\,dz = \iint\limits_{D} dx\,dy \int\limits_{z_1(x,\ y)}^{z_2(x,\ y)} f(x,\ y,\ z)\,dz \qquad (1)$$

where the functions $z_1(x, y)$ and $z_2(x, y)$ are the z-coordinates QM_1 and QM_2. The quantities x and y are held constant during computation of the integral

$$\int\limits_{z_1}^{z_2} f(x,\ y,\ z)\,dz$$

The result of the computation is regarded as a function of the arguments x, y.

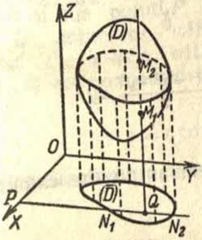

Fig. 457

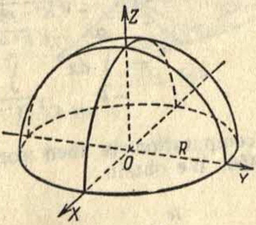

Fig. 458

After integration with respect to the variable z is performed, the right side of (1) becomes a double integral, which is calculated as in Sec. 456. Thus the triple integral is reduced to an iterated integral:

$$\iiint\limits_{D} f(x,\ y,\ z)\,dx\,dy\,dz =$$

$$= \int\limits_{a}^{b} dx \int\limits_{y_1(x)}^{y_2(x)} dy \int\limits_{z_1(x,\ y)}^{z_2(x,\ y)} f(x,\ y,\ z)\,dz \qquad (2)$$

Here the functions $y_1(x)$ and $y_2(x)$ are the ordinates PN_1, PN_2.

Example. Find the integral $I = \iiint\limits_{D} z\,dv$ extended over a hemisphere of radius R (Fig 458).

The integral I expresses the static moment of the hemisphere with respect to the plane of the base (the density μ of the hemisphere is taken as unity).

Solution. It is not necessary to partition the domain. The domain $\bar{D}$ is the circle

$$x^2+y^2\leqslant R^2$$

so that $a=-R$, $b=R$, $y_1(x)=-\sqrt{R^2-x^2}$, $y_2(x)=\sqrt{R^2-x^2}$. The z-coordinates of the lower and upper boundaries of the hemisphere are $z_1(x, y)=0$, $z_2(x, y)=\sqrt{R^2-x^2-y^2}$. Using formula (2), we find

$$I=\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{0}^{\sqrt{R^2-x^2-y^2}} z\,dz=$$

$$=\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{R^2-x^2-y^2}{2}\,dy$$

The computation is then continued as in the examples of Sec. 456. We obtain

$$I=\frac{1}{2}\int_{-R}^{R} dx\left[(R^2-x^2)\,y-\frac{y^3}{3}\right]_{y=-\sqrt{R^2-x^2}}^{y=\sqrt{R^2-x^2}}=$$

$$=\frac{2}{3}\int_{-R}^{R}(R^2-x^2)^{\frac{3}{2}}\,dx=\frac{2}{3}\cdot\left[\frac{1}{4}x(R^2-x^2)^{\frac{3}{2}}+\right.$$

$$\left.+\frac{3}{8}R^2x(R^2-x^2)^{\frac{1}{2}}+\frac{3}{8}R^4\arcsin\frac{x}{R}\right]_{-R}^{+R}=\frac{\pi R^4}{4}$$

Note. The mass of a hemisphere (for $\mu=1$) is numerically equal to its volume $\frac{2}{3}\pi r^3$. The quotient $I:\frac{2}{3}\pi R^3=\frac{3}{8}R$ is the altitude of the centre of gravity above the plane of the base. Hence, the centre of gravity divides the altitude of the hemisphere in the ratio 5:3.

463. Cylindrical Coordinates

The position of a point P (Fig. 459) in space may be determined by its z-coordinate

$$z=QP$$

and by the polar coordinates

$$r=OQ,\quad \varphi=\angle XOQ$$

of its projection Q on the xy-plane. The quantities r, φ, z are called the *cylindrical coordinates of the point* P. The rectangular and cylindrical coordinates of the point P (if the origin O coincides with the pole and the x-axis with the polar axis) are connected by the relations

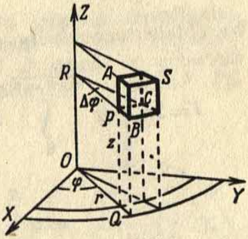

Fig. 459

$$x=r\cos\varphi,\qquad y=r\sin\varphi$$

(the z-coordinates are the same in both systems).

464. Expressing a Triple Integral in Cylindrical Coordinates

A triple integral $\iiint\limits_D f(P)\,dv$ is expressed in cylindrical coordinates of the point P by the formula

$$\iiint\limits_D f(P)\,dv=\iiint\limits_D F(r,\ \varphi,\ z)\,r\,dr\,d\varphi\,dz \tag{1}$$

Here, $F(r,\ \varphi,\ z)$ is the function of cylindrical coordinates which represents the function $f(P)$ of point P. The expression $r\,dr\,d\varphi\,dz$ is called the *element of volume in cylindrical coordinates*. It is equivalent to the volume of the solid PS (Fig. 459) where $PA=dz$, $PB=dr$, $\overset{\frown}{PC}=r\,d\varphi$.

The integral (1) is expressed in terms of an iterated integral as if r, φ, z were rectangular coordinates for the integrand $F(r, \varphi, z)\, r$.

Example. Using cylindrical coordinates, compute the integral found in the example of Sec. 462. We have

$$I=\iiint\limits_{D} zr\,dr\,d\varphi\,dz=\int\limits_{0}^{R} dz \int\limits_{0}^{\sqrt{R^2-x^2}} dr \int\limits_{0}^{2\pi} zr\,d\varphi \qquad (2)$$

We obtain successively

$$I=2\pi\int\limits_{0}^{R} dz \int\limits_{0}^{\sqrt{R^2-z^2}} zr\,dr=2\pi\int\limits_{0}^{R} z\,dz\left[\frac{r^2}{2}\right]_0^{\sqrt{R^2-z^2}}=$$

$$=\pi\int\limits_{0}^{R}(R^2-z^2)\,z\,dz=\frac{\pi R^4}{4} \qquad (3)$$

465. Spherical Coordinates

The position of a point P in space (Fig. 460) may be defined by the following three quantities: the distance

$$\rho=OP$$

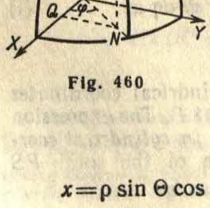

Fig. 460

from the point O, the angle

$$\Theta=\angle ZOP$$

between the rays OZ and OP, and the angle

$$\varphi=\angle XON$$

between the half-planes ZOX and ZOP. The quantities ρ, Θ, φ are called the *spherical* or *polar coordinates* of the point P. The rectangular and spherical coordinates (if the basic planes of both systems coincide) are connected by the relations

$$x=\rho\sin\Theta\cos\varphi, \quad y=\rho\sin\Theta\sin\varphi, \quad z=\rho\cos\Theta$$

466. Expressing a Triple Integral in Spherical Coordinates

The triple integral $\iiint\limits_D f(P)\,dv$ is expressed in spherical coordinates of the point P by the formula

$$\iiint\limits_D f(P)\,dv=\iiint\limits_D F(\rho,\ \Theta,\ \varphi)\,\rho^2\,d\rho\,\sin\Theta\,d\Theta\,d\varphi \qquad (1)$$

Here, $F(\rho,\ \Theta,\ \varphi)$ is that function of spherical coordinates which is the function $f(P)$ of the point P. The expression $\rho^2\,d\rho\,\sin\Theta\,d\Theta\,d\varphi$ is called the *element of volume in spherical coordinates*. It is equivalent to the volume of the solid[1] PS

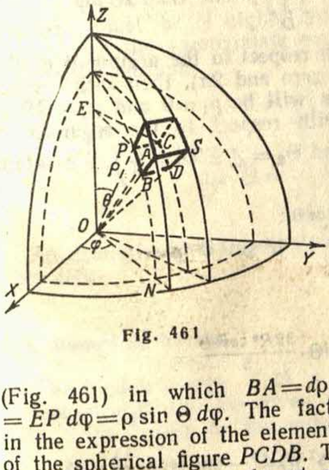

Fig. 461

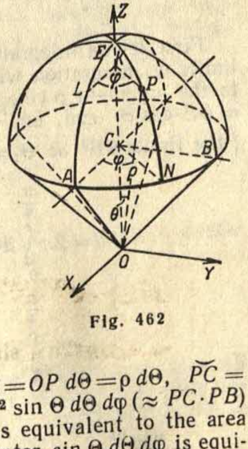

Fig. 462

(Fig. 461) in which $BA=d\rho$, $\breve{PB}=OP\,d\Theta=\rho\,d\Theta$, $\breve{PC}=EP\,d\varphi=\rho\sin\Theta\,d\varphi$. The factor $\rho^2\sin\Theta\,d\Theta\,d\varphi\ (\approx PC\cdot PB)$ in the expression of the element dv is equivalent to the area of the spherical figure $PCDB$. The factor $\sin\Theta\,d\Theta\,d\varphi$ is equivalent to the solid angle at which the rectangle $PCDB$ is seen from the centre.[2]

---

[1] This solid is bounded by two spherical surfaces (with radii r and $r+dr$), by two planes passing through the z-axis, and by two conical surfaces whose axes coincide with the z-axis.

[2] A solid angle is a portion of space contained inside a cavity of some conical surface (with closed directrix). For a measure of a solid angle we take the ratio of the area cut out by the solid angle in a sphere (with centre at the vertex of the solid angle) to the square of the radius of the sphere.

Example. Find the integral $I=\iiint\limits_D r^2\,dv$ where the function $f(P)=r^2$ of point P is the square of its distance from the z-axis (KP in Fig. 462), and the domain D is the solid bounded from below by a cone (whose altitude OC is equal to the radius of the base $CA=R$) and above by a hemisphere of radius R.

The integral I expresses the moment of inertia of the solid D about its z-axis (Sec. 468).

Solution. Let us introduce the spherical coordinates $\rho=OP$, $\Theta=\angle EOP$, $\varphi=\angle ACN=\angle LKP$. Since $r=KP=\rho\sin\Theta$, the desired integral is of the form

$$I=\iiint\limits_D \rho^2\sin^2\Theta\,dv=\iiint\limits_D \rho^4\sin^3\Theta\,d\rho\,d\Theta\,d\varphi$$

First let us integrate with respect to the argument φ (the limits of integration will be zero and 2π), then with respect to the argument ρ (the limits will be $\rho_1=0$ and $\rho_2=OP=OE\cdot\cos\Theta$) and, finally, with respect to the argument Θ $\left(\text{the limits will be } \Theta_1=0 \text{ and } \Theta_2=\angle EOA=\frac{\pi}{4}\right)$. We obtain

$$I=2\pi\int\limits_0^{\frac{\pi}{4}} d\Theta \int\limits_0^{2R\cos\Theta} \rho^4\sin^3\Theta\,d\rho=$$

$$=2\pi\int\limits_0^{\frac{\pi}{4}} \sin^3\Theta\,d\Theta\cdot\frac{32R^5\cos^5\Theta}{5}=$$

$$=\frac{64\pi R^5}{5}\int\limits_0^{\frac{\pi}{4}} \cos^5\Theta\,(1-\cos^2\Theta)\,d(-\cos\Theta)=\frac{11}{30}\pi R^5$$

467. Scheme for Applying Double and Triple Integrals

Multifarious geometrical and physical quantities can be expressed by double or triple integrals, depending on whether they refer to a surface (plane or curved) or to a solid.[1] The scheme is the same as for quantities expressed by means of ordinary (single) integrals, namely (cf. Sec 334):

(1) The desired quantity U is associated with some domain D (of a surface or space)

(2) The domain D is partitioned into subdomains $\Delta\sigma_k$ (or Δv_k); their number will subsequently tend to infinity and their diameters will tend to zero.

Let the sought-for quantity U be subdivided into parts $u_1, u_2, \ldots, u_n$, the sum of which yields U.[2]

(3) One of the parts $u_1, u_2, \ldots$ is taken as a representative and is expressed by an approximate formula of the type

$$u_k \approx f(P_k)\,\Delta\sigma_k$$

$$[\text{or } u_k \approx f(P_k)\,\Delta v_k]$$

The error must be of higher order than $\Delta\sigma_k$ (or than Δv_k).

(4) From the approximate equation we obtain the following exact equation:

$$U = \iint\limits_D f(P)\,d\sigma$$

$$[\text{or } U = \iiint\limits_D f(P)\,dv]$$

An example is the computation of the moment of inertia (Sec. 468).

468. Moment of Inertia

The kinetic energy T of a body rotating on an axis AB is proportional (for the given position of the axis with respect to the body) to the square of the angular velocity ω:

$$T = \frac{1}{2} I\omega^2 \tag{1}$$

The doubled coefficient of proportionality, that is the quantity I, is called the *moment of inertia* of the body about the axis AB. If the body consists of n mass points with masses $m_1, m_2, \dots, m_n$ distant from the axis $r_1, r_2, \dots, r_n$, then the moment of inertia is expressed by the formula

---

[1] Quantities which refer to a line are expressed by ordinary integrals.

[2] Quantities possessing this property are termed additive (see fine print on p. 491).

$$I = m_1 r_1^2 + m_2 r_2^2 + \dots + m_n r_n^2 \tag{2}$$

The expression of the moment of inertia of a solid body is obtained from (2) by applying the scheme of Sec. 467, namely:

(1) The moment of inertia I is associated with the domain D occupied by the body.

(2) The domain D is subdivided into parts $D_1, D_2, \dots, D_n$ and I is decomposed into the parts $I_1, I_2, \dots, I_n$, the sum of which yields I.

(3) Assume that in particle D_k the density μ_k is everywhere such as in one of its points P_k. We obtain the approximate equation

$$m_k \approx \mu_k \, \Delta v_k \tag{3}$$

and the moment of inertia I_k is expressed by the approximate formula

$$I_k \approx \mu_k r_k^2 \, \Delta v_k \tag{4}$$

(4) From the approximate equation (4) we obtain the exact equation

$$I = \iiint\limits_D \mu r^2 \, dv \tag{5}$$

See Example in Sec. 466.

If the axis AB is taken as the z-coordinate, then (5) takes the form

$$I = \iiint\limits_D \mu(x, y, z)(x^2 + y^2) \, dx \, dy \, dz \tag{6}$$

If the given body is a lamina whose plane is perpendicular to the axis AB, then in place of the triple integral (6) we get the double integral

$$I = \iint\limits_D \mu(x, y)(x^2 + y^2) \, dx \, dy \tag{7}$$

where $\mu(x, y)$ is the surface density of the lamina.

If the given body is a rectilinear rod intersecting the axis AB at right angles, then by bringing it to coincidence with the x-axis (we will then have $y = 0$), we get, in place

469. Expressing Certain Physical and Geometrical Quantities in Terms of Double Integrals

Quantity	General expression	In rectangular coordinates	In polar coordinates
Area of plane figure	$S=\iint\limits_D d\sigma$	$\iint dx\,dy$	$\iint r\,dr\,d\varphi$
Area of piece of surface (Sec. 459)[1]	$S=\iint\limits_D \frac{d\sigma}{\cos\nu}$	$\iint\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$	$\iint\sqrt{r^2+r^2\left(\frac{\partial z}{\partial \rho}\right)^2+\left(\frac{\partial z}{\partial \varphi}\right)^2}\,dr\,d\varphi$
Volume of cylindrical solid standing on xy-plane (Sec. 452)	$V=\iint\limits_D z\,d\sigma$	$\iint z\,dx\,dy$	$\iint zr\,dr\,d\varphi$

---

[1] Domain D is the projection on the xy-plane; only one point of the surface is projected on every point of the domain; γ is the angle between the tangent plane and the xy-plane.

Continued

Quantity	General expression	In rectangular coordinates	In polar coordinates
Moment of inertia of plane figure[1] about z-axis[2]	$I_z=\iint\limits_D r^2\,d\sigma$	$\iint (x^2+y^2)\,dx\,dy$	$\iint r^3\,dr\,d\varphi$
Moment of inertia of plane figure[1] about x-axis	$I_x=\iint\limits_D y^2\,d\sigma$	$\iint y^2\,dx\,dy$	$\iint r^3\sin^2\varphi\,dr\,d\varphi$
Coordinates of centre of gravity of homogeneous lamina[2]	$x_c=\dfrac{\iint\limits_D x\,d\sigma}{S}$ $y_c=\dfrac{\iint\limits_D y\,d\sigma}{S}$	$\dfrac{\iint x\,dx\,dy}{S}$ $\dfrac{\iint y\,dx\,dy}{S}$	$\dfrac{\iint r^2\cos\varphi\,dr\,d\varphi}{S}$ $\dfrac{\iint r^2\sin\varphi\,dr\,d\varphi}{S}$

---

[1] Coincident with the xy-plane.

[2] Or, what is the same, about the centre O.

470. Expressing Certain Physical and Geometrical Quantities in Terms of Triple Integrals

Quantity	General expression	In rectangular coordinates	In cylindrical coordinates	In spherical coordinates
Volume of body	$V=\iiint\limits_D dv$	$\iiint dx\,dy\,dz$	$\iiint r\,dr\,d\varphi\,dz$	$\iiint \rho^2 \sin\theta\,d\rho\,d\varphi\,d\theta$
Moment of inertia of geometrical body about z-axis	$I_z=\iiint\limits_D r^2\,dv$	$\iiint (x^2+y^2)\,dx\,dy\,dz$	$\iiint r^3\,dr\,d\varphi\,dz$	$\iiint \rho^4 \sin^3\theta\,d\rho\,d\varphi\,d\theta$
Mass of physical body [1]	$M=\iiint\limits_D \mu\,dv$	$\iiint \mu\,dx\,dy\,dz$	$\iiint \mu r\,dr\,d\varphi\,dz$	$\iiint \mu\rho^2 \sin\theta\,d\rho\,d\varphi\,d\theta$

---

[1] The density (point function) is denoted by μ.

Continued

Quantity	General expression	In rectangular coordinates	In cylindrical coordinates	In spherical coordinates
Coordinates of centre of gravity of homogeneous body	$x_c = \frac{\iiint\limits_D x\,dv}{V}$	$\frac{\iiint x\,dx\,dy\,dz}{V}$		
	$y_c = \frac{\iiint\limits_D y\,dv}{V}$	$\frac{\iiint y\,dx\,dy\,dz}{V}$		
	$z_c = \frac{\iiint\limits_D z\,dv}{v}$	$\frac{\iiint z\,dx\,dy\,dz}{V}$		

of the triple integral (6) the ordinary integral

$$I=\int_a^b \mu(x)\,x^2\,dx \tag{8}$$

where $\mu(x)$ is the linear density of the rod.

Note. The moment of inertia of a geometrical body is the moment of inertia of a material body occupying the same space and having everywhere density equal to unity.

Formulas (6), (7), (8) take the form

$$I=\iiint\limits_D (x^2+y^2)\,dx\,dy\,dz, \tag{6a}$$

$$I=\iint\limits_D (x^2+y^2)\,dx\,dy, \tag{7a}$$

$$I=\int_a^b x^2\,dx\left(=\frac{b^3-a^3}{3}\right) \tag{8a}$$

471. Line Integrals

Let there be given a function $P(x, y)$ that is continuous in some domain of the number plane XOY. Let us take in that domain some curve 1) with initial point at A (Figs. 463, 464) and terminal point at B (the initial and terminal points can coincide).

Partition AB (Fig. 463) into n subarcs $AA_1, A_1A_2, \ldots, A_{n-1}B$ and for the sake of uniformity assign the designations A_0, A_n to the points A and B. Take a point $M_i(x_i, y_i)$ on each subarc A_iA_{i+1} and form the sum

$$S_n=P(x_1, y_1)\,\Delta x_1+P(x_2, y_2)\,\Delta x_2+\ldots+P(x_n, y_n)\,\Delta x_n \tag{1}$$

where Δx_i is the increment in the abscissa corresponding to the motion of the point A_{i-1} to the point A_i. 2)

The following theorem holds.

---

1) It is assumed that the curve AB has a continuously varying tangent except for a finite number of separate points where the tangent can vary discontinuously, as at S and T in Fig. 464.

2) This increment may be positive (as on the segment AA_1) or negative (as on A_4A_5).

Theorem. If as n increases without bound the largest of the quantities $|\Delta x_i|$ tends to zero, then the sum (1) tends to a limit that is independent both of the mode of formation of the segments A_iA_{i+1} and of the choice of the intermediate points M_i.

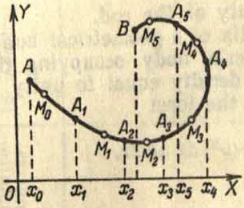

Fig. 463

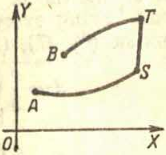

Fig. 464

Definition. The limit to which the sum S_n tends when the largest of the quantities $|\Delta x_i|$ tends to zero is called the *line (curvilinear) integral* of the expression $P(x, y)\,dx$ *taken along the path AB.*

Notation.

$$\int\limits_{AB} P(x, y)\,dx \tag{2}$$

The line integral of the expression $Q(x, y)\,dy$ is similarly defined:

$$\int\limits_{AB} Q(x, y)\,dy \tag{3}$$

and also the line integral of the expression $P(x, y)\,dx + Q(x, y)\,dy$ denoted by

$$\int\limits_{AB} P(x, y)\,dx + Q(x, y)\,dy \tag{4}$$

The integrals (2) and (3) are special cases of the integral (4) (for $Q=0$ and for $P=0$).

In similar fashion we define the line integral

$$\int\limits_{AB} P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz \tag{5}$$

along the space curve AB.

Note 1. If, while retaining the curve AB, we reverse the direction of the path, the line integral maintains its absolute value and reverses sign. When the points A and B are distinct, the direction of the path is indicated by the order of the letters A, B in the notations (2) to (5), and we have

$$\int\limits_{BA} P\,dx+Q\,dy=-\int\limits_{AB} P\,dx+Q\,dy,$$

$$\int\limits_{BA} P\,dx+Q\,dy+R\,dz=-\int\limits_{AB} P\,dx+Q\,dy+R\,dz$$

When the points A and B coincide, the direction of the path may be indicated by intermediate points in an appropriate order.

Such an indication is not needed when the path is a contour K in a plane domain. In that case the notation $\int\limits_{+K} P\,dx+Q\,dy$ states that the traversal is counterclockwise (for the conventional arrangement of the axes). But if the domain is traversed in the opposite direction, the line integral is written $\int\limits_{-K} P\,dx+Q\,dy$.

Note 2. The line integral is a generalization of the ordinary integral[1] and possesses all the properties of the latter (Sec. 315).

472. Mechanical Meaning of a Line Integral

Let a point M of mass m be in motion along a path AB in a field of force. Let $X(x, y, z)$, $Y(x, y, z)$, $Z(x, y, z)$ be the coordinates of the *vector of potential* at the point $M(x, y, z)$, that is, the vector of the force F acting at the point (x, y, z) on unit mass. Then the work performed by that force acting on point M is expressed by the line integral

---

[1] If the path of integration AB is a segment (a, b) of the x-axis, the line integral $\int\limits_{AB} P(x, y)\,dx+Q(x, y)\,dy$ turns into the ordinary integral $\int\limits_a^b P(x, 0)\,dx$.

$$\int\limits_{AB} m\,(X\,dx+Y\,dy+Z\,dz) \tag{1}$$

Explanation. Let A_iA_{i+1} be a small segment of the path AB. The work performed on that segment is approximately expressed[1] by the scalar product (Sec. 104a) $mF_i\overrightarrow{A_iA_{i+1}}$, where $\boldsymbol{F_i}$ is the vector of potential at the point A_i. In coordinate form (Sec. 107) we obtain $m\,[X_i\Delta x_i+Y_i\Delta y_i+Z_i\Delta z_i]$. Summing, we find the approximate value of the work along the path AB. The limit of the sum, that is the line integral (1), gives the exact value of the work.

473. Computing a Line Integral

To compute the line integral

$$\int\limits_{AB} P\,(x,\ y)\,dx+Q\,(x,\ y)\,dy \tag{1}$$

we have to represent the curve AB by parametric equations:

$$x=\varphi\,(t),\qquad y=\psi\,(t) \tag{2}$$

and substitute expression (2) into the integrand. The ordinary integral

$$\int\limits_{t_A}^{t_B}\{P\,[\varphi\,(t),\ \psi\,(t)]\,\varphi'\,(t)+Q\,[\varphi\,(t),\ \psi\,(t)]\,\psi'\,(t)\}\,dt \tag{3}$$

is equal to line integral (1).

Note. One of the functions $\varphi\,(t)$, $\psi\,(t)$ may be chosen at will, so long as both functions $\varphi\,(t)$, $\psi\,(t)$ have continuous derivatives throughout the interval $(t_A,\ t_B)$ with the exception of points where the tangent changes by a jump, as at S, T in Fig. 464. If there are such points, integral (3) is improper (Sec. 328).

A line integral taken along a space curve is computed in similar fashion.

---

[1] We replace the curvilinear segment $\overparen{A_iA_{i+1}}$ by the chord A_iA_{i+1} and assume that the field potential along this chord remains unchanged.

Example 1. Compute the line integral

$$I=\int\limits_{AB} -y\,dx+x\,dy \tag{4}$$

along the upper part of the semicircle $x^2+y^2=a^2$ (Fig. 465).

Solution. Represent the arc AB by the parametric equations

$$x=a\cos t, \quad y=a\sin t \tag{5}$$

(here t is the angle BOM so that $t_A=\pi$, $t_B=0$). Putting (5) into (4), we find

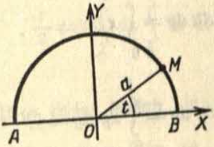

Fig. 465

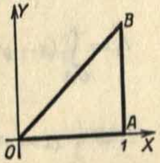

Fig. 466

$$\begin{aligned} I&=\int\limits_{\pi}^{0} -a\sin t\,d(a\cos t)+a\cos t\,d(a\sin t)= \\ &=a^2\int\limits_{\pi}^{0} dt=-\pi a^2 \end{aligned} \tag{6}$$

For the parameter we can take the abscissa x, i. e. we can take the equation of the semicircle in the form $y=\sqrt{a^2-x^2}$. Then $x_A=-a$, $x_B=a$ and we get

$$\begin{aligned} I&=\int\limits_{-a}^{a} -\sqrt{a^2-x^2}\,dx+x\,d\sqrt{a^2-x^2}= \\ &=-a^2\int\limits_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}=-\pi a^2 \end{aligned}$$

To take for the parameter the ordinate y, we first have to subdivide the arc AB into parts, otherwise x will not be a single-valued function of the ordinate.

Example 2. Compute the line integral

$$I=\int\limits_{OABO}(x-y^2)\,dx+2xy\,dy \tag{7}$$

along the perimeter of the triangle OAB (Fig. 466).

Solution. We divide the closed path $OABO$ into three segments OA, AB, BO. On segment OA we take the abscissa for the parameter (then $y=0$, $dy=0$), on segment AB, the ordinate (then $x=1$, $dx=0$), on BO, the abscissa (then $y=x$, $dy=dx$). We have

$$I_1=\int\limits_{OA}(x-y^2)\,dx+2xy\,dy=\int\limits_0^1 x\,dx=\frac{1}{2},$$

$$I_2=\int\limits_{AB}(x-y^2)\,dx+2xy\,dy=\int\limits_0^1 2y\,dy=1,$$

$$I_3=\int\limits_{BO}(x-y^2)\,dx+2xy\,dy=\int\limits_1^0 (x+x^2)\,dx=-\frac{5}{6};$$

$$I=I_1+I_2+I_3=\frac{1}{2}+1-\frac{5}{6}=\frac{2}{3}$$

474. Green's Formula

Let D be a plane domain bounded by a contour K (Fig 467) and let the functions $P(x, y)$, $Q(x, y)$ be everywhere continuous in this domain together with their partial derivatives $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$. Then the following *Green's formula*[1] holds true:

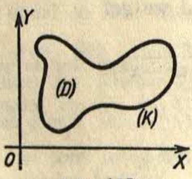

Fig. 467

$$\int\limits_{+K}P(x, y)\,dx+Q(x, y)\,dy=\iint\limits_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy \tag{1}$$

---

[1] George Green (1793-1841) English mathematician and physicist, contributed substantially to the mathematical theory of electricity and magnetism.

Example. Compute the line integral $I=\int (x-y^2)\,dx+2xy\,dy$ along the perimeter of triangle OAB (Fig. 466) (cf. Sec. 473, Example 2).

Solution. By formula (1), putting $P=x-y^2$, $Q=2xy$, we find

$$I=\iint_D\left[\frac{\partial}{\partial x}(2xy)-\frac{\partial}{\partial y}(x-y^2)\right]dx\,dy=\iint_D 4y\,dx\,dy$$

Here, domain D is the triangle OAB; computing the double integral, we find

$$I=\int_0^1 dx\int_0^x 4y\,dy=\int_0^1 2x^2\,dx=\frac{2}{3}$$

475. Condition Under Which Line Integral Is Independent of Path

Let the functions $P(x, y)$, $Q(x, y)$, and also their partial derivatives $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$ be continuous in a domain D (Fig. 468) bounded by some continuous (nonselfintersecting) closed curve. In D take two fixed points $A(x_0, y_0)$, $B(x_1, y_1)$ and consider all possible paths of integration leading from A to B and lying wholly in D (such are the paths ALB, ANB in Fig 468). Two cases are possible.

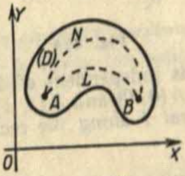

Fig. 468

Case 1 (exceptional). In domain D the following equation is identically satisfied:

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0 \tag{1}$$

Then the line integral

$$I=\int_{AB} P\,dx+Q\,dy \tag{2}$$

does not depend on the choice of path and, accordingly, is denoted by

$$\int_A^B P\,dx+Q\,dy$$

Case 2 (general). Eq. (1) is not an identity. Then the line integral (2) depends on the choice of path.

Explanation. The difference $I_1 - I_2$ of the line integrals $I_1 = \int\limits_{ALB} P\,dx + Q\,dy$, $I_2 = \int\limits_{ANB} P\,dx + Q\,dy$ is equal to the sum $I_1 + (-I_2)$, that is (Sec. 471, Note 1) to the sum $\int\limits_{ALB} P\,dx + Q\,dy + \int\limits_{BNA} P\,dx + Q\,dy$. The latter sum gives the integral along the contour $ALBNA$; it is equal (Sec. 474) to the double integral $I_3 = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$ over the domain $ALBNA$. If Eq. (1) is an identity, then $I_3 = 0$; hence, $I_1 = I_2$, i. e. the line integrals along the paths ALB and ANB are the same. But if Eq. (1) is not an identity, then it is possible to choose paths ALB and ANB so that $I_3 \neq 0$, and then $I_1 \neq I_2$.

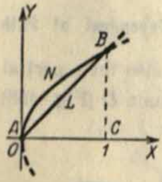

Fig. 469

Example 1. Consider the integral

$$I = \int\limits_{AB} y\,dx + x\,dy \tag{3}$$

The functions $P(x, y) = y$, $Q(x, y) = x$, $\frac{\partial Q}{\partial x} = 1$, $\frac{\partial P}{\partial y} = 1$ are everywhere continuous and Eq. (1) is satisfied identically. Hence, for fixed points A, B integral (3) is independent of the path. For example, let us take points $A(0, 0)$ and $B(1, 1)$ (Fig. 469) and let us compute the integral I along the rectilinear path ALB $(y = x)$. We get

$$I_{ALB} = \int_0^1 x\,dx + x\,dx = 1$$

If for the path we take an arc of the parabola ANB $(x = y^2)$, we again get $I_{ANB} = \int_0^1 y\,d(y^2) + y^2\,dy = 3\int_0^1 y^2\,dy = 1$. The same value is obtained by going along the polygonal line ACB. Along AC we have: $y = 0$, $dy = 0$ so that $I_{AC} =$

$=\int\limits_0^1 0\cdot dx=0$; along CB we have: $x=1$, $dx=0$ so that $I_{CB}=\int\limits_0^1 1\cdot dy=1$. Hence, $I_{ACB}=I_{AC}+I_{CB}=1$.

Notation: $I=\int\limits_{A(0,\,0)}^{B(1,\,1)} y\,dx+x\,dy=1$.

Example 2. Retaining the points $A(0,0)$, $B(1,1)$, consider the integral $I=\int\limits_{AB} y^2\,dx+x^2\,dy$. Eq. (1) takes the form $x-y=0$, i. e. it is not an identity. The integral I now depends on the path. Thus, along the path ALB (Fig. 469) we have $I=\int\limits_0^1 x^2\,dx+x^2\,dx=\frac{2}{3}$, along the path ANB the integral has a different value:

$$I=\int\limits_0^1 y^2\,d(y^2)+y^4\,dy=\int\limits_0^1 (2y^3+y^4)\,dy=\frac{7}{10}$$

We get the same value $\frac{7}{10}$ by going along the arc of the parabola $y=x^2$. Generally, in Case 2 it is always possible to choose two paths along which the integral will have the same values.

476. An Alternative Form of the Condition Given in Sec. 475

Theorem 1 (criterion of total differential). If the equation

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0 \tag{1}$$

is satisfied identically in a domain D, then for every point of D the expression $P\,dx+Q\,dy$ is the total differential of some function $F(x, y)$. But if Eq. (1) is not an identity, then the expression $P\,dx+Q\,dy$ is not a total differential of any function.

Example 1. For the expression $y\,dx+x\,dy$ (here $P=y$, $Q=x$) Eq. (1) is satisfied identically in any domain. Therefore, $y\,dx+x\,dy$ is the total differential of some function $F(x, y)$. In the given case, we can take $F(x, y)=xy$ or $xy+3$ and generally, $xy+C$.

Example 2. The expression $y^2\,dx+x^2\,dy$ cannot be the total differential of any function because Eq (1), which takes the form $2x-2y=0$, is not an identity.

Explanation. Suppose that $y^2\,dx+x^2\,dy$ is the differential of some function $F(x, y)$. Then we would have $\frac{\partial F}{\partial x}=y^2$, $\frac{\partial F}{\partial y}=x^2$. But this is impossible because the mixed derivatives $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$ (they are continuous) must be equal (Sec. 443), i. e. the equation $\frac{\partial}{\partial x}(x^2)-\frac{\partial}{\partial y}(y^2)=0$ must be satisfied identically, but this is not the case.

By virtue of Theorem 1 the condition of Sec. 475 takes the following form.

Case 1 (exceptional). The expression $P\,dx+Q\,dy$ is (in the given domain) the total differential of some function $F(x, y)$ (it is called an *antiderivative*). Then the line integral $\int\limits_{AB} P\,dx+Q\,dy$ is independent of the choice of path (lying in that domain).

Case 2 (general). The expression $P\,dx+Q\,dy$ is not a total differential. Then the line integral depends on the choice of path

In the former case, we can compute the value of the integral (if we know an antiderivative) on the basis of the following theorem.

Theorem 2. If the integrand $P\,dx+Q\,dy$ is the total differential of the function $F(x, y)$, then the line integral $\int\limits_A^B P(x, y)\,dx+Q(x, y)\,dy$ is equal to the difference between the values of the function at the points B and A:

$$\int\limits_{A(x_0, y_0)}^{B(x_1, y_1)} P\,dx+Q\,dy=\int\limits_{A(x_0, y_0)}^{B(x_1, y_1)} dF(x, y)=$$
$$=F(x_1, y_1)-F(x_0, y_0) \tag{2}$$

Example 3. The integral $I=\int\limits_{AB} 2xy\,dx+x^2\,dy$ for fixed points $A(1, 3)$, $B(2, 4)$ is independent of the choice of path

$\left[\text{because } \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=\frac{\partial (x^2)}{\partial x}-\frac{\partial (2xy)}{\partial y}\equiv 0\right]$. It is required to find the value of I.

Solution. The expression $2xy\,dx+x^2\,dy$ is the total differential of the function x^2y. By Theorem 2 we have

$$I=\int\limits_{A(1,\ 3)}^{B(2,\ 4)} d(x^2y)=2^2\cdot 4-1^2\cdot 3=13$$

Note. In the general case it is just as difficult to find an antiderivative as it is to compute the line integral directly.

However, in many cases, finding an antiderivative is simplified. Thus, if each of the functions $P(x, y)$, $Q(x, y)$ is the sum of terms of the type Ax^my^n (A is a constant, m and n are any real numbers), then the antiderivative is found in the following manner. We compute the indefinite integrals $\int P(x, y)\,dx$, $\int Q(x, y)\,dy$, holding y constant in the first integral, and x in the second. Then we combine the two expressions and take only once each of the terms entering into both expressions. The arbitrary constants which appear in integration may be omitted since it is sufficient to have one antiderivative.

Example 4. Find the line integral

$$I=\int\limits_{A(0,\ 0)}^{B(1,\ 1)} x(1+2y^3)\,dx+3y^2(x^2-1)\,dy$$

[Condition (1) is fulfilled].

Solution. We find $\int x(1+2y^3)\,dx=\frac{x^2}{2}+x^2y^3$ (y is treated as a constant), $\int 3y^2(x^2-1)\,dy=x^2y^3-y^3$ (x is treated as a constant).

Combine these expressions taking the term x^2y^3 once. We get tne antiderivative $F(x, y)=\frac{x^2}{2}-y^3+x^2y^3$. Formula (2) yields $I=F(1, 1)-F(0, 0)=\frac{1}{2}$.

DIFFERENTIAL EQUATIONS

477. Fundamentals

A *differential equation* is an equation containing derivatives of the unknown function (or of several unknown functions). Differentials may enter in place of derivatives.

If the unknown functions depend on one argument, the differential equation is called *ordinary*, if they depend on several, then the equation is termed a *partial differential equation*. We shall consider only ordinary differential equations.

The general form of a differential equation in one unknown function is

$$\Phi(x, y, y', y'', \ldots, y^{(n)})=0 \tag{1}$$

The *order of the differential equation* is the order of the highest derivative in the equation.

Examples. The equation $y'=\frac{y^2}{x}$ is a first-order differential equation; the differential equation $y''+y=0$ is of second order, the equation $y'^2=x^3$ is of first order.

A function $y=\varphi(x)$ is called a *solution* of a differential equation if, when substituted into the equation, it reduces the equation to an identity.

The basic task of the theory of differential equations is finding all the solutions of a given differential equation. In the simplest case, this reduces to evaluating an integral. For this reason, the solution of a differential equation is also called its *integral*, and the process of finding all the solutions is called *integrating* the differential equation.

Generally, the *integral of a given differential equation* is any equation, not containing derivatives, from which the given differential equation follows as a consequence.

Example 1. The function $y=\sin x$ is a solution (integral) of the second-order differential equation

$$y''+y=0 \tag{2}$$

because substitution of $y=\sin x$ turns Eq. (2) into

$$(\sin x)''+\sin x=0 \tag{3}$$

which is an identity.

The functions $y=\frac{1}{2}\sin x$, $y=\cos x$, $y=3\cos x$ are also solutions of Eq. (2), the function $y=\sin x+\frac{1}{2}$ is not a solution.

Example 2. Consider the differential equation of the first order

$$xy'+y=0 \tag{4}$$

The *function*

$$y=\frac{1.5}{x} \tag{5}$$

is a solution of (4) because substitution of (5) into (4) reduces (4) to an identity:

$$x\cdot\frac{1.5}{-x^2}+\frac{1.5}{x}=0$$

At the same time, Eq. (5) is an integral of the differential equation (4).

The equation

$$xy=0.2 \tag{6}$$

is also an integral of the differential equation (4). Indeed, from (6) it follows that $(xy)'=0$, whence (if we apply the formula of the derivative of a product) follows (4). Solving the integral (6) for y, we get

$$y=\frac{0.2}{x} \tag{7}$$

Function (7) is a solution of the differential equation (4). At the same time, Eq. (7) is an integral of equation (4).

The equations $xy=\sqrt{3}$, $xy=-2$, $xy=\pi$, etc. are integrals of the differential equation (4), and the functions $y=\frac{\sqrt{3}}{x}$, $y=-\frac{2}{x}$, $y=\frac{\pi}{x}$, etc. are solutions.

Example 3. Find all the solutions of the following first-order differential equation:

$$y'=\cos x \tag{8}$$

Solution. The unknown function $y=\varphi(x)$ is an antiderivative of the function $\cos x$. The most general form of such a function is the indefinite integral $\int \cos x\,dx$. Hence, all solutions are contained in the formula

$$y=\sin x+C \tag{9}$$

The function $y=\sin x+C$, containing an arbitrary constant C, is the *general solution*[1] of Eq. (8), the function $y=\sin x$ $\left(\text{or } y=\sin x+\frac{1}{2},\ y=\sin x-1, \text{ etc.}\right)$ is a *particular solution*.

478. First-Order Equation

The general form of a first-order differential equation is

$$\Phi(x, y, y')=0 \tag{1}$$

Solved for y', the equation takes the form

$$y'=f(x, y) \tag{2}$$

It is assumed that the function $f(x, y)$ is uniquely defined and is continuous in some domain: integrals are sought which lie in that domain.

479. Geometrical Interpretation of a First-Order Equation

A curve L (Fig. 470) which depicts some integral of the differential equation

$$y'=f(x, y) \tag{1}$$

is called an *integral curve* of that equation.

The derivative y' is the slope of the tangent line $T'T$ to the integral curve. Before the integral curve passing through a given point $M(x, y)$ is found, we can find y' from Eq. (1) and draw the straight line $T'T$ through M. $T'T$ will indicate the direction of the required integral curve. The collection of straight lines $T'T$ which correspond to all possible points of the domain in question is termed the *direction field* of Eq. (1).

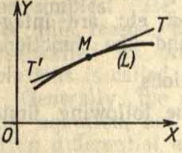

Fig. 470

Geometrically, the problem of integrating Eq. (1) is formulated as follows: *find the curves for which the direction of the tangent line coincides everywhere with the direction of the field.*

---

[1] The definitions of a general solution and a particular solution of a differential equation are given in Sec. 481 (for first-order equations) and in Secs. 493, 494 (for higher-order equations).

If the direction field is depicted by close-lying dashed lines (Figs. 471, 472), then the integral curves may be constructed (approximately) by eye.

Example 1. In Fig. 471 we have the direction field of the equation

$$\frac{dy}{dx}=-\frac{x}{y} \tag{2}$$

Eq. (2) expresses the fact that the direction of the field at point $M(x, y)$ is perpendicular to the straight line OM $\left(\text{the slope of the direction of the field is } \frac{dy}{dx}\text{, while the slope of the straight line } OM \text{ is } \frac{y}{x}\right)$. It is easy to see that the integral curves are circles centred at O. Hence, the integrals of Eq. (2) are of the form

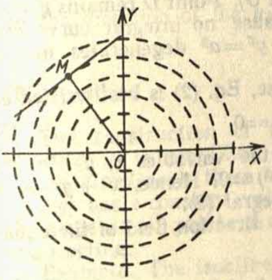

Fig. 471

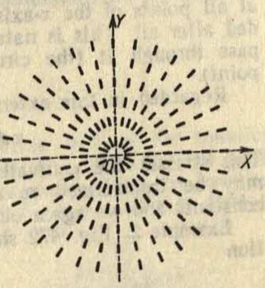

Fig. 472

$$x^2+y^2=a^2 \tag{3}$$

where a^2 is a constant which can take on any positive value. The functions

$$y=\sqrt{a^2-x^2}, \quad y=-\sqrt{a^2-x^2} \tag{4}$$

are solutions of Eq. (2). This can readily be verified.

Note. According to Sec. 478, the points of the x-axis must be excluded from our consideration because the function $f(x, y)=-\frac{x}{y}$ is not defined at these points. However, we depict the direction of the field (by vertical dashes) at these points as well. We thus *expand the meaning* of Eq. (2) (in accord with its geometrical interpretation).

Namely, the notation (2) is to be understood as a *combination of two equations:*

$$\frac{dy}{dx}=-\frac{x}{y}, \quad \frac{dx}{dy}=-\frac{y}{x} \tag{2a}$$

In the second equation, x is regarded as a function of the argument y. Accordingly, we consider as solutions not only the integrals (4) but also the integrals

$$x=\sqrt{a^2-y^2}, \quad x=-\sqrt{a^2-y^2} \tag{4a}$$

Eqs. (2a) are equivalent at all points not lying on the x- and y-axes. The second of the Eqs. (2a) replaces the first at all points of the x-axis (except O). Point O remains excluded after all. This is natural because no integral curve can pass through it (the circle $x^2+y^2=a^2$ degenerates into a point).

Regarded in this extended sense, Eq. (2) is best written as

$$x\,dx+y\,dy=0 \tag{5}$$

This stresses the equivalence of the variables x, y, Eq. (5) may be transformed to $d(x^2+y^2)=0$. Hence, x^2+y^2 is a constant, and we again obtain integral (3).

Example 2. Fig. 472 shows the direction field of the equation

$$\frac{dy}{dx}=\frac{y}{x} \tag{6}$$

The integral curves are the straight lines $y=Cx$. Taking Eq. (6) in the extended sense (see note above), we can depict the direction field at any point of the y-axis (except O). We get vertical dashes located along a vertical straight line. This means that the straight line ($x=0$) is adjoined to the integral curves $y=Cx$.

At point O the field direction remains indeterminate: there is an accumulation of integral curves of all possible directions.

The functions

$$y=Cx \quad (C \text{ a constant}) \tag{7}$$

and also the functions

$$x=C_1y \quad (C_1 \text{ a constant}) \tag{7a}$$

are solutions (integrals) of Eq. (6). The following equations

$$\frac{y}{x}=C, \quad \frac{x}{y}=C, \quad \frac{x^2}{y^2}=C, \quad \ln\left|\frac{y}{x}\right|=C \tag{8}$$

and others are also integrals.

Eq. (6) is written in the form

$$x\,dy-y\,dx=0 \tag{9}$$

If we divide (9) by x^2, we get $\frac{x\,dy-y\,dx}{x^2}=0$, i. e. $d\left(\frac{y}{x}\right)=0$. Whence we get the integral $\frac{y}{x}=C$. Dividing (9) by y^2, we get $\frac{x}{y}=C_1$ $\left(\text{here } C_1=\frac{1}{C}\right)$.

Example 3. The direction field of an equation of the form $y'=f(x)$ is considered in Sec. 295 (Examples 1 to 3). The integral curves $y=\int f(x)\,dx$ are equidistant from each other (in the direction of the y-axis).

480. Isoclines

The construction of the direction field of the equation $y'=f(x, y)$ is simplified if we first draw *lines of equal inclination (isoclines)*; these are curves along which the function $f(x, y)$ has a constant value. The direction of the field is the same at all points of any one isocline.

Example. The isoclines of the equation $y'=x^2+y^2$ are the circles $x^2+y^2=a^2$ (Fig. 473). At all points of the circle $x^2+y^2=1$ (radius OC is taken as the scale unit) the slope y' of the direction of the field is equal to unity; at all points of the circle $x^2+y^2=2$ (radius $OD=\sqrt{2}$) we have $y'=2$, etc. The integral curves are shown as heavy lines.

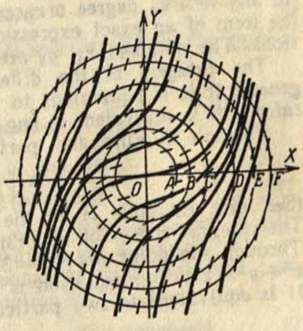

Fig. 473

Note. In practical cases, when using isoclines, there is no need to give the direction field in the form of dashed lines. It is sufficient to affix to each isocline a number stating the value of the slope. A drawing with isoclines also includes a dense pencil of rays with slope indicated on each ray. A solution is obtained by constructing dashes parallel to the corresponding rays.

481. Particular and General Solutions of a First-Order Equation

The first-order differential equation

$$y' = f(x, y) \tag{1}$$

has an infinity of solutions (see Examples in Sec. 479). As a rule, only *one integral curve*[1] passes through a given point in the domain under consideration (Sec. 478). The corresponding solution of Eq. (1) is called a *particular solution*; the collection of all particular solutions is called the *general solution*. One strives to represent the general solution of the differential equation (1) in the form of some function

$$y = \varphi(x, C) \quad (C \text{ a constant}) \tag{2}$$

which would yield any particular solution (for a suitably chosen value of C). Such a representation is sometimes not even possible theoretically; in practical cases, this is possible only for a few (but important) classes of equations (Secs. 482-486).

Now it is always possible to find a particular solution passing through a given point (x_0, y_0) at least approximately (to any desired degree of accuracy; Secs. 490, 491) if not in the form of an exact expression in terms of elementary functions. The numbers x_0, y_0 are called *initial values*.

The integral of the differential equation (1) is called *general* if it is equivalent to the general solution, and *particular* if it is equivalent to one particular solution or to several.

Example 1. Find the particular solution of the equation

$$x\,dx + y\,dy = 0 \tag{3}$$

(Sec. 479, Example 1) for the initial values $x_0 = 4$, $y_0 = -3$. The integral curves of Eq. (3) are circles with centre (0, 0). Through the point $M_0(4, -3)$ there passes the integral curve $x^2 + y^2 = 25$. This equation is a particular integral of Eq. (3). It is equivalent to two particular solutions:

$$y = \sqrt{25 - x^2},$$

$$y = -\sqrt{25 - x^2}$$

The latter is the desired solution (the former does not pass through M_0).

---

[1] The only possible exception is for points where the partial derivative $f'_y(x, y)$ is discontinuous or does not exist.

Example 2. The particular solution of Eq. (3) passing through the point (x_0, y_0) is of the form

$$y=\sqrt{x_0^2+y_0^2-x^2} \quad \text{if } y_0>0, \tag{4}$$

$$y=-\sqrt{x_0^2+y_0^2-x^2} \quad \text{if } y_0<0 \tag{5}$$

For cases when $y_0=0$, that is when the point (x_0, y_0) lies on the x-axis, the particular solution (in accord with the note pertaining to Example 1, Sec. 479) is of the form

$$x=\sqrt{x_0^2-y^2} \quad \text{if } x_0>0, \tag{6}$$

$$x=-\sqrt{x_0^2-y^2} \quad \text{if } x_0<0 \tag{7}$$

At the point $x_0=0$, $y_0=0$ (origin) there is no particular solution.

The collection of particular solutions (4), (5), (6), (7) forms the general solution of the differential equation (3).

If we denote the constant quantity $x_0^2+y_0^2$ by C^2, then the general solution may be written as

$$y=\pm\sqrt{C^2-x^2} \tag{8}$$

The equation

$$x^2+y^2=C^2 \tag{9}$$

which is equivalent to the general solution (8), is the general integral of Eq. (3).

482. Equations with Variables Separated

If a differential equation is of the form

$$P(x)\,dx+Q(y)\,dy=0 \tag{1}$$

(the coefficient P depends only on x, the coefficient Q only on y), then we say that the *variables are separated.*

The general integral of an equation with separated variables is represented by the equation[1]

$$\int P(x)\,dx+\int Q(y)\,dy=C \quad (C \text{ a constant}) \tag{2}$$

---

[1] Here and henceforward the symbol $\int$ denotes some one antiderivative; that is, the arbitrary constant term is disregarded. Incidentally, there will be no error if we include in the integral $\int P(x)\,dx$ the constant term C_1 and in the integral $\int Q(y)\,dy$ the term C_2. But the solution will needlessly be in a more involved form.

In order to find the particular integral for the initial values x_0, y_0, we can do as follows: substituting x_0, y_0 into (2) we find the corresponding value $C=C_0$. The desired particular integral will be $\int P(x)\,dx+\int Q(y)\,dy=C_0$. When we are not interested in the general solution, a particular solution is best sought directly from the formula

$$\int_{x_0}^{x} P(x)\,dx+\int_{y_0}^{y} Q(y)\,dy=0 \tag{3}$$

Example. Find the particular solution of the equation

$$\sin x\,dx+\frac{dy}{\sqrt{y}}=0 \tag{4}$$

for the initial data $x_0=\frac{\pi}{2}$, $y_0=3$.

Solution. The general integral of Eq. (4) is

$$\int \sin x\,dx+\int\frac{dy}{\sqrt{y}}=C \text{ or } -\cos x+2\sqrt{y}=C \tag{5}$$

Putting $x=\frac{\pi}{2}$, $y=3$, we get $C=2\sqrt{3}$; the desired particular solution is

$$y=\frac{(2\sqrt{3}+\cos x)^2}{4} \tag{6}$$

It may be obtained directly from the formula

$$\int_{\frac{\pi}{2}}^{x} \sin x\,dx+\int_{3}^{y}\frac{dy}{\sqrt{y}}=0$$

483. Separation of Variables. General Solution

An equation of the form $X_1Y_1\,dx+X_2Y_2\,dy=0$, where the functions X_1 and X_2 depend solely on x, [1] and the functions Y_1 and Y_2 only on y, can be reduced to the form (1), Sec. 482, via division by Y_1X_2. The process of this reduction is called *separation of variables*.

---

[1] One or both may be constants; the same goes for the functions Y_1, Y_2.

Example 1. Consider the equation

$$y\,dx-x\,dy=0 \tag{1}$$

Dividing by xy, we get the equation

$$\frac{dx}{x}-\frac{dy}{y}=0 \tag{2}$$

where the variables are separated. Integrating, we find

$$\int\frac{dx}{x}-\int\frac{dy}{y}=C \tag{3}$$

i. e.

$$\ln|x|-\ln|y|=C \tag{4}$$

or

$$\ln\left|\frac{x}{y}\right|=C \tag{4a}$$

If we introduce a new constant C_1 connected with C by the relation $C=\ln C_1$, then in place of (4a) we can write

$$\frac{x}{y}=C_1 \tag{4b}$$

(cf. Example 2, Sec. 479).

Note 1. Let the value $y=k$ serve as a root of the equation $Y_1=0$. Then the function $y=k$ (which reduces to the constant k) serves as one of the solutions of the differential equation $X_1Y_1\,dx+X_2Y_2\,dy=0$ (because for $y=k$ we have $dy=0$ and, by hypothesis, $Y_1=0$). This solution can be lost in the division by Y_1X_2. In the same way we can lose a solution of the form $x=l$, where l is a root of the equation $X_2=0$. Thus, in Example 1, when we obtained Eq. (4) we lost the particular solution $y=0$ of the differential equation (1) and also the particular solution $x=0$. The point is that Eq. (4) is meaningless both for $y=0$ and for $x=0$ (the number zero has no logarithm).

Having cleared Eq. (4a) of logarithms, we again introduced the solution $x=0$ (for $C_1=0$).

Example 2. Find all the solutions of the equation

$$\sqrt{1-y^2}\,dx-y\,dy=0 \tag{5}$$

Solution. Within the limits of a strip bounded by a pair of straight lines $y=\pm 1$, at least one of the functions $\frac{\sqrt{1-y^2}}{y}\left(=\frac{dy}{dx}\right)$, $\frac{y}{\sqrt{1-y^2}}\left(=\frac{dx}{dy}\right)$ is uniquely defined and continuous. Outside this strip not one of the indicated functions is defined. Hence (Sec. 478), all integrals of Eq. (5) lie in the strip bounded by the straight lines $y=\pm 1$.

Dividing Eq. (5) by $\sqrt{1-y^2}$, we get the equation

$$dx-\frac{y\,dy}{\sqrt{1-y^2}}=0$$

where the variables are separated. Integrating, we get

$$x-\sqrt{1-y^2}=C$$

or

$$x-C=\sqrt{1-y^2} \tag{6}$$

This equation represents a family of semicircles as depicted in Fig. 474. But it does not contain all the integral curves of Eq. (5): in dividing (5) by $\sqrt{1-y^2}$ we lost the solutions $y=1$ and $y=-1$ (the straight lines uv, $u'v'$ in Fig. 474).

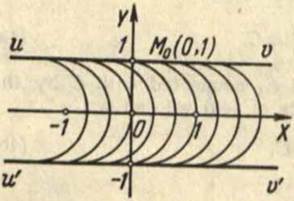

Fig. 474

Note 2. Here, the lost solutions *are not particular* solutions (in contrast to the solutions lost in Example 1). The point is that (Sec. 481) we use the term particular solution for a solution which is *unique* with respect to certain initial values But *two solutions* pass through every point of the solution $y=1$. For example, through the point $M_0(0, 1)$ (Fig. 474) there passes, in addition to the straight line $y=1$, the semicircle $x=\sqrt{1-y^2}$, which depicts yet another solution of Eq. (5); this solution is obtained from (6) for $C=0$.

Though (6) does not embrace all solutions, it contains all *particular* solutions (semicircles) and for this reason is the *general* integral of Eq. (5). The solutions $y=1$, $y=-1$ are called *singular solutions*.

Generally, the integral of a differential equation of first order is called *singular* if at least one more integral passes through every one of its points.

484. Total Differential Equation

If the coefficients $P(x, y)$, $Q(x, y)$ in the equation

$$P(x, y)\,dx+Q(x, y)\,dy=0 \tag{1}$$

satisfy the condition

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x} \tag{2}$$

then the left-hand side of (1) is the total differential of some function $F(x, y)$ (antiderivative function of the expression $P\,dx+Q\,dy$; see Sec. 476). The general integral of Eq. (1) is

$$F(x, y)=C \tag{3}$$

Example. Find the particular integral of the equation

$$\frac{x^2-y}{x^2}\,dx+\frac{x+1}{x}\,dy=0 \tag{4}$$

for the initial conditions $x_0=1$, $y_0=1$.

Solution. Condition (2) is fulfilled. Then the functions $P(x, y)=1-\frac{y}{x^2}$, $Q=1+\frac{1}{x}$ decompose into terms of the form Ax^my^n. Therefore, we find the antiderivative (Sec. 476, Note) as follows.

Perform the integration

$$\int\left(1-\frac{y}{x^2}\right)dx=x+\frac{y}{x} \text{ (for constant } y\text{)},$$
$$\int\left(1+\frac{1}{x}\right)dy=y+\frac{y}{x} \text{ (for constant } x\text{)}$$

Combine these expressions and retain the term $\frac{y}{x}$ only once. The function $x+y+\frac{y}{x}$ is the antiderivative. The general integral will be

$$x+y+\frac{y}{x}=C \tag{5}$$

Substituting the initial data $x=1$, $y=1$, we find $C=3$. The desired particular integral is $x+y+\frac{y}{x}=3$.

484a. Integrating Factor

If the coefficients $P(x, y)$, $Q(x, y)$ in the equation

$$P(x, y)\,dx+Q(x, y)\,dy=0 \tag{1}$$

do not satisfy the condition

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x} \tag{2}$$

then the left-hand side of (1) is not a total differential. But it is sometimes possible to find a factor $M(x, y)$ such that the expression $M(P\,dx+Q\,dy)$ becomes the total differential of some function $F_1(x, y)$. Then the general integral is

$$F_1(x, y)=C$$

The function $M(x, y)$ is called an *integrating factor*.

Example. The left-hand member of the equation $2y\,dx+x\,dy=0$ is not a total differential. But multiplying by x yields

$$x(2y\,dx+x\,dy)=d(x^2y)$$

The general integral of the given equation is

$$x^2y=C$$

Note. Every differential equation has integrating factors (even an infinity of them), but there are no general techniques for finding them.

485. Homogeneous Equation

The differential equation

$$M\,dx+N\,dy=0 \tag{1}$$

is called *homogeneous* if the ratio $\frac{M}{N}$ can be represented as a function of the ratio $\frac{y}{x}$. We denote this ratio by t:

$$t=\frac{y}{x} \tag{2}$$

Thus, the equation

$$(y+\sqrt{x^2+y^2})\,dx-x\,dy=0 \tag{3}$$

is homogeneous because

$$\frac{M}{N}=\frac{y+\sqrt{x^2+y^2}}{-x}=-\frac{y}{x}-\sqrt{1+\left(\frac{y}{x}\right)^2}=-t-\sqrt{1+t^2} \tag{4}$$

By means of the substitution

$$y=tx \quad (\text{whence } dy=t\,dx+x\,dt) \tag{5}$$

any homogeneous equation can be reduced to an equation with variables separable.

Example 1. Integrate Eq. (3) with the initial conditions $x_0=3$, $y_0=4$.

Solution. After the substitution (5), Eq. (3) takes the form

$$\sqrt{x^2+x^2t^2}\,dx-x^2\,dt=0 \tag{6}$$

or

$$|x|\sqrt{1+t^2}\,dx-x^2\,dt=0 \tag{7}$$

The variables can be separated, and we get

$$\frac{dx}{|x|}=\frac{dt}{\sqrt{1+t^2}} \tag{8}$$

In separating the variables we lost the solution $x=0$. However, it definitely does not satisfy the initial conditions.

Since we have to integrate for the initial conditions $x_0=3$, $t_0=\frac{y_0}{x_0}=\frac{4}{3}$, it follows that the abscissa x is positive (see note below), and we have to put

$$|x|=x \tag{9}$$

We get

$$\int_3^x \frac{dx}{x}=\int_{4/3}^t \frac{dt}{\sqrt{1+t^2}} \tag{10}$$

whence

$$\ln x-\ln 3=\ln\left(t+\sqrt{1+t^2}\right)-\ln 3 \tag{11}$$

Replacing t by $\frac{y}{x}$ and taking antilogarithms, we get the particular integral

$$x=\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}} \tag{12}$$

The corresponding particular solution is

$$y=\frac{x^2-1}{2} \tag{13}$$

Note. The left-hand member of formula (10) is meaningless when the upper limit is zero or takes on negative values. Therefore, when seeking a solution, we must confine ourselves to positive values of x. More investigation is required to see whether the function (13) also yields a solution of Eq. (3) for $x \leqslant 0$. Substituting (13) into the left-hand side of (3) shows that the function $y=\frac{x^2-1}{2}$ gives a solution for all values of x.

Example 2. Integrate Eq. (3) with the initial conditions $x_0=-3$, $y_0=4$.

Solution. The sequence of operations is the same as in Example 1. However, in place of (9) we have to put

$$|x|=-x \tag{9a}$$

so that in place of (10) we get

$$-\int_{-3}^{x}\frac{dx}{x}=\int_{-4/3}^{t}\frac{dt}{\sqrt{1+t^2}} \tag{10a}$$

whence

$$-\ln|x|+\ln 3=\ln\left(t+\sqrt{1+t^2}\right)-\ln\frac{1}{3} \tag{11a}$$

In place of (12) we obtain

$$\frac{1}{|x|}=\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}} \tag{12a}$$

or

$$-\frac{1}{x}=\frac{y}{x}-\sqrt{\frac{x^2+y^2}{x}} \tag{12b}$$

(the minus sign in front of the last fraction appeared because for $x<0$ we have $\sqrt{x^2}=-x$). From (12b) we get the desired particular solution

$$y=\frac{x^2-1}{2}$$

It coincides with the solution to Example 1 (cf. note pertaining to Example 1).

If, disregarding (9a), we had used (10) in place of (10a) we would have obtained an erroneous result.

486. First-Order Linear Equation

The first-order differential equation

$$M\,dx+N\,dy=0 \tag{1}$$

is called *linear* if the ratio $\frac{M}{N}$ contains y to the first power only. A linear equation is commonly given in the form

$$y'+P(x)\,y=Q(x) \tag{2}$$

where $P(x)$ and $Q(x)$ are any (continuous) functions of x.

If, in particular, $Q(x)=0$, then Eq. (2) is called a *linear equation with right-hand member zero.*[1] In this case the variables can be separated and the general solution is of the form

$$y=Ce^{-\int P\,dx} \tag{3}$$

---

[1] A linear equation with right-hand member zero is also called *homogeneous*. But this term has yet another meaning (Sec. 485).

Example 1. Find the general solution of the linear equation with right-hand member zero

$$y'-\frac{x}{1+x^2}y=0 \tag{4}$$

Solution. Separating the variables, we get

$$\frac{dy}{y}=\frac{x\,dx}{1+x^2} \tag{5}$$

whence

$$\ln|y|=\frac{1}{2}\ln(1+x^2)+C \tag{6}$$

or

$$y=C_1\sqrt{1+x^2} \tag{6a}$$

where $C_1=e^C$. We could have obtained the same result using formula (3) $\left(\text{for } P=-\frac{x}{1+x^2}\right)$:

$$y=Ce^{-\int-\frac{x\,dx}{1+x^2}}=Ce^{1/2\ln(1+x^2)}=C\sqrt{1+x^2}$$

Note 1. The particular solution $y=0$ which is obtained from (6a) for $C_1=0$ cannot be obtained from (6); this solution was lost in the division of (4) by y. Having cleared (6) of logarithms, we again introduced the solution $y=0$. Cf. Sec. 484, Example 1.

Note 2. In practical cases, use of the ready-at-hand formula (3) does not give us any essential advantage over the successive transformations indicated in Example 1.

A *linear equation with right-hand member* [in it $Q(x)\neq 0$] is integrated as follows: we find the general solution (3) of the corresponding equation with right-hand member zero; then in this solution we replace the constant C by the unknown function u. Substitute into (2) the expression obtained. After simplifications, the variables u, x are separated; integrating, we find the expression of u in terms of x. The function $y=ue^{-\int P\,dx}$ will be the general solution[1] of Eq. (2).

Example 2. Find the general solution of the equation

$$y'-\frac{x}{1+x^2}y=x \tag{7}$$

---

[1] This general solution is expressed by the formula

$$y=\left[\int dx\,Q(x)\,e^{\int P(x)\,dx}+C_1\right]e^{-\int P\,dx} \tag{A}$$

Solution. The general solution of the corresponding equation with right-hand member zero is (see Example 1) $y=C\sqrt{1+x^2}$. Replacing the constant C by the unknown function u, we get

$$y=u\sqrt{1+x^2} \tag{8}$$

whence

$$y'=\frac{du}{dx}\sqrt{1+x^2}+\frac{ux}{\sqrt{1+x^2}} \tag{9}$$

Substituting (8) and (9) into (7) and simplifying, we get

$$\frac{du}{dx}=\frac{x}{\sqrt{1+x^2}}$$

Whence we get an expression of u in terms of x:

$$u=\int\frac{x\,dx}{\sqrt{1+x^2}}=\sqrt{1+x^2}+C_1 \tag{10}$$

By virtue of (8) and (10), the general solution of the given equation will be

$$y=(\sqrt{1+x^2}+C_1)\sqrt{1+x^2}\ {}^{[1]} \tag{11}$$

Note. The equation

$$\frac{dx}{dy}+P(y)\,x=Q(y) \tag{12}$$

which is obtained from (2) by interchanging x and y is integrated in similar fashion.

437. Clairaut's Equation

Clairaut's equation is an equation of the form

$$y=xy'+\varphi(y') \tag{1}$$

The general integral is

$$y=xC+\varphi(C) \tag{2}$$

---

[1] We get the same result $\left[\text{for } P=-\frac{x}{1+x^2},\ Q=x\right]$ from formula (A):

$$y=\left[\int x\,dx\,e^{\int-\frac{x\,dx}{1+x^2}}+C_1\right]e^{-\int-\frac{x\,dx}{1+x^2}}=$$

$$=\left[\int x\,dx\,\frac{1}{\sqrt{1+x^2}}+C_1\right]\sqrt{1+x^2}=(\sqrt{1+x^2}+C_1)\sqrt{1+x^2}$$

Besides, Clairaut's equation has a singular integral (Sec. 483), which is obtained by eliminating the parameter t from the equations

$$x = -\varphi'(t), \quad y = -t\varphi'(t) + \varphi(t) \tag{3}$$

The general integral (2) is depicted as a family of straight lines tangent to some curve L. The singular integral is depicted by the curve L itself [Eqs. (3) represent it in parametric form].

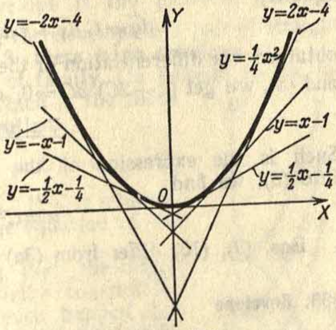

Fig. 475

Example. The equation

$$y = xy' - y'^2 \tag{1a}$$

is Clairaut's equation. Its general integral

$$y = Cx - C^2 \tag{2a}$$

is depicted as the collection of straight lines (Fig. 475) tangent to the parabola

$$y = \frac{1}{4}x^2 \tag{4}$$

Eq. (4) is a singular integral. It is obtained in the following manner. In the example we have $\varphi(t) = -t^2$, $\varphi'(t) = -2t$ and Eqs. (3) assume the form

$$x = 2t, \quad y = t^2 \tag{3a}$$

Eliminating t, we get (4).

Explanation. Using Eq. (1a) as an illustration, we shall show how the equation of a singular integral is obtained.

The curve L which is tangent to the integral curves (2a) will itself be an integral curve (because its direction is everywhere coincident with the field direction). To find the curve L, note that it must have one common point $N(\bar{x}, \bar{y})$ with each of the straight lines

$$y = Cx - C^2 \tag{5}$$

The quantity C is constant for each straight line (5) but varies from line to line so that the coordinates $\bar{x}$, $\bar{y}$ are functions of C. Let us find these functions. Since the point $N(\bar{x}, \bar{y})$ lies on the straight line (5), we must have an identity:

$$\bar{y} = C\bar{x} - C^2 \tag{6}$$

Since the directions of L and (5) coincide at the point N, the differentials $d\bar{y}$, $d\bar{x}$ must have the same ratio as the differentials dx, dy of the coordinates of the straight line (5), that is we should have

$$d\bar{y}=C\,d\bar{x} \tag{7}$$

At the same time, the differentials $d\bar{x}$, $d\bar{y}$ must satisfy the equality

$$d\bar{y}=C\,d\bar{x}+\bar{x}\,dC-2C\,dC \tag{8}$$

obtained via differentiation of the identity (6). Comparing (7) and (8), we get $(\bar{x}-2C)\,dC=0$, or

$$\bar{x}=2C \tag{9}$$

Such is the expression of the function $\bar{x}$. Substituting it into (6), we find

$$\bar{y}=C^2 \tag{10}$$

Eqs. (9), (10) differ from (3a) in notation alone.

488. Envelope

Definition 1. A set of lines is called a (one-parameter) family if every line can be associated with a definite number C *(parameter of the family)* so that a continuous change in the parameter C is associated with a continuous change in the line. An equation of the form

$$f(x,\ y,\ C)=0 \tag{1}$$

where $f(x,\ y,\ C)$ is a continuous function of three arguments x, y, C, represents a family of lines in a plane. The separate lines of the family correspond to separate values of C.

Eq. (1) is called the *equation of the family.*

Example 1. The equation

$$y=Cx-C^2$$

represents the family of straight lines shown in Fig. 475. The slope of the straight line is taken as the parameter of the family.

Example 2. The equation

$$(x-C)^2+y^2=1$$

represents a family of circles of radius 1 with centre on the x-axis (Fig. 474). The abscissa of the centre is taken as the parameter.

Example 3. The equation

$$x^2+y^2=C^2$$

represents a family of circles with centre at $O(0, 0)$. The radius is the parameter.

Definition 2. The *envelope* of a given family is a line tangent at each of its points to one of the lines of the family.

In Example 1, the envelope is the parabola $y=\frac{1}{4}x^2$ (cf. Sec. 487); in Example 2, the envelope is a pair of straight lines: $y=\pm 1$; in Example 3, there is no envelope.

Theorem. The envelope of family (1) belongs to the so-called *discriminant curve*, which is the locus of points satisfying the equations

$$f(x, y, C)=0, \quad f'_C(x, y, C)=0 \qquad (2)$$

for all possible values of C. If C is eliminated from Eqs. (2), we get the equation of the discriminant curve.

Note 1. It may happen that the discriminant curve is only partially covered by the envelope, and it may even happen that the discriminant curve exists, but the family (1) has no envelope.

Example 4. The discriminant curve of the family of straight lines $y=Cx-C^2$ is given by the system

$$y=Cx-C^2, \quad x-2C=0$$

Eliminating C, we get the equation $y=\frac{1}{4}x^2$. The discriminant curve is a parabola which coincides with the envelope of the family (cf. Example, Sec. 487).

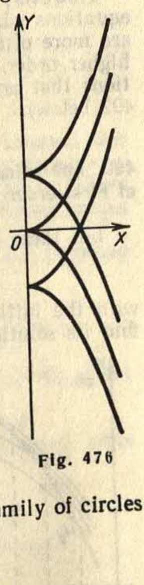

Fig. 476

Example 5. The discriminant curve of the family of circles

$$(x-C)^2+y^2=1$$

is given by the system

$$(x-C)^2+y^2=1, \quad -2(x-C)=0$$

Eliminating C, we get the equation $y^2=1$. The discriminant curve (the pair of straight lines $y=\pm 1$) coincides with the envelope (cf. Sec. 483, Example 2).

Example 6. The discriminant curve of the family of semicubical parabolas $(y-C)^2=x^3$ (Fig. 476) is the straight line $x=0$, but this family has no envelope.

Note 2. If family (1) portrays the general integral of some differential equation, then the envelope represents a singular integral. If there is no envelope, there is no singular integral.

489. On the Integrability of Differential Equations

In Secs. 482-487 we considered the most important types of first-order equations whose solution reduces to finding the integrals of known functions.[1] We say that such equations can be *reduced to quadratures*.

However, in practical situations we encounter first-order equations which are not reducible to quadratures. Such cases are more often encountered in the solution of equations of higher order. Approximate methods are used to solve equations that are not reducible to quadratures (see Secs. 490-492 below).

490. Approximate Integration of First-Order Equations by Euler's Method

Let there be given an equation

$$y' = f(x, y) \tag{1}$$

with the initial conditions $x = x_0$, $y = y_0$. It is required to find its solution in some interval (x_0, x). Divide this interval into n parts (equal or unequal) by a sequence of points $x_1, x_2, \ldots, x_{n-1}$ (Fig. 477).

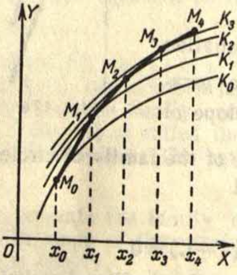

Fig. 477

On the subinterval (x_0, x_1) we put

$$y = y_0 + f(x_0, y_0)(x - x_0) \tag{2}$$

that is, in place of the desired integral curve M_0K_0 we take its tangent line M_0M_1.

At the point $x = x_1$ we obtain an approximate value of the required solution

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + f(x_0, y_0)\,\Delta x_0 \tag{3}$$

---

[1] These integrals may not be expressible in terms of elementary functions (Sec. 309).

On the subinterval (x_1, x_2) we put

$$y=y_1+f(x_1, y_1)(x-x_1)$$

That is, in place of the desired integral curve M_0K_0 we take the tangent line M_1M_2 to the integral curve M_1K_1 (a double error thus occurs: the tangent M_1M_2 departs from the curve M_1K_1 and this curve does not coincide with the desired line M_0K_0). Continuing this process, we obtain a succession of approximate values:

$$\left.\begin{aligned} y_2&=y_1+f(x_1, y_1)\,\Delta x_1,\\ y_3&=y_2+f(x_2, y_2)\,\Delta x_2,\\ &\cdots\cdots\cdots\cdots\cdots\\ y_n&=y_{n-1}+f(x_{n-1}, y_{n-1})\,\Delta x_{n-1} \end{aligned}\right\} \tag{4}$$

Sufficient refinement in partitioning the given interval will yield any required degree of accuracy, but at the expense of much work. For this reason, Euler's method is employed only for rough approximations. Mostly, it is advantageous to divide the interval (x_0, x) into equal parts.

Example. Find an approximate solution to the equation

$$y'=\frac{1}{2}xy$$

in the interval (0, 1) with the initial data $x_0=0$, $y_0=1$ $\left[\text{here } f(x, y)=\frac{1}{2}xy\right]$.

Solution. Divide the interval (0, 1) into 10 equal parts so that

$$\Delta x_0=\Delta x_1=\ldots=\Delta x_9=0.1$$

Using formulas (3) and (4), we find successively:

$$y_1=y_0+\frac{1}{2}x_0y_0\,\Delta x_0=1+\frac{1}{2}\cdot 0\cdot 1\cdot 0.1=1,$$

$$y_2=y_1+\frac{1}{2}x_1y_1\,\Delta x_1=1+\frac{1}{2}\cdot 0.1\cdot 1\cdot 0.1=1.005$$

and so on. The computations are arranged according to the following scheme:

x	y	$\Delta y=\frac{1}{2}xy\,\Delta x$	True value of y
0	1	0	1
0.1	1	0.005	1.0025
0.2	1.005	0.0101	1.0100
0.3	1.0151	0.0152	1.0227
0.4	1.0303	0.0206	1.0408
0.5	1.0509	0.0263	1.0645
0.6	1.0772	0.0323	1.0942
0.7	1.1095	0.0392	1.1303
0.8	1.1487	0.0459	1.1735
0.9	1.1946	0.0538	1.2244
1.0	1.2484		1.2840

A table of the approximate solution is built up from the first two columns. The given equation also admits an exact solution via the formula $\int\limits_1^y \frac{dy}{y}=\int\limits_0^x \frac{1}{2}x\,dx$, whence $y=e^{\frac{1}{4}x^2}$. The corresponding values of y are given in the last column. A comparison with the first column shows that the error progressively increases and at $x=1$ reaches 2.9%.

491. Integration of Differential Equations by Means of Series

The solution of the equation

$$y'=f(x,\ y) \tag{1}$$

with initial conditions $x=x_0$, $y=y_0$ may be sought in the form of a series arranged in powers of $x-x_0$; that is, in the form

$$y=y_0+c_1(x-x_0)+c_2(x-x_0)^2+\ldots+c_n(x-x_0)^n+\ldots \tag{2}$$

the factors $c_1, c_2, \ldots, c_n, \ldots$ are found by the method of undetermined coefficients (Sec. 307) or by other methods.

The series method in application to differential equations was systematically employed by Newton (Sec. 292). In contrast to Euler's method, which gives the solution in the form of a table (Sec. 490), the solution here is obtained as a formula. However, the formula is useless outside the interval of convergence of the series. Theoretically, certain cases are possible when the solution cannot be expressed by a series (cf. Sec. 400). The problem was theoretically investigated by Cauchy. S. Kovalevskaya[1] investigated the analogous problem for partial differential equations.

Despite the above-indicated restrictions, the power-series method is of great practical significance.

Example. Find the solution of the equation

$$y' = \frac{1}{2} xy \tag{3}$$

with initial conditions $x_0 = 0$, $y_0 = 1$.

Solution. According to formula (2) we put

$$y = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots \tag{4}$$

The coefficients $c_1, c_2, c_3, \ldots$ are as yet unknown. Differentiating (4), we get

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots \tag{5}$$

Substituting (4) and (5) into (3), we obtain

$$c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots = \frac{1}{2} x + \frac{1}{2} c_1 x^2 + \frac{1}{2} c_2 x^3 + \ldots \tag{6}$$

Now equate the coefficients of like powers of x. This yields the relations

$$c_1 = 0, \quad 2c_2 = \frac{1}{2}, \quad 3c_3 = \frac{1}{2} c_1, \quad 4c_4 = \frac{1}{2} c_2, \ \ldots \tag{7}$$

From them we successively find the coefficients

$$c_1 = 0, \quad c_2 = \frac{1}{4}, \quad c_3 = 0, \quad c_4 = \frac{1}{32}, \quad c_5 = 0, \ \ldots \tag{8}$$

The required solution is of the form

$$y = 1 + \frac{1}{4} x^2 + \frac{1}{32} x^4 + \frac{1}{384} x^6 + \ldots \tag{9}$$

For $x = 1$ we get $y \approx 1.2839$ (cf. Table, Sec. 490). The expansion (9) coincides with the expansion of the function $e^{\frac{x^2}{4}}$:

$$e^{\frac{x^2}{4}} = 1 + \frac{x^2}{4} + \frac{1}{2!}\left(\frac{x^2}{4}\right)^2 + \frac{1}{3!}\left(\frac{x^2}{4}\right)^3 + \ldots \tag{10}$$

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[1] S. V. Kovalevskaya (1850-1891), celebrated Russian mathematician. She obtained important results in mathematics, mechanics and theoretical physics; her writings also include works of fiction and journalism.

Alternative solution. Differentiating successively (3), we find

$$y''=\frac{1}{2}(xy)'=\frac{1}{2}y+\frac{1}{2}xy', \tag{11}$$

$$y'''=\left(\frac{1}{2}y+\frac{1}{2}xy'\right)'=y'+\frac{1}{2}xy'', \tag{12}$$

$$y^{\mathrm{IV}}=\left(y'+\frac{1}{2}xy''\right)'=\frac{3}{2}y''+\frac{1}{2}xy''' \tag{13}$$

and so forth. Substituting into (3) the initial values $x_0=0$, $y_0=1$, we find $y_0'=0$, then from (11) we get

$$y_0''=\frac{1}{2}y_0+\frac{1}{2}x_0y_0'=\frac{1}{2}$$

In the same way we find

$$y_0'''=0, \quad y_0^{\mathrm{IV}}=\frac{3}{4}$$

etc. Substituting the values found into the Taylor series

$$y=y_0+y_0'x+\frac{y_0''}{2!}x^2+\frac{y_0'''}{3!}x^3+\frac{y_0^{\mathrm{IV}}}{4!}x^4+\dots$$

we again get the series (9).

492. Forming Differential Equations

The procedure of forming a differential equation according to the conditions of a problem (geometrical, physical or technical) is that we express mathematically a *relationship between variable quantities and infinitesimal increments in the quantities.* At times a differential equation is obtained without considering increments, since they have been considered earlier. For instance, in representing velocity by the expression $v=\frac{ds}{dt}$ we do not invoke the increments Δs, Δt, but they are actually taken into account because

$$\frac{ds}{dt}=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$$

When forming first-order differential equations, infinitesimal increments are immediately replaced by the corresponding differentials. The error committed in so doing is automatically eliminated in the passage to the limit, as will be shown in the note that follows. Generally, any infinitesimal can be replaced by its equivalent; say, an infinitesimal arc by the corresponding chord or vice versa.

There are no exhaustive rules for the formation of differential equations. As in the forming of algebraic equations, ingenuity is often required. Much depends on the skill acquired in doing exercises.

Example 1. A tank contains 100 litres of brine with 10 kg of dissolved salt. Every minute, two litres of brine flow out of the tank and three litres of fresh water are added. Mixing maintains the same concentration of salt throughout the tank. How much salt remains in the tank after one hour?

Solution. Denote by x the quantity of salt in the tank (in kg), by t the time reckoned from an initial instant (in min).

During a time interval dt a total of $(-dx)$ kg of salt leaves the tank [x is a decreasing function of time, hence, dx is a negative quantity and $(-dx)$ is a positive quantity].

In order to form the equation, let us compute the loss of salt in a different way. At time t the tank has $(100+t)$ litres of liquid ($3t$ litres flowed in and $2t$ litres flowed out); the brine contains x kg of dissolved salt. Thus, one litre of the brine contains $\frac{x}{100+t}$ kg of salt. During time dt $2dt$ litres of brine leave the tank; hence, the quantity of salt diminishes by

$$\frac{x}{100+t}\cdot 2dt \text{ kg}$$

We get the differential equation

$$-dx=\frac{2x\,dt}{100+t} \tag{1}$$

Separating the variables and taking into account the initial conditions $t_0=0$, $x_0=10$, we get

$$\int_{10}^{x}-\frac{dx}{x}=\int_{0}^{t}\frac{2dt}{100+t} \tag{2}$$

that is

$$\ln\frac{10}{x}=2\ln\frac{100+t}{100} \tag{3}$$

or

$$\frac{10}{x}=\left(\frac{100+t}{100}\right)^2 \tag{3a}$$

Substituting $t=60$ into (3a), we find the required quantity of salt: $x\approx 3.91$ (kg).

For less rounded data it is best to take formula (3). Multiplying both sides by the modulus M (Sec. 242), change from natural logarithms to common logarithms.

Note. When we formed Eq. (1) we allowed for two errors: first, we took dx and dt in place of Δx and Δt, second, we assumed that during time dt the loss of salt amounted to $\frac{x}{100+t}\cdot 2dt$ kg, i. e. that the concentration of the solution of brine is equal to $\frac{x}{100+t}$ during the entire interval $(t, t+dt)$. Actually it is $\frac{x}{100+t}$ only at the onset of the interval, and then proceeds to decrease. But these two errors are automaticaly compensated for.

Indeed, during a *small* interval of time $(t, t+\Delta t)$ the concentration of the brine solution differs but slightly from $\frac{x}{100+t}\,\frac{\text{kg}}{\text{litre}}$; hence, during this time the amount of salt diminishes *roughly* by the amount $\frac{2x\,\Delta t}{100+t}$ litres. We thus have the approximate equation

$$-\Delta x\approx\frac{2x\,\Delta t}{100+t}$$

or

$$\frac{\Delta x}{\Delta t}\approx-\frac{2x}{100+t}$$

This approximate equation is the more exact, the less is Δt; in other words, $-\frac{2x}{100+t}$ is the limit of the ratio $\frac{\Delta x}{\Delta t}$ as $\Delta t\to 0$. This limit is the derivative $\frac{dx}{dt}$. Hence, the derivative $\frac{dx}{dt}$ is *exactly* equal to $-\frac{2x}{100+t}$:

$$\frac{dx}{dt}=-\frac{2x}{100+t}$$

This exact equation is equivalent to Eq. (1).

Example 2. A pier is being constructed for a bridge. It is to be 12 metres high and have circular horizontal sections. The pier is calculated for a load of $P=90$ tons (in addition to its own weight). The density of the material $\gamma=2.5\,\frac{\text{tons}}{\text{m}^3}$. The admissible pressure is $k=300\,\frac{\text{tons}}{\text{m}^2}$. Find the areas of the upper and lower bases, and also the shape of the axial section of the pier (for maximum saving in materials used).

Solution. For a permissible pressure of $k=300\ \frac{\text{tons}}{\text{m}^2}$ the area s_0 of the upper base can withstand a load ks_0; by hypothesis, $ks_0=P$. Consequently,

$$s_0=\frac{P}{k}=\frac{90}{300}=0.3\ \text{(sq. m)} \qquad (4)$$

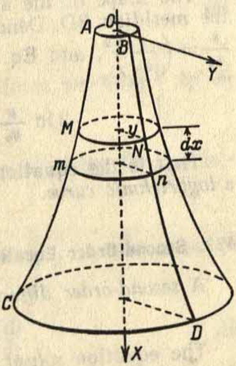

Fig. 478

The area s of the horizontal section increases with falling level because, aside from the load P, the area s is acted upon by the above-lying part of the pier.

Denote by x the distance of a section s (MN in Fig. 478) from the upper base. Isolate infinitesimal horizontal layer $MNnm$. The area of its lower base mn exceeds the area of the upper base MN by ds. Therefore, at the lower base the limiting load is $k\,ds$ times that of the upper base. On the other hand, the load mn is greater than the load of section MN by an amount equal to the weight of the layer $MNnm$, that is, by $\gamma s\,dx$.[1] We get the differential equation

$$k\,ds=\gamma s\,dx \qquad (5)$$

Separating the variables and integrating (initial conditions: $x=0$, $s=s_0$), we get

$$\int_{s_0}^{s}\frac{ds}{s}=\frac{\gamma}{k}\int_{0}^{x}dx \qquad (6)$$

whence

$$\ln\frac{s}{s_0}=\frac{\gamma}{k}x \qquad (7)$$

In order to find the area s_1 of the lower base, it is necessary to substitute $x=12$ (for $s_0=0.3$, $\gamma=2.5$, $k=300$). Going over to common logarithms (Sec. 242), we get

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[1] We assume that the layer $MNnm$ is cylindrical (the error is of higher order than dx).

$$\log\frac{s_1}{0.3}=M\cdot\frac{2.5}{300}\cdot 12 \tag{8}$$

whence $s_1=0.33$ (sq. m).

The shape of the axial section is given by the equation of the meridian BD. Denote the radius of section MN by y; then $\frac{s}{s_0}=\left(\frac{y}{y_0}\right)^2$, and Eq. (7) yields

$$2\ln\frac{y}{y_0}=\frac{\gamma}{k}x \quad\text{or}\quad y=y_0e^{\frac{\gamma}{2k}x} \tag{9}$$

That is the equation of the meridian. Curve (9) is called a *logarithmic curve*.

493. Second-Order Equations

A *second-order differential equation* has the general form

$$\Phi(x, y, y', y'')=0 \tag{1}$$

The equation *solved for* y'' is of the form

$$y''=f(x, y, y') \tag{2}$$

It is assumed that the function $f(x, y, y')$ of the three arguments x, y, y' is uniquely defined and continuous in some range of these arguments.

As a rule,[1] specification of the initial values $x=x_0$, $y=y_0$, $y'=y_0'$ (lying in the range under consideration) defines a unique solution of Eq. (2).

Geometrically, a unique interval curve passes through the given point $M(x_0, y_0)$ in a *given direction*.

The corresponding solution of Eq. (2) is called a *particular* solution. The collection of all particular solutions is termed the *general solution*. An attempt is made to represent the general solution in the form of some function

$$y=\varphi(x, C_1, C_2) \quad (C_1 \text{ and } C_2 \text{ are constants}) \tag{3}$$

capable of yielding any particular solution (for appropriately chosen values of C_1, C_2).

Note. An infinity of integral curves (one in each of the possible directions) pass through the given point $M(x_0, y_0)$.

---

[1] The only possible exception is the case when at least one of the derivatives $f_y'(x, y, y')$, $f_{y'}'(x, y, y')$ is discontinuous or does not exist.

Example. For the initial values $x_0=1$, $y_0=1$, $y_0=2$, find the particular solution of the equation

$$y''=x \tag{4}$$

Solution. Rewrite the given equation as

$$\frac{dy'}{dx}=x \tag{5}$$

Taking into account the initial conditions, we have $\int\limits_2^{y'} dy' = \int\limits_1^x x\,dx$, that is, $y'=\frac{x^2}{2}+\frac{3}{2}$. Again allowing for the initial conditions, we get $\int\limits_1^y dy=\int\limits_1^x \left(\frac{x^2}{2}+\frac{3}{2}\right) dx$. The required particular solution is

$$y=\frac{x^3}{6}+\frac{3}{2}x-\frac{2}{3} \tag{6}$$

Alternative method. From (5) we find

$$y'=\frac{x^2}{2}+C_1 \tag{7}$$

and from this

$$y=\frac{x^3}{6}+C_1x+C_2 \tag{8}$$

The function (8) is the general solution because for appropriately chosen values of C_1, C_2 it yields any particular solution. Thus, substituting into (7) and (8) the given initial values, we get

$$2=\frac{1}{2}+C_1, \qquad 1=\frac{1}{6}+C_1+C_2 \tag{9}$$

whence we find

$$C_1=\frac{3}{2}, \qquad C_2=-\frac{2}{3}$$

Substituting these values into (8), we again get the particular solution (6).

Caution. By far not every solution containing two arbitrary constants is a general solution. For example, the function

$$y=\frac{x^3}{6}+C_3x-C_4\left(x-\frac{1}{C_4}\right) \tag{10}$$

is a solution of Eq. (4) but does not contain all the particular solutions; thus, the solution (6) is not obtainable from (10) for any values of C_3, C_4. Hence, the solution (10) is not a general solution. This is already evident from the fact that the two constants C_3, C_4 "are not essential", that is, they may be replaced by one. Indeed, formula (10) may be written as

$$y=\frac{x^3}{6}+(C_3-C_4)\,x+1$$

Denoting C_3-C_4 by C_1, we get

$$y=\frac{x^3}{6}+C_1x+1$$

This solution is obtained from the general solution (8) for $C_2=1$.

494. Equations of the nth Order

An *nth order equation* solved for $y^{(n)}$,

$$y^{(n)}=f\,(x,\ y,\ y',\ y'',\ \ldots,\ y^{(n-1)})$$

has as a rule (cf. Sec. 493) a unique solution for given initial values x_0, y_0, y_0', $\ldots$, $y_0^{(n-1)}$. Such a solution is called a *particular* solution. The set of all particular solutions is called the *general* solution. We try to represent the general solution in the form

$$y=\varphi\,(x,\ C_1,\ C_2,\ \ldots,\ C_n)$$

Not every solution containing n constants is a general solution (cf. Sec. 493, Caution).

495. Reducing the Order of an Equation

Occasionally, a differential equation of second or higher order allows for reducing the order. The following two cases are the most important.

Case 1. The equation does not contain y. Then for the unknown function we take y'.

Example 1. Integrate the second-order equation

$$(1+x)\,y''+y'=0 \qquad (1)$$

Solution. Taking y' for the unknown function, we rewrite (1) as

$$(1+x)\frac{dy'}{dx}+y'=0 \qquad (2)$$

This is a first-order equation (in the unknown function y'). Multiplying by dx, we obtain a total differential equation (Sec. 484) so that the general integral of Eq. (2) is

$$(1+x)\,y' = C_1 \tag{3}$$

Now let us return to the earlier unknown function y and write Eq. (3) as

$$(1+x)\,\frac{dy}{dx} = C_1 \tag{3a}$$

Integrating (3a), we find

$$y = C_1 \ln(1+x) + C_2 \tag{4}$$

This is the general solution of Eq. (1).

Case 2. The equation does not contain x. Then we again take y' for the unknown function, but for the argument we take y (in place of x). The derivatives of second and higher order are thereby transformed by the formulas

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dy}\cdot\frac{dy}{dx} = \frac{dy'}{dy}\,y', \tag{5}$$

$$y''' = \frac{dy''}{dx} - \frac{d}{dy}\left(\frac{dy'}{dy}\,y'\right)\frac{dy}{dx} = \frac{d}{dy}\left(\frac{dy'}{dy}\,y'\right)y' \tag{6}$$

and so forth.

Example 2. Integrate the second-order equation

$$y'' + y = 0 \tag{7}$$

Solution. Applying formula (5), we transform (7) to

$$y'\,dy' + y\,dy = 0 \tag{8}$$

This is a first-order equation (the variables are y and y'). The general integral of Eq. (8) is

$$y'^2 + y^2 = C_1^2 \tag{9}$$

Returning to the earlier variables x, y, write (9) as

$$\frac{dy}{\sqrt{C_1^2 - y^2}} = \pm dx \tag{10}$$

Integrating, we find

$$\arcsin\frac{y}{C_1} = \pm(x + C_2)$$

whence

$$y = C_1 \sin(x + C_2)$$

(the sign ± is included in the constant C_1).

This is the general solution of Eq. (8); it may be transformed to

$$y=C_3\sin x+C_4\cos x$$

where

$$C_3=C_1\cos C_2, \quad C_4=C_1\sin C_2$$

496. Second-Order Linear Differential Equations

A *second-order linear equation* is one of the form

$$y''+P(x)\,y'+Q(x)\,y=R(x) \tag{1}$$

where the functions $P(x)$, $Q(x)$, $R(x)$ do not depend on y.

If $R(x)=0$, then Eq. (1) is called an *equation with right-hand member zero*;[1] if $R(x)\neq 0$, then it is termed an *equation with right-hand member.*

The homogeneous equation

$$y''+P(x)\,y'+Q(x)\,y=0 \tag{2}$$

has the following properties.

Theorem 1. If a function $\varphi_1(x)$ is a solution of Eq. (2), then the function $C_1\varphi_1(x)$ (C_1 a constant) is also a solution.

Theorem 2. If the functions $\varphi_1(x)$ and $\varphi_2(x)$ are two solutions of Eq (2), then the function $\varphi_1(x)+\varphi_2(x)$ is also a solution.

Corollary. If $\varphi_1(x)$, $\varphi_2(x)$ are two solutions of Eq. (2), then $C_1\varphi_1(x)+C_2\varphi_2(x)$ (C_1 and C_2 constants) is also a solution.

Example 1. Consider the homogeneous linear equation

$$y''+\frac{1}{x}\,y'-\frac{1}{x^2}\,y=0 \tag{3}$$

Having convinced oneself by verification that the functions x and $\frac{1}{x}$ are solutions, we conclude that the function

$$y=C_1x+C_2\,\frac{1}{x}$$

is also a solution of Eq. (3).

Note 1. The solution $y=C_1\varphi_1(x)+C_2\varphi_2(x)$ *will not always be a general solution.* Thus, the functions $\varphi_1(x)=3x$ and $\varphi_2(x)=5x$ are solutions of Eq. (3), the function $y=C_1\varphi_1(x)+C_2\varphi_2(x)=(3C_1+5C_2)\,x$ is also a solution, but not a general solution (the two constants C_1, C_2 are not essential; cf. Sec. 493, Caution).

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[1] A linear equation with right-hand member zero is also called a *homogeneous* equation. See footnote on p. 720.

Note 2. The solution $y = C_1\varphi_1(x) + C_2\varphi_2(x)$ will not be a general solution if the functions $\varphi_1(x)$, $\varphi_2(x)$ are ***linearly dependent***, that is if they can be connected by the relation

$$a_1\varphi_1(x) + a_2\varphi_2(x) = 0 \tag{4}$$

where at least one of the constants a_1, a_2 is nonzero.

But if the solutions $\varphi_1(x)$, $\varphi_2(x)$ are ***linearly independent***, i. e. if the relation (4) is possible only when both constants a_1, a_2 are zero, then the function

$$y = C_1\varphi_1(x) + C_2\varphi_2(x)$$

yields a general solution.

Example 2. The solutions $\varphi_1(x) = 3x$ and $\varphi_2(x) = 5x$ of Eq. (3) are linearly dependent because for $a_1 = 5$, $a_2 = -3$ or for $a_1 = 10$, $a_2 = -6$, or for $a_1 = 15$, $a_2 = -9$, etc., we get $a_1\varphi_1(x) + a_2\varphi_2(x) = 0$.

The solutions $\varphi_1(x) = 3x$ and $\varphi_2(x) = -\frac{1}{2x}$ are linearly independent because the relation (4) is possible only for $a_1 = a_2 = 0$. Accordingly, the solution $y = 3C_1x + 5C_2x$ is not a general solution but the solution $y = 3C_1x - \frac{C_2}{2x}$ is a general solution.

The foregoing refers solely to ***homogeneous*** **linear equations.**

The nonhomogeneous equation

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \tag{5}$$

has the following property.

Theorem 3. If a function $f(x)$ is one of the solutions of Eq. (5), then its general solution is

$$y = C_1\varphi_1(x) + C_2\varphi_2(x) + f(x) \tag{6}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are two linearly independent solutions of Eq. (2), i. e. of the corresponding homogeneous equation.

Example 3. Consider the equation

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 8x \tag{7}$$

Convincing ourselves by verification that the function $f(x) = x^3$ is its solution, we conclude that the general solution of Eq. (7) is (cf. Example 1)

$$y = C_1x + C_2\frac{1}{x} + x^3$$

Theorem 3 may also be stated as follows: *the general solution of a nonhomogeneous linear equation is the sum of some particular solution and the general solution of the corresponding homogeneous equation.*

Note 3. A second-order (homogeneous or nonhomogeneous) linear equation can be reduced to quadratures only in special cases, which include the practically important case when the coefficients $P(x)$ and $Q(x)$ are both constants (see Secs. 497-499).

497. Second-Order Linear Equations with Constant Coefficients

The equation

$$y'' + py' + qy = R(x) \tag{1}$$

where p and q are constants and $R(x)$ depends solely on x (or is a constant) is called a *second-order linear differential equation with constant coefficients.* Eq. (1) can always be reduced to quadratures. And when $R(x) = 0$ (homogeneous equation), the solution is not only reducible to quadratures, but is also always expressible in terms of elementary functions (see Sec. 498).

498. Second-Order Homogeneous Linear Equations with Constant Coefficients

Let us consider the equation

$$y'' + py' + qy = 0 \tag{1}$$

where p, q are constants. We shall seek a solution of the form

$$y = e^{rx} \tag{2}$$

Substituting (2) into (1), we find that the number r must satisfy the equation

$$r^2 + pr + q = 0 \tag{3}$$

This is called a *characteristic equation.*

There can be three cases.

Case 1. $\left(\frac{p}{2}\right)^2 - q > 0$. The characteristic equation (3) has two distinct real roots r_1, r_2 $\left(r_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}\right)$.

In this case we have two linearly independent (Sec. 496, Note 2) solutions: $y = e^{r_1x}$, $y = e^{r_2x}$. The general solution will be

$$y = C_1e^{r_1x} + C_2e^{r_2x} \tag{4}$$

Example 1. Find the general solution of the equation

$$8y''+2y'-3y=0 \tag{5}$$

and also the particular solution for the initial conditions $x_0=0$, $y_0=-6$, $y_0'=7$.

Solution. The characteristic equation

$$8r^2+2r-3=0 \tag{6}$$

has two distinct real roots:

$$r_1=\frac{1}{2}, \quad r_2=-\frac{3}{4}$$

The functions $y=e^{\frac{1}{2}x}$, $y=e^{-\frac{3}{4}x}$ give two linearly independent solutions. The general solution of Eq. (5) is

$$y=C_1e^{\frac{1}{2}x}+C_2e^{-\frac{3}{4}x} \tag{7}$$

To find the particular solution, compute the derivative y':

$$y''=\frac{1}{2}C_1e^{\frac{1}{2}x}-\frac{3}{4}C_2e^{-\frac{3}{4}x} \tag{7a}$$

Substituting the initial data into (7) and (7a), we get the system

$$-6=C_1+C_2, \quad 7=\frac{1}{2}C_1-\frac{3}{4}C_2$$

From it we find $C_1=2$, $C_2=-8$. The desired particular solution is

$$y=2e^{\frac{1}{2}x}-8e^{-\frac{3}{4}x}$$

Case 2. $\left(\frac{p}{2}\right)^2-q=0$. The characteristic equation has two equal roots $\left(r_1=r_2=-\frac{p}{2}\right)$.

In this case, the solutions $y=e^{r_1x}$, $y=e^{r_2x}$ are linearly dependent (they coincide). But now, besides the solution $y=e^{-\frac{p}{2}x}$, there is the linearly independent solution

$y=xe^{-\frac{p}{2}x}$. The general solution will be

$$y=(C_1+C_2x)\,e^{-\frac{p}{2}x} \tag{8}$$

Example 2. Find the general solution of the equation

$$y''+4y'+4y=0 \tag{9}$$

and the particular solution for the initial conditions $x_0=0.5$, $y_0=0.5$, $y_0'=-4$.

Solution. The characteristic equation

$$r^2+4r+4=0$$

has the equal roots $r_1=r_2=-2$. The functions $y=e^{-2x}$, $y=xe^{-2x}$ yield linearly independent solutions. The general solution of Eq. (9) is

$$y=(C_1+C_2x)\,e^{-2x} \tag{10}$$

Differentiating, we find

$$y'=[-2C_1+C_2(1-2x)]\,e^{-2x} \tag{10a}$$

Substituting the initial data into (10) and (10a), we get the system

$$0.5=(C_1+0.5C_2)\,e^{-1}, \qquad -4=-2C_1e^{-1}$$

From this we find $C_1=2e$, $C_2=-3e$. The desired particular solution is

$$y=(2e-3ex)\,e^{-2x}$$

or

$$y=(2-3x)\,e^{1-2x}$$

Case 3. $\left(\frac{p}{2}\right)^2-q<0$. The characteristic equation has a pair of complex-valued roots:

$$r_{1,2}=-\frac{p}{2}\pm\beta i \tag{11}$$

where

$$\beta=\sqrt{q-\left(\frac{p}{2}\right)^2}$$

In this case, the expressions

$$e^{r_1x}, \qquad e^{r_2x} \tag{12}$$

do not have real values for any real value of x, except $x=0$. But we can now use the functions

$$y=e^{-\frac{p}{2}x}\cos\beta x,\qquad y=e^{-\frac{p}{2}x}\sin\beta x \tag{13}$$

Substituting them into Eq. (1), we are convinced that each of the functions (13) is a solution of Eq. (1).

In Sec. 498a it is shown how the solutions (13) are derived from *complex-valued* solutions of the form (12).

The solutions (13) are linearly independent and therefore the general solution will be

$$y=e^{-\frac{p}{2}x}(C_1\cos\beta x+C_2\sin\beta x) \tag{14}$$

or, in an alternative form,

$$y=C_3e^{-\frac{p}{2}x}\sin(C_4+\beta x) \tag{14a}$$

(where $C_3\sin C_4=C_1$, $C_3\cos C_4=C_2$).

Example 3. Find the general solution of the equation

$$y''+y'+y=0 \tag{15}$$

Solution. The characteristic equation

$$r^2+r+1=0 \tag{16}$$

has the imaginary roots $r_{1,\,2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$. The functions

$$y=e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x \text{ and } y=e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$$

yield linearly independent solutions. The general solution of Eq. (1) is

$$y=e^{-\frac{1}{2}x}\left(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right) \tag{17}$$

or

$$y=C_3e^{-\frac{1}{2}x}\sin\left(C_4+\frac{\sqrt{3}}{2}x\right) \tag{17a}$$

Example 4. Find the general solution of the equation

$$y''+y=0$$

Solution. The characteristic equation $r^2+1=0$ has the imaginary roots $r_{1,2}=\pm i$ $\left(\text{here } \beta=1,\ -\frac{p}{2}=0\right)$. The general solution is (cf. Sec. 495, Example 2)

$$y=C_1\cos x+C_2\sin x$$

498a. Connection Between Cases 1 and 3 in Sec. 498

The particular solutions of the form

$$\varphi_1(x)=e^{r_1x},\quad \varphi_2(x)=e^{r_2x}\left[r_{1,2}=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}\right] \tag{1}$$

which were used in Sec. 498 for Case 1 can also be used for Case 3 if we introduce complex-valued numbers into the discussion and define the complex power of the number e, as was done in Sec. 409. Then formulas (1) will be written as

$$\varphi_1(x)=e^{\left(-\frac{p}{2}+\beta i\right)x},\quad \varphi_2(x)=e^{\left(-\frac{p}{2}-\beta i\right)x} \tag{2}$$

where $\beta\left[=\sqrt{q-\left(\frac{p}{2}\right)^2}\right]$ and $-\frac{p}{2}$ are real numbers. The expressions (2) represent a pair of *complex-valued* functions of the *real* argument x. Since these functions are differentiated by the ordinary rules (Sec. 408), they are solutions of the equation $y''+py'+qy=0$ in Case 3 as well. These solutions do not satisfy us since they are not real. *But from them we can derive real solutions*. Indeed, applying Euler's formula (Sec. 410) we can represent the solutions (2) in the form

$$\varphi_1(x)=e^{\alpha x}(\cos\beta x+i\sin\beta x), \tag{3}$$

$$\varphi_2(x)=e^{\alpha x}(\cos\beta x-i\sin\beta x) \tag{4}$$

The function $C_1\varphi_1(x)+C_2\varphi_2(x)$ is a solution for any constant values of C_1, C_2 (Sec. 496). Putting first $C_1=\frac{1}{2}$, $C_2=\frac{1}{2}$ and then $C_1=-\frac{i}{2}$, $C_2=\frac{i}{2}$ another time, we get two real solutions:

$$e^{\alpha x}\cos\beta x \text{ and } e^{\alpha x}\sin\beta x$$

They were the ones which were used in Case 3, Sec. 498.

499. Second-Order Nonhomogeneous Linear Equations with Constant Coefficients

The general solution of a nonhomogeneous equation

$$y''+py'+qy=R(x) \tag{1}$$

is obtained with the aid of quadratures from the general solution of the corresponding homogeneous equation

$$y''+py'+qy=0 \tag{2}$$

by the general method given below in Sec. 501. But in many practical cases of importance the aim is attained in a much simpler fashion.

First find some *particular solution* $f(x)$ of the given Eq. (1), then *add* $f(x)$ *to the general solution of the corresponding homogeneous Eq.* (2). The sum is (Sec. 496, Theorem 3) the general solution of the given equation.

The following three rules are used to find the function $f(x)$.

Rule 1. If the right-hand member $R(x)$ of Eq. (1) is of the form

$$R(x)=P(x)e^{kx} \tag{3}$$

where $P(x)$ is some polynomial of degree m, and if the number k is not a root of the characteristic equation

$$r^2+pr+q=0 \tag{4}$$

then Eq. (1) has a particular solution of the form

$$y^*=Q(x)\,e^{kx} \tag{5}$$

where $Q(x)$ is some polynomial of the same degree m [the asterisk on y is used to distinguish the particular solution $y^*=f(x)$ of Eq. (1) from its general solution].

The coefficients and the constant term of the polynomial $Q(x)$ may be found by the method of undetermined coefficients.

Note 1. If the factor $P(x)$ is a constant (a polynomial of degree zero), then $Q(x)$ is also a constant.

Note 2. The rule can be extended to the case when $R(x)$ is a polynomial (i.e. $k=0$). Then the solution (5) is also a polynomial.

Example 1. Find the general solution of the equation

$$y''-\frac{1}{2}y'-\frac{1}{2}y=3e^{\frac{1}{2}x} \tag{6}$$

Solution. The characteristic equation

$$r^2-\frac{1}{2}r-\frac{1}{2}=0 \tag{7}$$

has the roots $r_1=1$, $r_2=-\frac{1}{2}$ so that the general solution of the corresponding homogeneous equation is

$$\bar{y}=C_1e^x+C_2e^{\frac{1}{2}x} \tag{8}$$

[the bar over y is used to distinguish the general solution of Eq. (2) from the general solution y of Eq. (1)].

It remains to find some particular solution y^* of Eq. (6). The right-hand member of (6) has the form (3), and $P(x)=3$ (polynomial of degree zero) and the number $k=\frac{1}{2}$ is not a root of the characteristic equation (7). By Rule 1, Eq. (6) has a solution of the form

$$y^*=Ae^{\frac{1}{2}x} \quad (A \text{ a constant}) \tag{9}$$

Substituting (9) into (6), we find

$$\left(\frac{1}{4}A-\frac{1}{2}\cdot\frac{1}{2}A-\frac{1}{2}A\right)e^{\frac{1}{2}x}=3e^{\frac{1}{2}x} \tag{10}$$

Equating the coefficients of $e^{\frac{1}{2}x}$, we get

$$A=-6 \tag{11}$$

The desired solution y^* is

$$y^*=-6e^{\frac{1}{2}x} \tag{12}$$

The general solution of Eq. (6) is

$$y=\bar{y}+y^*=C_1e^x+C_2e^{-\frac{1}{2}x}-6e^{\frac{1}{2}x} \tag{13}$$

Example 2. Find the general solution of the equation

$$y''-3y'+2y=x^2+3x \tag{14}$$

The characteristic equation

$$r^2-3r+2=0$$

has the roots $r_1=1$, $r_2=2$ so that (with the notation of Example 1)

$$\bar{y}=C_1e^x+C_2e^{2x} \tag{15}$$

The right-hand member of Eq. (14) is of the form (3), and $P(x)=x^2+3x$ and the number $k=0$ is not a root of the characteristic equation. We seek a solution of the form

$$y^*=Ax^2+Bx+C \tag{16}$$

Substituting into (14), we get the equation

$$2Ax^2+(2B-6A)x+2C-3B+2A=x^2+3x \tag{17}$$

Equating the coefficients of like powers of x, we get the system

$$2A=1,\quad 2B-6A=3,\quad 2C-3B+2A=0 \tag{18}$$

from which we find $A=\frac{1}{2}$, $B=3$, $C=4$ so that

$$y^*=\frac{1}{2}x^2+3x+4 \tag{19}$$

The general solution of Eq. (14) is

$$y=\bar{y}+y^*=C_1e^x+C_2e^{2x}+\frac{1}{2}x^2+3x+4 \tag{20}$$

Note 3. For the equation $y''-3y'=x^2+3x$ [1] it would be useless [2] to search for a particular solution of the form (16) because the number $k=0$ is now a root of the characteristic equation ($r^2-3r=0$). The conditions of Rule 1 break down and we have to apply Rule 2.

Rule 2. Let the right-hand member of Eq. (1) be of the form

$$R(x)=P(x)\,e^{kx} \tag{21}$$

where $P(x)$ is a polynomial of degree m, and let the number k be a root of the characteristic equation $r^2+pr+q=0$. If this root is single (i.e. one of the distinct roots), then Eq. (1) has a particular solution of the form

$$y^*=xQ(x)\,e^{kx} \tag{22}$$

where $Q(x)$ is a polynomial of degree m; if k is a double

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[1] The equation is solved in Example 3 below.

[2] Though no error would occur. In an attempt to find a solution of the form $y^*=Ax^2+Bx+C$, we get the following equation in place of (17):

$$-6Ax+(2A-3B)=x^2+3x$$

It is impossible to equate coefficients of like powers of x, since the right-hand member contains a second-degree term which is absent in the left-hand member. The attempt failed but no error was committed.

root of the characteristic equation (i.e. one of two equal roots), then Eq. (1) has a solution of the form

$$y^* = x^2 Q(x) e^{kx} \tag{23}$$

Notes 1 and 2 remain valid.

Example 3. Find the general solution of the equation

$$y'' - 3y' = x^2 + 3x \tag{24}$$

and also the particular solution for the initial conditions

$$x_0 = 0, \quad y_0 = 1, \quad y_0' = 3$$

Solution. Here, $P(x) = x^2 + 3x$ and the number $k=0$ serves as a single root of the characteristic equation

$$r^2 - 3r = 0$$

($r_1 = 3$, $r_2 = 0$). Eq. (24) has a particular solution of the form

$$y^* = x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx \tag{25}$$

Proceeding as in Example 2, we obtain the system

$$-9A = 1, \quad -6B + 6A = 3, \quad -3C + 2B = 0$$

from which we find $A = -\frac{1}{9}$, $B = -\frac{11}{18}$, $C = -\frac{11}{27}$ so that

$$y^* = -\frac{1}{9}x^3 - \frac{11}{18}x^2 - \frac{11}{27}x \tag{26}$$

The general solution of Eq. (24) is

$$y = C_1 e^{3x} + C_2 - \frac{1}{9}x^3 - \frac{11}{18}x^2 - \frac{11}{27}x \tag{27}$$

Differentiating, we get

$$y' = 3C_1 e^{3x} - \frac{1}{3}x^2 - \frac{11}{9}x - \frac{11}{27} \tag{27a}$$

Substituting the initial values into (27) and (27a), we obtain the system

$$1 = C_1 + C_2, \quad 3 = 3C_1 - \frac{11}{27}$$

which yields $C_1 = \frac{92}{81}$, $C_2 = -\frac{11}{81}$; the desired particular solution is $y = \frac{92}{81}e^{3x} - \frac{1}{9}x^3 - \frac{11}{18}x^2 - \frac{11}{27}x - \frac{11}{81}$.

Example 4. Find the general solution of the equation

$$y'' - 2y' + y = xe^x \tag{28}$$

Here, $P(x)=x$ and the number $k=1$ is a double root of the characteristic equation $r^2-2r+1=0$. Eq. (28) has a particular solution of the form

$$y^*=x^2(Ax+B)e^x=(Ax^3+Bx^2)e^x \tag{29}$$

Substitute (29) into (28); the terms in x^3 and x^2 vanish of themselves and we get the equation

$$(6Ax+2B)e^x=xe^x \tag{30}$$

Equating coefficients of like powers of x, we get the system $6A=1$, $2B=0$ so that

$$y^*=\frac{1}{6}x^3e^x$$

The general solution of Eq (28) (see Sec. 498, Case 2) is

$$y=(C_1+C_2x)e^x+\frac{1}{6}x^3e^x \tag{31}$$

Rule 3. Let the right side of Eq. (1) have the form

$$R(x)=e^{\alpha x}[P_1(x)\cos\beta x+P_2(x)\sin\beta x] \tag{32}$$

where $P_1(x)$ and $P_2(x)$ are polynomials of degree m_1 and m_2.

Two cases are possible:

(1) the complex numbers $\alpha\pm\beta i$ are not roots of the characteristic equation $r^2+pr+q=0$;

(2) the numbers $\alpha\pm\beta i$ are roots of this equation. 1)

In the first case, Eq. (1) has a solution of the form

$$y^*=e^{\alpha x}[Q_1(x)\cos\beta x+Q_2(x)\sin\beta x] \tag{33}$$

where $Q_1(x)$, $Q_2(x)$ are polynomials the degrees of which do not exceed the highest degrees of m_1, m_2.

In the second case, Eq (1) has a solution of the form

$$y^*=xe^{\alpha x}[Q_1(x)\cos\beta x+Q_2(x)\sin\beta x] \tag{34}$$

Example 5. Find the general solution of the equation

$$y''+y=10e^x\sin 2x \tag{35}$$

Here, $P_1(x)=0$, $P_2(x)=10$ (i. e. $P_1(x)$ and $P_2(x)$ are polynomials of degree zero), $\alpha=1$, $\beta=2$. The complex numbers $\alpha\pm\beta i=1\pm 2i$ are not roots of the characteristic equation $r^2+1=0$. Eq (35) has a particular solution of the form

$$y^*=e^x(A\cos 2x+B\sin 2x) \tag{36}$$

Substituting (36) into (35) we arrive at the equation

$$[(-2A+4B)\cos 2x+(-4A-2B)\sin 2x]e^x=10e^x\sin 2x \tag{37}$$

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1) The case when only one of the numbers $\alpha\pm\beta i$ is a root of the equation $r^2+pr+q=0$ is impossible (for real values of the coefficients p, q).

and get the system

$$-2A+4B=0, \quad -4A-2B=10$$

which yields $A=-2$, $B=-1$ so that

$$y^*=-e^x(2\cos 2x+\sin 2x)$$

The general solution of Eq. (35) is

$$y=C_1\cos x+C_2\sin x-e^x(2\cos 2x+\sin 2x) \tag{38}$$

Example 6. Find the general solution of the equation

$$y''+y=4x\sin x \tag{39}$$

Here, $P_1(x)=0$, $P_2(x)=4x$ [the highest degree of the polynomials $P_1(x)$ and $P_2(x)$ is first], $\alpha=0$, $\beta=1$. The complex numbers $\alpha\pm\beta i=\pm i$ are roots of the characteristic equation $r^2+1=0$. Eq. (39) has a particular solution of the form

$$\begin{aligned} y^* &= x\,[(A_1x+B_1)\cos x+(A_2x+B_2)\sin x]= \\ &=(A_1x^2+B_1x)\cos x+(A_2x^2+B_2x)\sin x \end{aligned} \tag{40}$$

Substituting (40) into (39), we get the equation

$$[4A_2x+(2B_2+2A_1)]\cos x+[-4A_1x+(-2B_1+2A_2)]\sin x=4x\sin x \tag{41}$$

and obtain the system

$$4A_2=0, \quad 2B_2+2A_1=0, \quad -4A_1=4, \quad -2B_1+2A_2=0$$

which yields $A_1=-1$, $B_1=0$, $A_2=0$, $B_2=1$ so that

$$y^*=-x^2\cos x+x\sin x$$

The general solution of Eq. (39) is

$$y=C_1\cos x+C_2\sin x+x(-x\cos x+\sin x) \tag{42}$$

Note 4. If the right side of Eq. (1) contains a sum where each term is of the form (21) or (32), then Eq. (1) has a particular solution constituting a sum of expressions of the form (5), (22), (23), (33), (34). The coefficients are found as in Examples 1 to 6.

500. Linear Equations of Any Order

A *linear equation of order n* is an equation of the form

$$y^{(n)}+P_1(x)\,y^{(n-1)}+\ldots+P_n(x)\,y=R(x) \tag{1}$$

If $R(x)=0$, then (1) is called an *equation with right-hand member zero* (or *homogeneous equation*), if $R(x)\neq 0$, then it is an *equation with right-hand member* (or *nonhomogeneous equation*).

The properties of second-order linear equations (Secs. 496 to 499) are extended to higher-order linear equations in the following manner.

If $\varphi_1(x)$, $\varphi_2(x)$, ..., $\varphi_n(x)$ are solutions of the homogeneous equation

$$y^{(n)}+P_1(x)\,y^{(n-1)}+\ldots+P_n(x)\,y=0 \tag{2}$$

then the function

$$y=C_1\varphi_1(x)+C_2\varphi_2(x)+\ \ldots+C_n\varphi_n(x) \tag{3}$$

is also a solution. This solution will not be a general solution if the solutions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ are *linearly dependent*, i.e. are connected by the relation

$$a_1\varphi_1(x)+a_2\varphi_2(x)+\ldots+a_n\varphi_n(x)=0 \qquad (4)$$

where, of the constants $a_1, a_2, \ldots, a_n$, there is at least one nonzero constant.

But if the solutions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ are *linearly independent*, i.e. if Eq. (4) is possible only when all the constants $a_1, a_2, \ldots, a_n$ are zero, then (3) is the general solution of Eq. (2).

The general solution of Eq. (1) is obtained from some particular solution by adjoining it to the general solution of Eq. (2).

A *homogeneous linear equation with constant coefficients*

$$y^{(n)}+p_1y^{(n-1)}+p_2y^{(n-2)}+\ldots+p_ny=0 \qquad (5)$$

is solved with the aid of the *characteristic equation*

$$r^n+p_1r^{n-1}+p_2r^{n-2}+\ldots+p_n=0 \qquad (6)$$

I. If all the roots $r_1, r_2, \ldots, r_n$ of the characteristic equation are real and single, then the general solution of Eq. (5) is

$$y=C_1e^{r_1x}+C_2e^{r_2x}+\ldots+C_ne^{r_nx} \qquad (7)$$

II. If some one real root r has multiplicity k $(r_1=r_2=\ldots=r_k)$, then in formula (7) the corresponding k terms are replaced by the term

$$(C_1+C_2x+\ldots+C_kx^{k-1})\,e^{rx} \qquad (8)$$

III. If the characteristic equation has a pair of single complex-conjugate roots $(r_{1,2}=\alpha\pm\beta i)$, then the corresponding pair of terms in formula (7) is replaced by the term

$$e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x) \qquad (9)$$

IV. If some pair of complex conjugate roots has multiplicity k, then the corresponding k pairs of terms in (7) are replaced by the summand

$$e^{\alpha x}\,[(C_1+C_2x+\ldots+C_kx^{k-1})\cos\beta x+ \qquad (10)$$
$$+(D_1+D_2x+\ldots+D_kx^{k-1})\sin\beta x]$$

Example. Consider the equation

$$y^{\mathrm{V}}+y^{\mathrm{IV}}+2y'''+2y''+y'+y=0 \qquad (11)$$

Its characteristic equation

$$r^5+r^4+2r^3+2r^2+r+1=0 \qquad (12)$$

has a single real root $r=-1$ and a pair of double conjugate imaginary roots $r=\pm i$. The general solution of Eq. (11) is

$$y=C_1e^{-x}+(C_2+C_3x)\cos x+(C_4+C_5x)\sin x \qquad (13)$$

For a *nonhomogeneous linear equation with constant coefficients*

$$y^{(n)}+p_1y^{(n-1)}+\ldots+p_ny=R(x) \qquad (14)$$

the general solution is obtained, by means of quadratures, from the general solution of the corresponding homogeneous equation by a method explained in Sec. 501. But if the right-hand member $R(x)$ has the form $P(x)e^{kx}$ [$P(x)$ is a polynomial], or the more general form

$$e^{\alpha x}[P_1(x)\cos\beta x+P_2(x)\sin\beta x]$$

or represents a sum of terms of similar form, then the solution is simplified.

I. Let

$$R(x)=P(x)e^{kx} \tag{15}$$

where $P(x)$ is a polynomial of degree m. Then Eq. (14) has a particular solution of the form

$$y^*=Q(x)e^{kx} \tag{16}$$

where $Q(x)$ is a polynomial of degree m, provided that the number k is not a root of the characteristic equation (6). Otherwise, Eq. (14) has a particular solution of the form

$$y^*=x^lQ(x)e^{kx} \tag{17}$$

where l is the multiplicity with which k enters into the number of roots of the characteristic equation (cf. Sec. 499, Rules 1, 2 and Examples 1 to 4).

II. Let

$$R(x)=e^{\alpha x}[P_1(x)\cos\beta x+P_2(x)\sin\beta x] \tag{18}$$

where $P_1(x)$ and $P_2(x)$ are polynomials of degree m_1 and m_2. Then Eq. (14) has a particular solution of the form

$$y^*=e^{\alpha x}[Q_1(x)\cos\beta x+Q_2(x)\sin\beta x] \tag{19}$$

where $Q_1(x)$, $Q_2(x)$ are polynomials the degrees of which do not exceed the highest of the degrees m_1 and m_2, provided that the complex numbers $\alpha\pm\beta i$ are not roots of the characteristic equation (6), otherwise Eq. (1) has a particular solution of the form

$$y^*=x^le^{\alpha x}[Q_1(x)\cos\beta x+Q_2(x)\sin\beta x] \tag{20}$$

where l is the multiplicity with which the pair of roots $\alpha\pm\beta i$ enters into the number of roots of the characteristic equation (cf. Sec. 499, Rule 3 and Examples 5, 6).

501. Method of Variation of Constants (Parameters)

The general solution of a nonhomogeneous linear equation is obtained from the general solution of the corresponding homogeneous equation with the aid of quadratures. To do this we employ the following device.

In the general solution of the homogeneous equation we replace all arbitrary constants by unknown functions, then differentiate the expression obtained and subject the unknown functions to supplementary conditions that simplify the form of the successive derivatives. Substituting the expressions of the derivatives y', y'', y''' and so forth into the given equation, we get yet another condition imposed on the unknown functions. Then it proves possible to find the first derivatives of all unknown functions and it remains to perform the quadratures.

This method is applicable to linear equations of any order both with constant and with variable coefficients. In Sec. 486 it was applied to a first-order linear equation. Here, we consider an equation of the second order:

$$y''+P(x)\,y'+Q(x)\,y=R(x) \tag{1}$$

Let the general solution of the corresponding homogeneous equation be

$$y=C_1\varphi_1(x)+C_2\varphi_2(x) \tag{2}$$

We seek the general solution of Eq. (1) in the form of (2), now treating C_1 and C_2 as unknown functions of x.

Differentiating (2), we find

$$y'=C_1\varphi_1'(x)+C_2\varphi_2'(x)+C_1'\varphi_1(x)+C_2'\varphi_2(x) \tag{3}$$

We introduce the supplementary condition

$$C_1'\varphi_1(x)+C_2'\varphi_2(x)=0 \tag{4}$$

Then the form of the first derivative is simplified and we have

$$y'=C_1\varphi_1'(x)+C_2\varphi_2'(x) \tag{5}$$

Differentiating once again, we get

$$y''=C_1\varphi_1''(x)+C_2\varphi_2''(x)+C_1'\varphi_1'(x)+C_2'\varphi_2'(x) \tag{6}$$

After substitution of expressions (2), (5) and (6) into Eq. (1), all terms in C_1 cancel [because the function $y=\varphi_1(x)$ is a solution of the equation $y''+Py'+Qy=0$]; in the same way all the terms containing C_2 cancel and we get one more condition:

$$C_1'\varphi_1'(x)+C_2'\varphi_2'(x)=R(x) \tag{7}$$

Conditions (4) and (7) permit finding the expressions of the derivatives C_1', C_2' and it remains to perform the quadratures.

Example. Consider the equation

$$y''+y=\tan x \tag{1a}$$

The general solution of the corresponding homogeneous equation is

$$y=C_1 \cos x+C_2 \sin x \tag{2a}$$

where C_1 and C_2 are arbitrary constants. We seek a solution of (1a) in the form (2a), now treating C_1 and C_2 as unknown functions.

Conditions (4) and (7) become

$$C_1' \cos x+C_2' \sin x=0, \quad -C_1' \sin x+C_2' \cos x=\tan x \tag{3a}$$

Whence we find

$$C_1'=-\tan x \sin x, \qquad C_2'=\sin x;$$

$$C_1=\int -\tan x \sin x\, dx+C_3, \quad C_2=\int \sin x\, dx+C_4$$

(C_3, C_4 are constants). In the given case, the integration can be performed in terms of elementary functions. Substituting into (2a), we get the general solution

$$y=\left(\ln \frac{\cos x}{1+\sin x}+\sin x+C_3\right) \cos x+(-\cos x+C_4) \sin x=$$

$$=\cos x \ln \frac{\cos x}{1+\sin x}+C_3 \cos x+C_4 \sin x$$

502. Systems of Differential Equations. Linear Systems

A *system of differential equations* is a collection of equations in several unknown functions and their derivatives, each equation having at least one derivative. In practical cases, one deals with systems where the number of equations is equal to the number of unknowns.

A system is called *linear* if the unknown functions and their derivatives enter each of the equations only to the first power. A linear system is of *normal form* when it is solved for all derivatives.

Example 1. The system of differential equations

$$\frac{dx}{dt}=x-y+\frac{3}{2} t^2, \tag{1}$$

$$\frac{dy}{dt}=-4x-2y+4t+1 \tag{2}$$

is linear and is of normal form.

In this example we have a *linear system with constant coefficients* (the coefficients of the unknown functions and their derivatives are constant).

We can eliminate from the linear system all unknowns (and their derivatives), except one, by adjoining to it the equations derived by differentiation. The resulting equation will contain one unknown function and its derivative of first and higher order. This equation will also be linear and if the original system was a system with constant coefficients, then the higher-order equation thus found will have constant coefficients.

Finding the unknown function of this equation, we substitute its expression into the given equations and find the remaining unknown functions.

Example 2. Solve the linear system of Example 1.

Solution. To eliminate y and $\frac{dy}{dt}$, differentiate (1). This yields

$$\frac{d^2x}{dt^2}=\frac{dx}{dt}-\frac{dy}{dt}+3t \tag{3}$$

From Eq. (1) we find the expression of y in terms of t, x and $\frac{dx}{dt}$; substituting into (2), we get the expression $\frac{dy}{dt}$ in terms of the same quantities. Substituting this expression into (3), we get a second-order linear equation:

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}-6x=3t^2-t-1 \tag{4}$$

By the method given in Sec. 499 we find its general solution:

$$x=C_1e^{2t}+C_2e^{-3t}-\frac{1}{2}t^2 \tag{5}$$

This expression is substituted into Eq. (1) and we find the second unknown function

$$y=-\frac{dx}{dt}+x+\frac{3}{2}t^2=-C_1e^{2t}+4C_2e^{-3t}+t^2+t \tag{6}$$

SOME REMARKABLE CURVES

503. Strophoid

1. Definition and construction. The *right strophoid*[1] (or simply *strophoid*) is defined as follows: take two mutually perpendicular straight lines AB, CD (Fig. 479) and on one of them take a point A; through this point draw an arbitrary straight line AL intersecting CD at point P. On AL lay off segments PM_1, PM_2 equal to OP (O is the point of intersection of AB and CD). The (right) strophoid is the locus of points M_1, M_2.

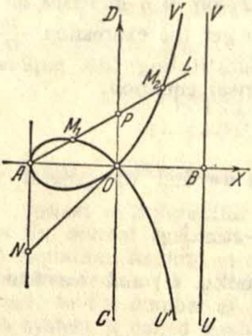

Fig. 479

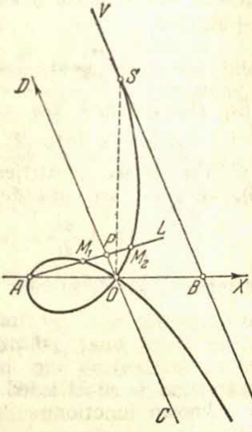

Fig. 480

The *oblique strophoid* (Fig. 480) is constructed in a similar manner but AB and CD intersect at an angle.

The strophoid was probably first considered by Roberval[2] in 1645 under the name *pteroid*.[3] The present name was introduced by Midi in 1849.

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[1] From the Greek word meaning "to turn, twist".

[2] Roberval is the pseudonym of the French scholar G. Persone (1602-1675) from the village of Roberval. He is one of the founders of the method of infinitesimals, invented scales named after him.

[3] From the Greek word meaning "wing".

2. **Stereometric construction.** Imagine a cylindrical surface with axis *CD* (Fig. 479) and radius *AO*. Through point *A* draw an arbitrary plane *K* perpendicular to the plane of the drawing (the straight line *AL* is the trace of this plane). At the intersection we get an ellipse, the foci M_1, M_2 of which describe a right strophoid.

The oblique strophoid is built in similar fashion; but here the cylindrical surface is replaced by a conic surface, the axis of the cone (*OS* in Fig. 480) passes through *O* perpendicular to *AB*; the straight line *UV* passing through *B* parallel to *CD* is one of the generatrices. The points M_1, M_2 are foci of the corresponding conic section; the oblique strophoid lies on both sheets of the conical surface and passes through the vertex *S* of that surface.

3. The **equation in Cartesian coordinates** (with origin at *O*, *x*-axis directed along the ray *OB*; $AO=a$, $\angle AOD=\alpha$; when the strophoid is oblique, the coordinate system is oblique, the *y*-axis is directed along the ray *OD*):

$$y^2(x-a)-2x^2y\cos\alpha+x^2(a+x)=0 \qquad (1)$$

For the right strophoid, Eq (1) reduces to the form

$$y=\pm x\sqrt{\frac{a+x}{a-x}} \qquad (2)$$

The equation **in polar coordinates** (with pole *O* and polar axis *OX*):

$$\rho=-\frac{a\cos 2\varphi}{\cos\varphi}$$

Rational **parametric** representation ($u=\tan\varphi$):

$$x=a\left(\frac{u^2-1}{u^2+1}\right), \quad y=au\left(\frac{u^2-1}{u^2+1}\right)$$

4. **Peculiarities of shape.** Point *O* is the node;[1] the tangents to the two branches passing through *O* are mutually perpendicular (both for the right and the oblique strophoid). For the oblique strophoid (Fig. 480) the straight line *UV* serves as an asymptote (in the case of infinite recession downwards). Besides, *UV* is tangent to the oblique strophoid at the point *S*, which is equidistant from *A* and *B*.

In the right strophoid, the point of tangency *S* goes off to infinity (recession upwards) so that the straight line *UV* (Fig. 479) serves as asymptote to both branches.

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[1] The *node* of a curve is a point through which the curve passes two or more times in different directions.

5. The **radius of curvature** at the node of the right strophoid is

$$R_0 = a\sqrt{2} = ON \quad \text{(Fig. 479)}$$

6. **Areas and volumes** of a right strophoid. The area S_1 of loop AOM_1 is

$$S_1 = 2a^2 - \frac{1}{2}\pi a^2$$

The volume V_1 of a solid generated by rotation of the loop about the x-axis is

$$V_1 = \pi a^3\left(2\ln 2 - \frac{4}{3}\right) \approx 0.166a^3$$

The area S_2 between the branches OU', OV' and the asymptote (this area extends to infinity but has a finite magnitude) is

$$S_2 = 2a^2 + \frac{1}{2}\pi a^2$$

The volume of the solid generated by rotation of the figure $U'OV'VU$ about the x-axis is of infinite magnitude.

504. Cissoid of Diocles

1. **Definition and construction.** On the line segment $OA = 2a$ as diameter, construct a circle C (Fig. 481) and draw through A a tangent UV. Through O draw an arbitrary straight line OF intersecting UV at F; this straight line will again intersect the circle C at point E. Lay off segment FM, equal to chord OE, on the straight line OF from point F in the direction of O. The curve described by point M when OF is rotated about O is called the *cissoid of Diocles* after the Greek scholar who lived in the second century B. C. and introduced this curve as a graphical solution of the problem of doubling (duplicating) the cube.[1]

2. **Historical background.** Diocles defined the cissoid by means of a different construction. He drew the diameter BD perpendicular to OA; point M was found at the intersection of the *chord* OE and the straight line GG' parallel to BD drawn from point G symmetric to E about BD. The Diocles curve therefore lay wholely *within* the circle C. It consisted of the arcs OB and OD. If curve BOD is closed by the se-

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[1] In this problem it is required to find the edge of a cube whose volume is twice that of a given cube.

micircle BAD described by point E, we have a figure shaped like an ivy leaf. Whence the name "cissoid" (which in Greek means "ivy-shaped").

In about 1640 Roberval (and, later, Sluze [1]) noticed that the cissoid continues without bound beyond the limits of the circle if the point E also describes the other semicircle BOD; then M lies *on the continuation* of the chord OE. However, the name "cissoid of Sluze" that Huyghens suggested did not stick.

3. **Equation in rectangular coordinates** (with origin at O, axis of abscissas the x-axis):

$$y^2=\frac{x^3}{2a-x}$$

In polar coordinates (with pole O, polar axis OX) we have

$$\rho=\frac{2a\sin^2\varphi}{\cos\varphi}$$

The rational **parametric** representation ($u=\tan\varphi$) is

$$x=\frac{2a}{1+u^2},\qquad y=\frac{2a}{u(1+u^2)}$$

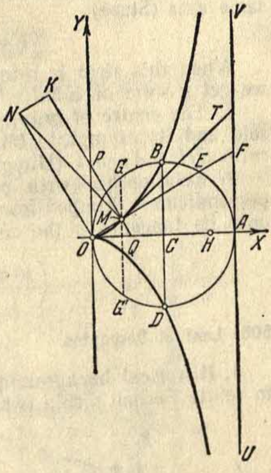

Fig. 481

4. **Peculiarities of shape.** The cissoid is symmetric about OA, passes through points B, D and has asymptote UV ($x=2a$); O is a cusp-point [2] (radius of curvature $R_0=0$).

Construction of tangent. To construct a tangent to the cissoid at its point M, draw $MP\perp OM$. Let P, Q be points of intersection of MP and the straight lines OX, OY. Lay off from point P on the continuation of segment QP a segment $PK=PQ$. Construct $KN \parallel MO$ and $ON \parallel QP$. Join M and the point N of intersection of KN and ON. The straight line MN is the normal to the cissoid. The desired tangent MT is perpendicular to MN.

5. The **area** S of the strip between the cissoid and its asymptote (this strip extends to infinity) is finite; it is three

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[1] René F. W. de Sluze (1622-1685), Dutch scholar, follower of Descartes.

[2] For a definition of cusp-point see Sec. 507, Item 4.

times the area of the generating circle C:

$$S=3\pi a^2$$

6. The **volume** V of the solid of revolution of the above-indicated strip about the asymptote UV is equal to the volume V' of the solid of revolution of the circle C about the same axis (Sluze):

$$V=V'=2\pi^2a^3$$

When this strip is rotated about the axis of symmetry we get a solid of infinite volume.

7. The **centre of gravity** H of the strip between the cissoid and its asymptote UV divides the segment OA in the ratio $OH:HA=5:1$ (Huyghens).

8. **Relationship with parabola.** The locus of the feet of perpendiculars dropped from the vertex of a parabola ($y^2=2px$) onto its tangents is the cissoid

$$\left(y^2=-\frac{x^3}{\frac{p}{2}-x}\right)$$

505. Leaf of Descartes

1. **Historical background.** In 1638 Descartes, in an attempt to refute Fermat's rule (which he misunderstood) for finding tangents, suggested that Fermat find the tangent to the line $x^3+y^3=nxy$. In our ordinary interpretation of negative coordinates, this curve, which in the 18th century was given the name of the folium of Descartes (leaf of Descartes), consists of a loop $OBAC$ (Fig. 482) and two infinite branches (OI, OL). But it was Huyghens who first (1692) represented it in this form. The curve $x^3+y^3=nxy$ had up till then been depicted in the form of four petals (one of these being $OBAC$) symmetrically arranged in the four quadrants. It was thus given the name "flower of jasmine".

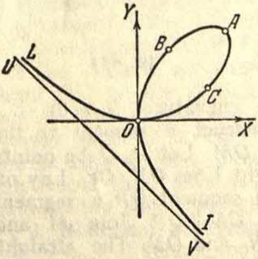

Fig. 482

2. **The equation of the leaf of Descartes** is usually written as

$$x^3+y^3=3axy \tag{1}$$

The coefficient $3a$ expresses the diagonal of the square whose side is equal to the largest chord OA of the loop so that

$$OA=\frac{3a}{\sqrt{2}} \tag{2}$$

In polar coordinates (with pole O and polar axis OX) the equation is

$$\rho=\frac{3a\cos\varphi\sin\varphi}{\cos^3\varphi+\sin^3\varphi} \tag{3}$$

The rational **parametric** representation ($u=\tan\varphi$) is

$$x=\frac{3au}{1+u^3}, \quad y=\frac{3au^2}{1+u^3} \tag{4}$$

Peculiarities of shape. Point O is a node. The tangent lines passing through O coincide with the coordinate axes.

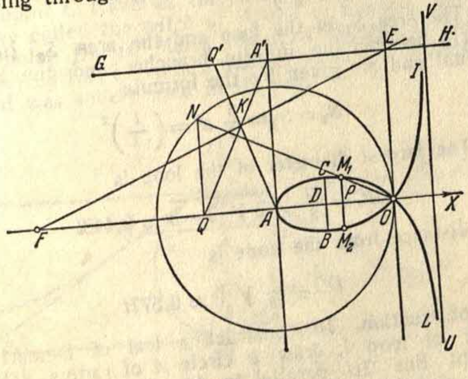

Fig 483

The straight line OA ($y=x$) is the axis of symmetry. The point $A\left(\frac{3a}{2}, \frac{3a}{2}\right)$ farthest from the node is called the *vertex*. The straight line UV ($x+y+a=0$) is the asymptote of both infinite branches.

3. **Equation with respect to the axis of symmetry.** If the axis of symmetry OA is taken as the x-axis and if we direct the x-axis from the node O (origin) to the asymptote UV (Fig. 483), then the leaf of Descartes will be given by the

equation

$$y = \pm x \sqrt{\frac{l+x}{l-3x}} \tag{5}$$

where $l = \frac{3a}{\sqrt{2}} = OA$.

The corresponding equation in polar coordinates is

$$\rho = \frac{l(\sin^2\varphi - \cos^2\varphi)}{3\sin^3\varphi + \cos^3\varphi}$$

The rational **parametric** representation ($u = \tan\varphi$) is

$$x = l\frac{u^2-1}{3u^2+1}, \quad y = l\frac{u(u^2-1)}{3u^2+1}$$

4. **Radius of curvature:** at the vertex, $R_A = \frac{3a}{8\sqrt{2}} = \frac{l}{8}$; at the node, $R_0 = \frac{3a}{2} = \frac{l}{\sqrt{2}}$.

5. The **area** S_1 of the loop and the area S_2 (infinite) of the strip between the infinite branches and the asymptote are equal and are given by the formula

$$S_1 = S_2 = \frac{3}{2}a^2 = \left(\frac{l}{3}\right)^2$$

6. **The largest diameter** of the loop is

$$BC = \frac{2l}{3}\sqrt{2\sqrt{3}-3} \approx 0.448l$$

Its distance from the node is

$$DO = \frac{l}{3}\sqrt{3} \approx 0.577l$$

7. **Construction.** To construct a leaf of Descartes with diameter of loop l, draw a circle A of radius $AO = l$ and a straight line GH parallel to AO. Then draw straight lines AA' and OE perpendicular to AO and mark the points A', E of their intersection with GH. Finally, lay off on ray OA the segment $OF = 3OA$ and draw the straight line FE. The required line is now constructed from points as follows.

Through O draw any straight line ON and through point N where this line meets (a second time) the circle draw $NQ \parallel AA'$. Join point Q, where NQ intersects the straight line OF, to A' and mark the point K where QA' intersects FE. Draw the straight line AK to intersection with the straight line GH at the point Q'. Finally, lay off on the straight line OA the line segment OP, equal to segment $A'Q'$

and in the same direction. The straight line M_1M_2 drawn through P parallel to AA' will intersect the straight line ON at point M_1. This point (and also point M_2 which is symmetric to it about AO) belongs to the sought-for line.

When point N emanating from O traverses circle A counterclockwise, point M_1 describes the trajectory $LOCABOI$.

506. Versiera

1. **Definition.** Let there be constructed (see Fig. 484) on a segment $OA=a$ (as diameter) a circle and let the half-chord BC be continued to point M defined by the proportion

$$BM:BC=OA:OB$$

When point C traverses the circle OC_1C_2, point M describes a curve called the *witch of Agnesi* or *versiera*, after the Italian scholar Donna Maria Gaetana Agnesi (1718-1799) who considered this curve in a manual on higher mathematics (1748) that was widely used in its day.

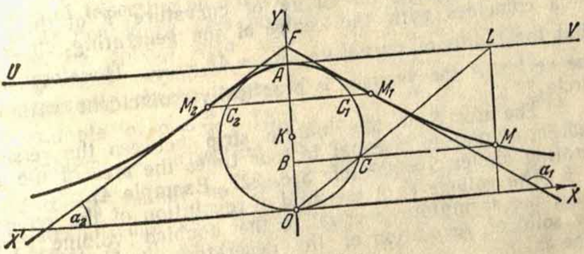

Fig. 484

2. **Construction.** Agnesi suggested the following simple construction of the versiera. Let L be the point of intersection of the straight line OC and the straight line UV tangent to the given circle at the point A (*vertex* of the versiera). Draw straight lines $LM \parallel AO$ and $CB \parallel AL$. The point M of intersection of LM and CB lies on the versiera. In doing the construction, it is well to bear in mind the peculiarities of the shape of the versiera (see below).

3. The **equation** (with origin at O; the tangent line $X'X$ to the generating circle at point O is the x-axis) is

$$y=\frac{a^3}{a^2+x^2}$$

($a=OA$ is the diameter of the generating circle).

4. **Peculiarities of shape.** The diameter OA is the axis of symmetry of the versiera, which lies entirely on one side of the straight line $X'X$. This line is asymptotic to the versiera. The versiera has two points of inflection: $M_1\left(\frac{a}{\sqrt{3}}, \frac{3a}{4}\right)$, $M_2\left(-\frac{a}{\sqrt{3}}, \frac{3a}{4}\right)$. They are constructed as indicated above if point C is brought to coincidence with one of the points C_1, C_2 $\left(\breve{AC}_1=\breve{AC}_2=\frac{\pi}{3}\right)$. The angles α_1, α_2 made by the tangents M_1F, M_2F with the axis $X'X$ are found from the formula $\tan\alpha_{1,2}=\mp\frac{3\sqrt{3}}{8}$ To construct the tangent lines M_1F, M_2F, it is sufficient to lay off the segment $AF=\frac{a}{8}$ on the continuation of the diameter OA.

At the vertex A, the centre of curvature K of the versiera coincides with the centre of the generating circle so that the radius of curvature $R_A=AK=\frac{a}{2}$. Therefore near the vertex A the versiera is practically coincident with the circle.

5. The **area** S of the infinite strip between the versiera and its asymptote is equal to four times the area of the generating circle: $S=\pi a^2$ (cf. Sec. 327, Example 4).

6. The **volume** V of the solid of revolution of the versiera about the asymptote is equal to the doubled volume V_1 of the solid of revolution of the generating circle about the same axis:

$$V=\frac{\pi^2a^3}{2}, \quad V_1=\frac{\pi^2a^3}{4}$$

The solid of revolution of the versiera about the axis of symmetry has infinite volume.

7. **Historical background.** The curve given by the equation $y=\frac{a^3}{a^2+x^2}$ is first encountered in Fermat's works. In the thirties of the 17th century he found the area bounded by an arc of this curve, by two ordinates and the axis of abscissas (at that time the problem was of considerable dif-

ficulty since methods of integration were just being developed). A construction of the versiera and a number of its properties were given in 1718 by the Italian scholar G. Grandi, who gave it the name "versiera". In Italian this word means "witch", and Grandi, undeterred by the ambiguity of the word, generated the term *sinus versus* (versed sine): in Grandi's day, the segment BC was called the sine of the arc OC, and the segment BA was called the versed sine. The curious name of witch of Agnesi which is still encountered in mathematical works apparently has no historical justification.

507. Conchoid of Nicomedes

1. **Historical background.** Nicomedes, an ancient Greek scholar who lived between 250 and 150 B. C., gave the name conchoid to the curve (PAQ in Fig. 485) because of its similarity to the shape of a mussel shell. He introduced this curve as a graphical solution to the problem of trisecting an angle α.

As we now know, this problem can be solved with straightedge and compass only for a specially chosen angle α $\left(\text{for example for } \alpha=\frac{\pi}{2}\right)$. Thus, the problem of trisecting an angle $\alpha=\frac{\pi}{3}$ cannot be solved with straightedge and compass alone (that is, if we construct only straight lines and circles). However, the problem can be solved if we invoke other curves, for instance the conchoid. To construct it, Nicomedes built a special instrument called a *conchoidograph*.

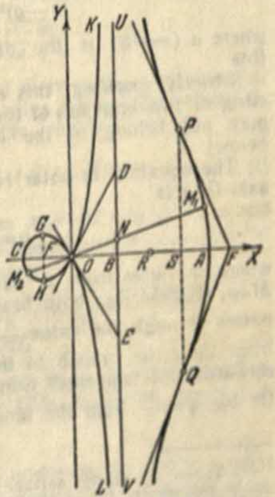

Fig. 485

2. **Definition and construction.** Given: point O *(pole)*, a straight line UV *(base line)* and a segment l. From the pole O (Fig. 485) draw an arbitrary straight line ON which intersects the base line at point N. On ON lay off, on

either side of N, the line segments $NM_1=NM_2=l$. The locus of the points M_1, M_2 is now called the *conchoid of Nicomedes*. The curve described by the point lying on the continuation of the segment ON beyond point N (point M_1 in Fig. 485) is called the *outer branch* of the conchoid; the curve described by the other point (M_2 in Fig. 485) is called the *inner branch*.

Note. Nicomedes himself (and later mathematicians up to the end of the 17th century) gave the name conchoid to the curve which is now termed the outer branch. The inner branch was regarded as a special curve and was called the second, third or fourth conchoid, depending on the peculiarities of its shape (see below).[1]

3. The **equation** (with origin at the pole O, the x-axis directed along the ray OB, and point B as the projection of the pole on the base line) is

$$(x-a)^2(x^2+y^2)=l^2x^2 \tag{1}$$

where a $(=OB)$ is the distance from the pole to the base line.

Strictly speaking, this equation represents a figure consisting of two branches of the conchoid and the pole O, which may not belong to the locus defined above (see Fig 487 below).

The equation **in polar coordinates** (with pole O and polar axis OX) is

$$\rho=\frac{a}{\cos\varphi}+l \tag{2}$$

where φ varies from a value φ_0 to $\varphi_0+2\pi$, and the point $M(\rho, \varphi)$ describes both branches of the conchoid. When φ passes through the value $\frac{\pi}{2}$, the point M makes a jump from the outer branch to the inner branch (goes to infinity upwards and "appears" from below). The transition is similar for $\varphi=\frac{3\pi}{2}$ from the inner branch to the outer branch.[2]

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[1] Taken separately, neither the outer branch nor the inner branch can be represented by an algebraic equation.

[2] In Eq. (2), the radius vector ρ takes on both positive and negative values (see Sec. 73, Note 2). To avoid this, we can use the equation $\rho=\frac{a}{\cos\varphi}\pm l$ in place of (2). However, when $l>a$ (Fig. 485) positive values of ρ on the inner branch are attained because when M_2 passes through the nodal point its polar angle φ jumps by $\pm\pi$. As

Unlike Eq. (1), Eq. (2) represents a figure containing only those points which satisfy the definition of the conchoid.

The **parametric** equations are

$$x=a+l\cos\varphi,\quad y=a\tan\varphi+l\sin\varphi \tag{3}$$

4. **Peculiarities of shape.** The conchoid is symmetric about the straight line OB; this line intersects the conchoid at point O and at two other points: A, C (*vertices*). The base line UV is asymptotic both to the inner and the outer branch. The shape of the conchoid (its inner branch) is essentially dependent on the relationship between the segments $a\,(=OB)$ and $l\,(=BA)$.

(1) When $l:a>1$ (Fig. 485), the inner branch has a loop (OCM_2); the point O is a node.

The slope of the tangent lines OD, OE at the node is

$$\tan\alpha=\pm\frac{\sqrt{l^2-a^2}}{a}$$

To construct tangents at the point O it suffices, at points D and E on UV, to strike arcs of radius l from the centre O.

The greatest diameter GH of the loop is

$$GH=2\,(la^{1/3}-al^{1/3}):(l^{2/3}+a^{2/3})^{1/2} \tag{4}$$

It is associated with the abscissa $x_G=OF=(l^2a)^{1/3}-l$ and the polar angle φ_G, defined by the formula $\cos\varphi_G=-(a:l)^{1/3}$. In Fig. 485, where $l:a=2$, we have

$$GM\approx 1.11a,\quad x_G\approx -0.59a,\quad \cos\varphi_G=-\sqrt[3]{0.5},$$

$$\varphi_G\approx 142°30'$$

(2) When $l:a=1$ the loop of the inner branch shrinks to the pole O and becomes a cusp[1] (Fig. 486); the tangent at this point coincides with OX.

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a result, the range of the angle φ consists of the interval $\left(-\frac{\pi}{2},\ +\frac{\pi}{2}\right)$ and of some portion of the interval $\left(+\frac{\pi}{2},\ +\frac{3\pi}{2}\right)$. Besides, for certain values of φ we have to take both signs $\pm$ in front of l, for other values, only the plus sign.

[1] A *cusp* of a curve is a point on the curve such that the direction of motion along the curve is reversed in the form of a jump.

(3) When $l:a < 1$, the inner branch does not pass through pole O (Fig 487); this point is an isolated point[1] of curve (1).

5. **Points of inflection.** There are two points of inflection P, Q on the outer branch (Figs. 485-487). On the inner branch, there are points of inflection (P', Q' in Fig. 487) only when the pole is an isolated point. The abscissa x_1 of the pair of points P, Q and the abscissa x_2 of the pair of points P', Q' may be found from the equation

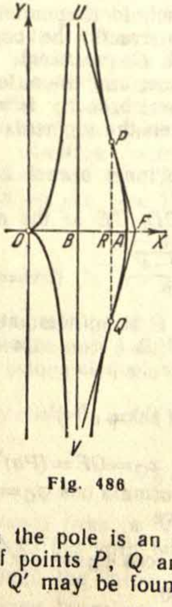

Fig. 486

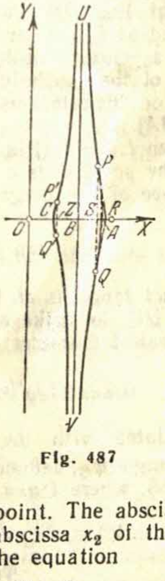

Fig. 487

$$x^3-3a^2x+2a(a^2-l^2)=0 \tag{5}$$

(1) For $l:a>1$ (Fig. 485), Eq. (5) has a unique root x_1, lying between $a\sqrt{3}\,(=OR)$ and $a+l\,(=OA)$ and the closer to $a\sqrt{3}$, the less $l:a$ differs from 1. Thus, for $l:a=2$ (Fig. 485), Eq. (5) is of the form $\left(\frac{x}{a}\right)^3-3\left(\frac{x}{a}\right)-6=0$. The root x_1 lies between $a\sqrt{3}$ and $3a$. Utilizing these

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[1] A point of a locus is called an *isolated point* (with respect to that locus) if it is the centre of a circle within which there are no points of the given locus.

boundaries and applying (twice) the formulas of Sec. 291, we find

$$x_1 \approx 2.35a\,(=OS)$$

(2) For $l:a=1$ (Fig. 486), Eq. (5) assumes the form $x^3-3a^2x=0$. It has three real roots $x_1=a\sqrt{3}$, $x_2=0$, $x_3=-a\sqrt{3}$. The first yields the abscissa OR of the points of inflection P, Q; the second is associated with the cusp O; the third is not associated with any point of the conchoid.

(3) For $l:a<1$ (Fig. 487), Eq. (5) has three real roots, of which the first $x_1\,(=OS)$ lies between $a\,(=OB)$ and $a\sqrt{3}\,(=OR)$; neither does it exceed the line segment $a+l\,(=OA)$. The second root $x_2\,(=OZ)$ lies between $a-l$ $(=OC)$ and a (both roots are the closer to a, the less $l:a$ differs from zero). The third root x_3 is negative. The root x_1 gives the abscissa of the points P, Q; the root x_2, the abscissa of the points P', Q'. The root x_3 is not associated with any point of the conchoid. Thus, for $l:a=0.5$ (Fig. 487) we have the equation

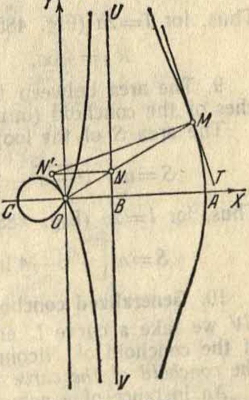

Fig. 488

$$\left(\frac{x}{a}\right)^3-3\left(\frac{x}{a}\right)+1.5=0$$

Between a and $a+l=1.5a$ lies the root $x_1\approx 1.38a\,(=OS)$, which yields the points of inflection P, Q. Between $a-l=0.5a$ and a lies the root $x_2\approx 0.57a\,(=OZ)$; it yields the points P', Q'. The third root $(x_3\approx -1.9a)$ is negative.

6. **Property of the normal.** The normal to a conchoid at a point M (Fig. 488) passes through the point N' of intersection of two straight lines, one of which is a perpendicular to OM drawn through the pole O and the other is a perpendicular to the base line UV drawn through the point N, where UV meets OM.

7. **Construction of a tangent.** In order to build a tangent to a conchoid at a point M, join M and the pole O. Through the point N of intersection of the straight lines OM, UV

draw a straight line $NN' \perp UV$ and through the pole O a straight line $ON' \perp OM$. Join the point N' of intersection of these straight lines to M. The straight line $N'M$ will be the normal to the conchoid. Drawing $MT \perp N'M$, we get the desired tangent line.

8. **Radii of curvature at the points A, C, O:**

$$R_A = \frac{(l+a)^2}{l}, \quad R_C = \frac{(l-a)^2}{l}, \quad R_O = \frac{l\sqrt{l^2-a^2}}{2a}$$

Thus, for $l=2a$ (Fig. 485)

$$R_A = 4.5a, \quad R_C = 0.5a, \quad R_O = a\sqrt{3}$$

9. The **area** between the asymptote and one of the branches of the conchoid (outer or inner) is infinite.

The area S of the loop is

$$S = a\sqrt{l^2-a^2} - 2al \ln \frac{l+\sqrt{l^2-a^2}}{a} + l^2 \arccos \frac{a}{l}$$

Thus, for $l=2a$ (Fig. 485)

$$S = a^2\left[\sqrt{3} - 4\ln(2+\sqrt{3}) + \frac{4}{3}\pi\right] \approx 0.65a^2$$

10. **Generalized conchoids.** If in place of the straight line UV we take a curve L and otherwise retain the definition of the conchoid of Nicomedes, we get a new curve called the *conchoid of the curve L with respect to the pole O*

An instance of a generalized conchoid is the limaçon of Pascal (see Sec. 508).

508. Limaçon. Cardioid

1. **Definition and construction.** Given: point O (*pole*), a circle K of diameter $OB=a$ (Fig. 489) passing through the pole (*base-circle*; it is shown dashed in the drawing), and the line segment l. From the pole O draw an arbitrary straight line OP. From point P, where the straight line OP intersects the circle a second time, lay off, on both sides of P, the line segments $PM_1 = PM_2 = l$. The locus of the points M_1, M_2 (heavy line in Fig. 489) is called the *limaçon of Pascal*, in honour of Etienne Pascal (1588-1651), father of the celebrated French scholar Blaise Pascal (1623-1662).

The term "limaçon (i.e. snail) de monsieur Pascal" (Pascal's limaçon) was suggested by Roberval, a contemporary and friend of Pascal. Roberval regarded this curve as one of the types of generalized conchoid (see Sec. 507).

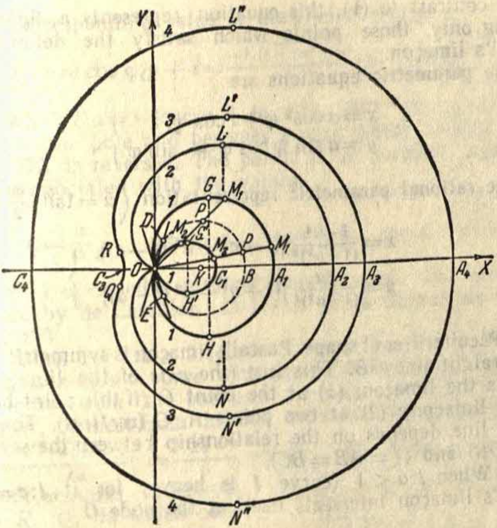

Fig. 489

2. The **equation** (with origin at the pole O, x-axis directed along the ray OB) is

$$(x^2+y^2-ax)^2=l^2(x^2+y^2) \tag{1}$$

Strictly speaking, this equation is a figure consisting of Pascal's limaçon and of the pole O, which may not belong to the above-defined locus (such is the case for curves *3* and *4* in Fig. 489).

The equation **in polar coordinates** (with O as pole and OX as polar axis) is

$$\rho=a\cos\varphi+l \tag{2}$$

where φ varies from a value φ_0 to $\varphi_0+2\pi$.[1]

---

[1] For $l<a$ (heavy line in Fig. 489), the radius vector ρ can assume both positive and negative values. To avoid this, we can take advantage of the equation $\rho=l\pm a\cos\varphi$. However, this entails certain inconveniences similar to those pointed out in the footnote relating to Item 3 of Sec. 507.

In contrast to (1), this equation represents a figure containing only those points which satisfy the definition of Pascal's limaçon.

The **parametric** equations are

$$\left.\begin{aligned} x&=a\cos^2\varphi+l\cos\varphi,\\ y&=a\sin\varphi\cos\varphi+\ \sin\varphi \end{aligned}\right\} \qquad (3)$$

The **rational parametric** representation $\left(u=\tan\frac{\varphi}{2}\right)$ is

$$\left.\begin{aligned} x&=\frac{1-u^2}{(1+u^2)^2}[(l+a)+u^2(l-a)],\\ y&=\frac{2u}{(1+u^2)^2}[(l+a)+u^2(l-a)] \end{aligned}\right\} \qquad (4)$$

3. **Peculiarities of shape.** Pascal's limaçon is symmetric about the straight line OB. This line (the *axis* of the limaçon) intersects the limaçon: (1) at the point O (if this point belongs to the limaçon); (2) at two points A, C (*vertices*). The shape of the line depends on the relationship between the segments a $(=OB)$ and l $(=AB=BC)$.

(1) When $l:a<1$ (curve I is heavy; for it $l:a=1:3$), Pascal's limaçon intersects itself at the node O

$$\left(\rho_{1,2}=0,\quad \cos\varphi_{1,2}=-\frac{l}{a},\quad \sin\varphi_{1,2}=\pm\frac{\sqrt{a^2-l^2}}{a}\right)$$

forming two loops: an outer loop OHA_1GO and an inner loop $OH'C_1G'O$. The slope of the tangent lines OD, OE at the nodal point is

$$\tan\alpha=\pm\frac{\sqrt{a^2-l^2}}{l}\left(=\pm\frac{2}{3}\sqrt{2}\right)$$

To construct tangents it suffices to draw chords OD, OE of length l in the circle K. The points G, H of the outer loop most distant from the axis are associated with the value

$$\cos\varphi=\frac{\sqrt{l^2+8a^2}-l}{4a}\ (\approx 0.62)$$

To the most distant points G', H' of the inner loop there corresponds the value

$$\cos\varphi=\frac{-\sqrt{l^2+8a^2}-l}{4a}\ (\approx 0.80)^{1)}$$

---

1) Thus, the polar angle of point G' is the angle between OX and the ray *opposite* the ray OG', and not the angle XOG' (see Sec. 73, Note 2).

The corresponding value of the radius vector is

$$\rho_{G'} = a\cos\varphi_{G'} + l = \frac{-\sqrt{l^2+8a^2}+3l}{4} \ (\approx -0.45a)$$

(2) When $l:a=1$ (curve *2* in Fig. 489), the inner loop shrinks to the pole and becomes a cusp where motion along the ray OX is reversed. The points L, N farthest from the axis are associated with the values

$$\varphi=\frac{\pi}{3},\qquad \rho=\frac{3}{2}a,\quad x=\frac{3}{4}a,\quad y=\pm\frac{3\sqrt{3}}{4}a$$

Curve 2 is called a *cardioid* (heart-shaped, the term was first used by de Castillon in 1741). It is shown separately in Fig. 490.

(3) When $1 < l:a < 2$ (curve *3*; for it $l:a=4:3$), Pascal's limaçon is a closed curve without self-intersection; it tears itself away from the pole and encloses the latter. Points L' and N', farthest from the axis, are associated with the value $\cos\varphi=\frac{\sqrt{l^2+8a^2}-l}{4a}\left(=\frac{\sqrt{22}-2}{6}\approx 0.45\right)$. The limaçon loses the cusp and acquires inflection points R, Q, to which corresponds the value $\cos\varphi_R = -\frac{2a^2+l^2}{3al}$. The angle ROQ $(=2\pi-2\varphi_R)$, at which the segment RQ is seen from the pole, at first increases from zero to 2 arccos $\frac{2\sqrt{2}}{3}$ ($\approx 39°40'$) as $l:a$ increases. To this value there corresponds $l:a=\sqrt{2}$. As $l:a$ increases further, the angle ROQ decreases and tends to zero as $l:a \longrightarrow 2$.

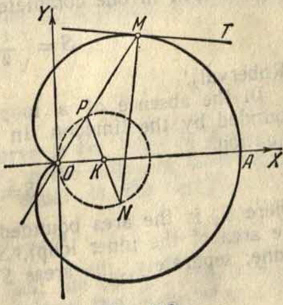

Fig. 490

(4) For $l:a=2$, the inflection points vanish, merging with the vertex C (the curvature at C becomes zero). The limaçon becomes oval in shape and retains that shape for all values of $l:a > 2$ (curve *4*; for it $l:a=7:3$). The points L'', N'', which are farthest from the axis, are associated with the value

$$\cos\varphi=\frac{\sqrt{l^2+8a^2}-l}{4a}\left(=\frac{1}{3}\right)$$

4. **Property of the normal.** The normal to Pascal's limaçon at a point M (Fig. 490) passes through the point N of the base-circle K, which is diametrically opposite to the point P, where OM meets the base-circle.

5. **Construction of a tangent.** To construct a tangent line to Pascal's limaçon at a point M, join M and pole O. Join N of the base-circle K, which is diametrically opposite to point P, to M. The straight line MN will be the normal to the limaçon. Drawing $MT \perp MN$, we get the desired tangent line.

6. The **radius of curvature** at points A, C, O is

$$R_A=\frac{(l+a)^2}{l+2a}, \quad R_C=\frac{(l-a)^2}{|l-2a|}, \quad R_O=\frac{1}{2}\sqrt{a^2-l^2}$$

The last expression assumes that $l \leqslant a$ (for $l > a$, the point O is isolated from the limaçon). In particular, for the cardioid ($l=a$, points O and C coincide) we have

$$R_A=\frac{4}{3}a, \quad R_C=R_O=0$$

7. **Areas.** The area S described by the radius vector of the limaçon in one complete rotation is

$$S=\left(\frac{1}{2}a^2+l^2\right)\pi \tag{5}$$

(Roberval).

In the absence of a loop ($l \geqslant a$), S expresses the area bounded by the limaçon. In the case of a loop, we have the equation

$$S=S_1+S_2$$

where S_1 is the area bounded by the outer loop (including the area of the inner loop), S_2 is the area of the inner loop alone; separately, the areas S_1, S_2 are expressed as

$$S_1=\left(\frac{1}{2}a^2+l^2\right)\varphi_1+\frac{3}{2}l\sqrt{a^2-l^2} \tag{5a}$$

where $\varphi_1=\arccos\left(-\frac{l}{a}\right)$;

$$S_2=\left(\frac{1}{2}a^2+l^2\right)\varphi_2-\frac{3}{2}l\sqrt{a^2-l^2} \tag{5b}$$

where $\varphi_2=\arccos\frac{l}{a}$.

For the cardioid

$$S\,(=S_1)=\frac{3}{2}\,\pi a^2$$

That is, the *area of the cardioid is equal to six times the area of the base-circle.*

8. In the general case, the **arc length** of Pascal's limaçon is not expressible in terms of elementary functions. For the cardioid, the arc length s reckoned from vertex A ($\varphi=0$) is

$$s=4a\sin\frac{\varphi}{2}$$

The length of the entire cardioid is $8a$, i. e. it is equal to eight times the diameter of the base-circle.

9. **Relationship with circle.** The locus of the feet of perpendiculars dropped, from some point O, onto tangents to a circle of radius r centred at B is Pascal's limaçon. If point O lies in the plane of the circle B, then O is the pole of the limaçon, the base-circle is constructed on the segment $OB=a$ as diameter; the constant line-segment l, which is laid off on the polar ray, is equal to the radius r of circle B.

When point O lies on circle B, Pascal's limaçon is a cardioid.

509. Cassinian Curves

1. **Definition.** A Cassinian curve is the locus of points M for which the product $MF_1 \cdot MF_2$ of the distances to the ends of a given segment $F_1F_2=2c$ is equal to the square of the given segment a:

$$MF_1\cdot MF_2=a^2$$

The points F_1, F_2 are called *foci*; the straight line F_1F_2 is called the *axis* of the Cassinian curve; the midpoint O of the segment F_1F_2 is the *centre.*

2. **Historical background.** The celebrated astronomer Giovanni Domenico (Jean Dominique) Cassini (1625-1712) believed that the curve bearing his name was capable of representing the earth's orbit better than an ellipse. This became known in 1749 from a publication of Cassini, Junior (also a notable astronomer). Although the Cassini hypothesis was not vindicated, the curve he discovered became the subject of numerous investigations. It is often called the *oval of*

Cassini, though actually it is not always oval in shape[1] (see below).

3. **Construction.** On $F_1F_2=2c$ as a diameter (Fig. 491) construct a circle O. On its tangent F_1K take a segment $F_1K=a$. Laying off from point O segments OA_1 and OA_2, equal to OK, on the axis F_1F_2, we get the points A_1, A_2 of the Cassinian curve which are farthest from the centre $(OA_1=OA_2=\sqrt{c^2+a^2})$.

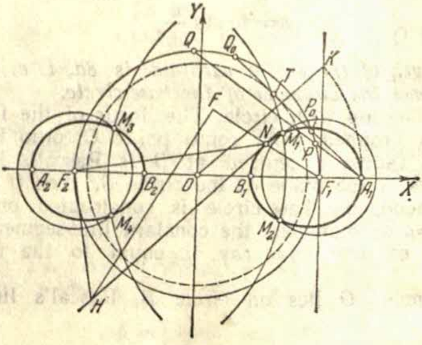

Fig. 491

If $a < c$, as in Fig. 491, then we additionally construct a circle of radius a with centre at O (shown dashed in Fig. 491) and draw from A_1 to it a tangent line A_1T. At the intersection with the base-circle $O(c)$ we get points P_0, Q_0. From one of the foci, say F_1, lay off, in the direction of O, line-segments $F_1B_1=A_1P_0$ and $F_1B_2=A_1Q_0$. We get points B_1, B_2 which are least distant from the centre $(OB_1=OB_2=\sqrt{c^2-a^2})$.

But if $a \geqslant c$, the least distant points C_1, C_2 (Fig. 492) lie on the axis of symmetry OY of segment F_1F_2 at a distance of $F_1C_1=F_2C_1=a$ from the foci F_1, F_2 $(OC_1=OC_2=\sqrt{a^2-c^2})$.

The points A_1, A_2 and B_1, B_2 (or C_1, C_2) are called the *vertices* of the Cassinian curve.

Through the point A_1 (or A_2) draw (Fig. 491) an arbitrary secant line A_1PQ of the base-circle $O(c)$; in the case of $a < c$, we confine ourselves to those secants which also intersect the supplementary circle $O(a)$. From focus F_1 as centre describe a circle of radius $r = A_1P$, and from F_2 a circle of radius $r' = A_1Q$. Their points M_1, M_2 of intersection belong to the Cassinian curve. Interchanging points F_1 and F_2, we get one more pair of points M_3, M_4. The desired curve is the locus of the points M_1, M_2, M_3, M_4.

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[1] An *oval* is a plane closed curve with the property that a straight line cannot have more than two common points with it. An oval curve cannot have points of inflection, cuspidal points, or nodal points.

4. The **equation** (with O as origin and F_2F_1 as axis of abscissas) is

$$(x^2+y^2)^2-2c^2(x^2-y^2)=a^4-c^4 \tag{1}$$

The equation **in polar coordinates** (with O as pole and OX as polar axis) is

$$\rho^4-2c^2\rho^2\cos 2\varphi+c^4-a^4=0 \tag{2}$$

or

$$\rho^2=c^2\cos 2\varphi \pm \sqrt{a^4-c^4\sin^2 2\varphi} \tag{3}$$

The double sign is taken when $a < c$, otherwise we take the plus sign only (or ρ would be imaginary).

5. **Peculiarities of shape.** The Cassinian curve is symmetric with respect to the straight lines OX and OY and, hence, about the point O.

For $a < c$, the Cassinian curve consists of a pair of separated ovals. (In Fig. 492, the pair of ovals L_1, L_1' correspond to the value $a=0.8c$; the pair L_2, L_2', to the value $a=0.9\,c$.) For $a > c$, this is a closed curve (for $a=1.1c$ the curve L_4, for $a=c\sqrt{2}$ the curve L_5, for $a=c\sqrt{3}$ the curve L_6). In the boundary case, $a=c$, the Cassinian curve is the *lemniscate* L_3 (cf. definition of the lemniscate). When increasing a tends to c, the vertices A_1, A_2 tend to coincidence with the vertices N_1, N_2 of the lemniscate, and the vertices B_1, B_2 with the nodal point O; the right oval converts to the right loop of the lemniscate, and the left one to the left loop of the lemniscate.

The segment a increases; when it exceeds c, but is less than $c\sqrt{2}$ $(c < a < c\sqrt{2})$, the Cassinian curve (L_4 in Fig. 492) acquires four symmetrical points of inflection D_1, D_2, D_3, D_4; though closed, it is not an oval.[1] The curvature at the vertices C_1, C_2 is infinitely large for infinitesimal $a-c$. But when a increases and tends to $c\sqrt{2}$, the curvature at the points C_1, C_2 tends to zero.

---

[1] Some straight lines, like, for example, D_1D_4, intersect the Cassinian curve at four points.

The boundary Cassinian curve corresponding to the relation $a=c\sqrt{2}$ (L_5 in Fig. 492), and all the other curves ($a>c\sqrt{2}$) are ovals. But the boundary oval has zero curvature at the vertices E_1, E_2 (at these points, the points of inflection of the curve L_4 are pairwise coincident, while at the points of inflection the curvature is always zero).

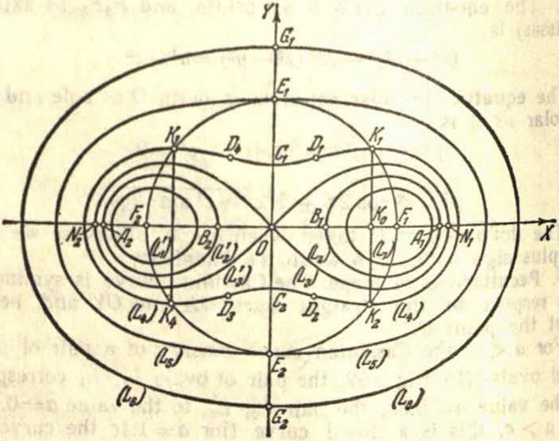

Fig. 492

6. Greatest diameter. For $a\geqslant c\sqrt{2}$, that is, for all ovals exterior to the boundary oval L_5, the greatest diameter $G_1G_2=2\sqrt{a^2-c^2}$ lies on the y-axis. Now any Cassinian curve lying inside the boundary oval (both interior and exterior to the lemniscate) has two greatest diameters $K_1K_2=K_3K_4=\frac{a^2}{2c}$. They are symmetric about OY and are distant

$$OK_0=\frac{\sqrt{4c^4-a^4}}{2c}$$

from the centre O. Their extremities K_1, K_2, K_3, K_4 lie on the base-circle O, which is the locus of those points at which tangents to the Cassinian curves are parallel to the x-axis. Each such tangent is "double", i. e. it touches the Cassinian curve at two points K_1, K_3 symmetric about OY.

7. **Radius of curvature:**

$$R=\frac{2a^2\rho^3}{c^4-a^4+3\rho^4}=\frac{a^2\rho}{\rho^2+c^2\cos 2\varphi} \tag{4}$$

In particular, at the vertices $A\left(\rho=\sqrt{c^2+a^2},\ \varphi=0\right)$, $B\left(\rho=\sqrt{c^2-a^2},\ \varphi=0\right)$, $C\left(\rho=\sqrt{a^2-c^2},\ \varphi=\frac{\pi}{2}\right)$:

$$R_A=\frac{a^2\sqrt{c^2+a^2}}{2c^2+a^2},\quad R_B=\frac{a^2\sqrt{c^2-a^2}}{2c^2-a^2},\quad R_C=\frac{a^2\sqrt{a^2-c^2}}{|a^2-2c^2|}$$

8. **Points of inflection.** The polar coordinates of the inflection points D_1, D_2, D_3, D_4 are defined by the formulas

$$\rho_D=\sqrt[4]{\frac{a^4-c^4}{3}},\quad \cos 2\varphi_D=-\sqrt{\frac{1}{3}\left(\frac{a^4}{c^4}-1\right)} \tag{5}$$

The locus of the points of inflection is a lemniscate with vertices E_1, E_2 (not indicated in the figure).

9. **Construction of a tangent.** In order to construct a tangent line to a Cassinian curve at a point N (Fig. 491), extend the segment F_1N beyond point N to a distance $NF=NF_1$. Through points F and F_2 draw straight lines FH and F_2H, perpendicular respectively to F_1N and F_2N. Then join their intersection point H to N. The straight line NH is the desired tangent.

If the straight lines FH, F_2H intersect at an inaccessible point, then the segments NF, NF_2 may be decreased proportionally.

510. Lemniscate of Bernoulli

1. **Historical background.** In 1694, James Bernoulli, 1) in a paper devoted to the theory of tides, utilized as an auxiliary device a curve which he specified by the equation $x^2+y^2=a\sqrt{x^2-y^2}$. He noted the similarity of this curve (Fig. 493) and the figure eight or the bow of a ribbon to which he gave the name *lemniscus* (a pendent ribbon), whence the name *lemniscate*. The lemniscate became popular in 1718 when the Italian mathematician G.C.Fagnano (1682-1766) established that the integral which represents the arc length of the lemniscate

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1) James Bernoulli (1654-1705), famous Swiss mathematician, pupil and associate of Leibniz in elaborating the infinitesimal calculus and its applications. The founder of the theory of probability in which he formulated and proved the theorem bearing his name ("law of large numbers").

is not expressible in terms of elementary functions, and nevertheless the lemniscate may be divided (by straightedge and compass) into n equal arcs, provided that $n=2^m$ or $3 \cdot 2^m$ or $5 \cdot 2^m$, where m is any positive integer.

The lemniscate is a particular case of the Cassinian curve (Sec. 509, Item 6). However, though the Cassinian curves were widely known from 1749 onwards, it was only in 1806 that the Italian mathematician Saladini established the identity of "Cassini's figure-of-eight" and Bernoulli's lemniscate.

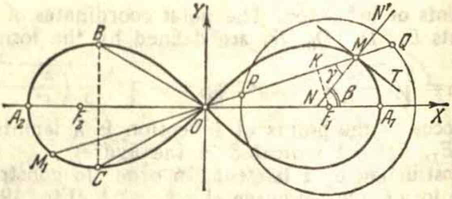

Fig. 493

2. **Definition.** The lemniscate is the locus of points for which the product of the distances to the ends of a given segment $F_1F_2=2c$ is equal to c^2. The points F_1, F_2 are called the *foci* of the lemniscate; the straight line F_1F_2 is called its axis.

3. The **equation** (with origin O at the midpoint of segment F_1F_2 and axis OX directed along F_2F_1) is

$$(x^2+y^2)^2=2c^2(x^2-y^2) \tag{1}$$

The **polar** equation (with O as pole and OX as the polar axis) is

$$\rho^2=2c^2\cos 2\varphi \tag{2}$$

The angle φ varies in the intervals $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$.

Rational parametric representation:

$$x=c\sqrt{2}\,\frac{u+u^3}{1+u^4}, \quad y=c\sqrt{2}\,\frac{u-u^3}{1+u^4}\,(-\infty<u<+\infty) \tag{3}$$

where the parameter u is connected with φ by the relation $u^2=\tan\left(\frac{\pi}{4}-\varphi\right)$.

4. **Construction.** We can use the general method of constructing Cassinian curves, but the method given below (Maclaurin's) is simpler and better. Construct (Fig. 493) a circle of radius $\frac{c}{\sqrt{2}}$ with centre at point F_1 (or F_2). Draw an arbitrary secant line OPQ and lay off on it, on either side of O, segments OM and OM_1 equal to the chord PQ. Point M will describe one of the loops of the lemniscate, point M_1 the other.

5. **Peculiarities of shape.** The lemniscate has two axes of symmetry: the straight line $F_1F_2\,(OX)$ and the straight line $OY \perp OX$. Point O is the node; both branches have an inflection here. The tangents at this point form with the x-axis angles $\pm\frac{\pi}{4}$. Points A_1, A_2 of the lemniscate which are farthest from the node O (*vertices* of the lemniscate) lie on the axis F_1F_2 at a distance $c\sqrt{2}$ from the node.

6. **Property of the normal.** The radius vector OM of the lemniscate forms with the normal MN an angle $\gamma(\angle OMN=\gamma)$, which is twice the polar angle $\varphi\,(=\angle XOM)$:

$$\gamma=\angle OMN=2\varphi$$

In other words, angle $\angle XNM=\beta$ between the x-axis and the vector NN' of the outer normal of the lemniscate at point M is equal to three times the polar angle of point M:

$$\beta=3\varphi$$

7. **Constructing a tangent.** To construct a tangent line to the lemniscate at point M, draw a radius vector OM and build $\angle OMN=2\angle XOM$. The perpendicular MT to the straight line MN is the desired tangent line.

8. The **greatest diameter** $BC=\frac{1}{2}F_1F_2=c$ (Fig. 493) serves as the base of an equilateral triangle with vertex O.

9. The **radius of curvature** is

$$R=\frac{2c^2}{3\rho}$$

10. The **area** S **of a polar sector** A_1OM is

$$S(\varphi)=\frac{c^2}{2}\sin 2\varphi=OK\cdot F_1K$$

(K is the projection of focus F_1 on the vector radius OM).

In other words, a perpendicular F_1K dropped from a focus of the lemniscate on an arbitrary vector radius OM bisects the area of the sector A_1OM.

The **area of each loop** of the lemniscate is $2S\left(\frac{\pi}{4}\right)=c^2$.

11. **Relationship with hyperbola.** The locus of the feet of perpendiculars dropped from the centre O of an equilateral hyperbola with vertices A_1, A_2 on its tangents is a lemniscate with those vertices.

511. Spiral of Archimedes [1]

1. **Construction.** To construct an Archimedean spiral with given parameter k, draw from centre O (Fig. 494) an arbitrary circle, say a circle of radius $ON=k$. [2]

Divide it by the points b_0, b_1, b_2, b_3, ... [3] into an arbitrary number n of equal arcs (we took $n=12$). On ray Ob_0 lay off segment $OA_1=2\pi k$ (the lead of the spiral). Divide it into the same number of equal parts. On rays Ob_1, Ob_2, Ob_3, ... lay off segments $OD_1=\frac{1}{n}OA_1$; $OD_2=\frac{2}{n}OA_1$, ... We get points D_1, D_2, D_3, ... of the first revolution of the spiral. Points E_1, E_2, E_3, ... of the second revolution are obtained by laying off, on extensions of the segments OD_1, OD_2, OD_3, ... , the segments D_1E_1, D_2E_2, ... equal to the lead OA_1. We get the points of subsequent circuits in similar fashion.

2. **Peculiarities of shape.** Any ray OQ with origin at the pole O has, besides O, an infinity of points Q_1, Q_2, ... which are common to the spiral. Two successive points Q_i, Q_{i+1} are spaced by the lead a $(=2k\pi)$. The tangent to the spiral at point O coincides with the initial straight line OX (this is good to bear in mind when constructing a spiral). The tangent line MT, at an arbitrary point M of the spiral, is obtained from the straight line MO by revolving MO through an (acute) angle $OMT=\alpha$, for which

$$\tan\alpha=\frac{OM}{k}=\frac{\rho}{k}=\varphi$$

For $\rho\longrightarrow\infty$, the angle α tends to 90° and near point M the arc of the spiral becomes more and more like the arc of a circle.

---

[1] First read Sec. 75.

[2] It is more convenient to take a circle of greater radius; we took a circle of radius k simply because it will be needed in the sequel.

[3] Point b_2 is not marked in the figure since it lies inside the circle labelled D_2 (the distance b_2D_2 amounts to about 5% of the radius k).

3. **The property of the normal.** The normal MN drawn through point M of an Archimedean spiral with lead a intersects the straight line ON, perpendicular to the radius vector OM, at point N which is distant $ON=\frac{a}{2\pi}\ (=|k|)$ from O.

4. **Constructing a tangent.** To construct a tangent at a point M of an Archimedean spiral (see Fig. 494), turn ray

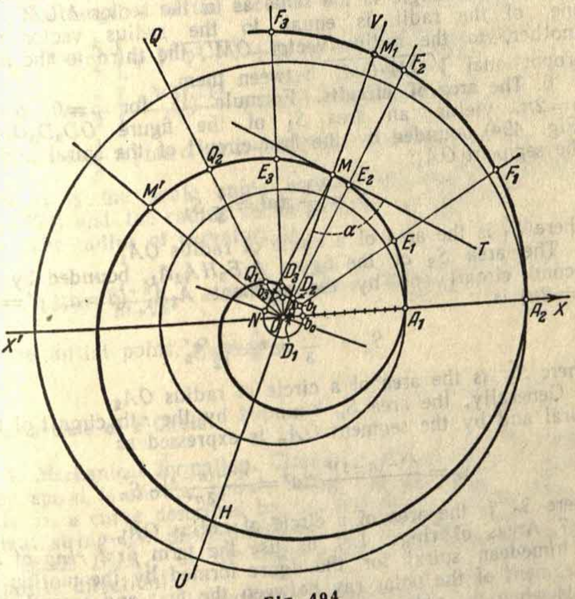

Fig. 494

OM about point O through an angle $+\frac{\pi}{2}$. Join point N, where the turned ray intersects a circle of radius k centred at O, to M. The straight line MN is normal to the spiral. Constructing $MT\perp MN$, we get the desired tangent line. The tangent to the left spiral (see Sec. 75 and Fig. 106) is built in a similar manner, with the sole difference, however, that the ray OM is rotated through an angle $-\frac{\pi}{2}$.

5. **The area** S **of sector** MOM' (if the polar angles of the points M, M' differ at most by 2π) is

$$S=\frac{1}{6}\omega(\rho^2+\rho\rho'+\rho'^2) \tag{1}$$

where $\rho=OM$, $\rho'=OM'$, $\omega=\angle MOM'$.

Geometrically, the sector of an Archimedean spiral is equal in area to the arithmetic mean of three circular sectors in which the angle is the same as in the sector MOM', and one of the radii is equal to the radius vector OM, another, to the radius vector OM', the third to the mean proportional $\sqrt{OM\cdot OM'}$ between them.

6. **The area of circuits.** Formula (1), for $\rho=0$, $\rho'=a$, $\omega=2\pi$, yields an area S_1 of the figure $OD_2D_3Q_1A_1O$ (Fig. 494) bounded by the first circuit of the spiral and by the segment OA_1:

$$S_1=\frac{1}{3}\pi a^2=\frac{1}{3}S_1' \tag{2}$$

where S_1' is the area of a circle of radius OA_1.

The area S_2 of the figure $A_1E_3HA_2A_1$, bounded by the second circuit and by the segment A_2A_1 ($\rho=a$, $\rho'=2a$, $\omega=2\pi$) is

$$S_2=\frac{7}{3}\pi a^2=\frac{7}{12}S_2' \tag{3}$$

where S_2' is the area of a circle of radius OA_2.

Generally, the area S_n bounded by the nth circuit of the spiral and by the segment OA_n is expressed as

$$S_n=\frac{n^3-(n-1)^3}{3}\pi a^2=\frac{n^3-(n-1)^3}{3n^2}S_n' \tag{4}$$

where S_n' is the area of a circle of radius OA_n.

7. **Areas of rings.** Let us use the term *first ring* of an Archimedean spiral for the figure formed by the motion of a segment of the polar ray between the first and second circuits when the polar ray turns through 360° from its initial position. To traverse this figure along its perimeter, we have to trace out segment A_1O, then the first circuit OQ_1A_1 of the spiral, then the segment A_1A_2 and, finally, the second circuit $A_2HQ_2A_1$ (retrograde motion).

The *second ring* is similarly formed by the segment of the polar ray between the second and third circuits. It is bounded by: (1) segment A_2A_1, (2) the second circuit, (3) segment A_2A_3, (4) the third circuit (traversed in retrograde fashion).

The third, fourth, etc. rings are defined similarly.

The area F_n of the nth ring is given by

$$F_n = S_{n+1} - S_n = 6nS_1$$

Here $S_1 = \frac{1}{3}\pi a^2$ is the area of the first circuit (zeroth ring).

The properties in this and the preceding items were discovered by Archimedes.

8. The **length** l **of the arc** OM is

$$l = \frac{k}{2}\left[\varphi\sqrt{\varphi^2+1} + \ln\left(\varphi + \sqrt{\varphi^2+1}\right)\right] =$$

$$= \frac{1}{2}\left[\frac{\rho\sqrt{\rho^2+k^2}}{k} + k\ln\frac{\rho+\sqrt{\rho^2+k^2}}{k}\right] =$$

$$= \frac{1}{2}k\left[\tan\alpha\sec\alpha + \ln(\tan\alpha + \sec\alpha)\right]$$

where α is the acute angle between the tangent line MT (Fig. 494) and the radius vector OM, or $\alpha = \angle ONM$.

9. The **radius of curvature** is

$$R = \frac{(\rho^2+k^2)^{3/2}}{\rho^2+2k^2} = k\frac{(\varphi^2+1)^{3/2}}{\varphi^2+2} = k\frac{(\tan^2\alpha+1)^{3/2}}{\sec^2\alpha+1}$$

At the initial point, $R_0 = \frac{k}{2}$.

512. Involute of a Circle

1. **Mechanical formation.** Closely related to the Archimedean spiral is another spiral, called the *involute of a circle*. This is a curve described by the extremity M (Fig. 495) of a taut string LM unwinding from (or winding onto) a circular spool D_0LL_1; in the latter case, point M moves in the opposite direction).

Geometrically, this property is expressed as follows:

2. **Definition.** Let point L start from an initial position D_0 and repeatedly describe a circle of radius k (k is the *parameter* of the involute of the circle). On the tangent line LH lay off, in the direction opposite to that of rotation, segment LM, equal to arc D_0L traversed by the point L. The involute of the circle is the curve described by point M. The same circle has an infinity of involutes (corresponding to all possible positions of the initial point D_0).

Depending on whether the point L is rotated clockwise or counterclockwise, we get a *right* involute of the circle

(D_0MP in Fig. 495) or a *left* involute (D_0Q). Ordinarily, the two involutes of a given circle are regarded as two branches of a *single* curve.

3. **Construction.** Divide the given circle into n equal arcs $D_0b_1=b_1b_2=b_2b_3=\ldots=b_{n-1}D_0$. On the tangent drawn through D_0 lay off a segment $D_0E_0=2\pi k$. Partition it into the same number of equal parts:

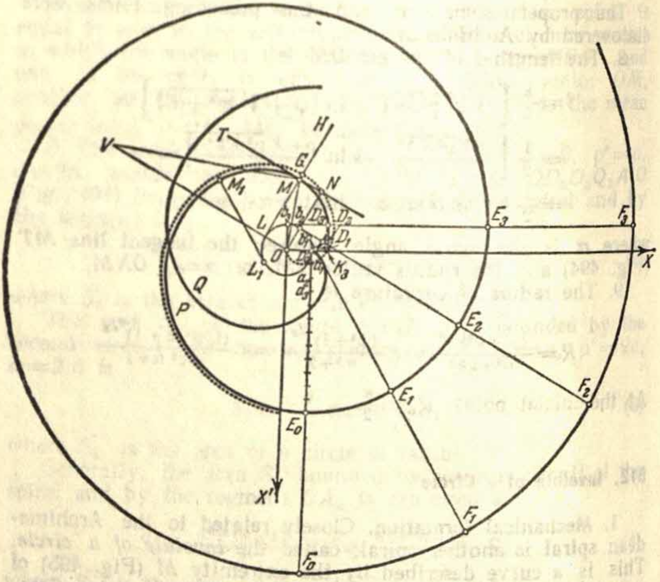

Fig. 495

$$D_0a_1=a_1a_2=\ldots=a_{n-1}E_0$$

On the tangents drawn through the successive points b_1, b_2, b_3, ..., lay off (in the direction opposite to the displacement of the point of tangency) segments b_1D_1, b_2D_2, b_3D_3, ..., respectively equal to the segments D_0a_1, D_0a_2, D_0a_3, ... We get the points D_1, D_2, D_3, ... of the first circuit D_0PE_0 of the involute of the circle. The points E_1, E_2, E_3, ... of the second circuit are obtained by laying off on extensions of the segments b_1D_1, b_2D_2, b_3D_3, ... the segments D_1E_1, D_2E_2, D_3E_3, ... equal to D_0E_0. The points of the subsequent circuits are found in similar fashion.

4. **Peculiarities of shape.** By virtue of the general properties of the involute of any curve (cf. Sec. 347, also Sec. 346) the involute of a circle possesses the following properties.

(a) The involute of a circle intersects all tangents to the circle at right angles. In particular, the involute makes a right angle, at the initial point D_0, with the tangent D_0F_0.

(b) Conversely, the normal MH to the involute serves as a tangent to the circle. Then the point L of tangency is the centre of curvature of the involute so that the segment ML is the radius of curvature of the involute:

$$R=ML \tag{1}$$

In particular, at the initial point D_0 the radius of curvature of the involute is zero:

$$R_0=0 \tag{2}$$

(c) The radius R of curvature of an involute increases with recession of M from the initial point; its increment $R_1-R=M_1L_1-ML$ is equal to the length of the corresponding arc $\overset{\frown}{LL_1}$ of the circle:

$$R_1-R=\overset{\frown}{LL_1} \tag{3}$$

In particular, on the segment D_0M of the involute the increment in the radius of curvature is equal to $R_M-R_0=R_M$, and

$$R_M=\overset{\frown}{D_0L}=k\alpha \tag{4}$$

where $\alpha=\angle D_0OL$ is the angle of rotation of the radius OL from the initial position OD_0.

(d) By construction, an involute does not go inside circle O. Therefore, when point M passes through the initial point D_0, the direction of motion is reversed, that is, D_0 is a cusp of the involute.

5. **Relationship with the Archimedean spiral.** Let us compare the right (left) branch of the involute of a circle with the right (left) Archimedean spiral having the same parameter $k=OD_0$ (i. e. with lead $2\pi a=D_0E_0$) as the involute of the circle. Let this spiral (shown dotted in Fig. 495) emanate from the centre O of the given circle in the direction of the ray OX' obtained by rotation of the initial radius OD_0 through the angle $-90°$ $(+90°)$. The point G describing the spiral approaches the involute indefinitely:

the shortest distance of point G to the involute (it is measured by the segment GM of the normal LH to the involute) constitutes only 1% of the lead of the spiral at the end of the first circuit.

On the other hand, the radius vector ON of the spiral, which forms an angle of $-90°$ $(+90°)$ with radius OL, is of the same length $k\alpha$ as the segment LM. This means that the *foot of a perpendicular dropped from centre O onto the tangent MT to the involute describes an Archimedean spiral.*

6. **The polar equation of the involute of a circle** (with pole O as centre of the given circle and with the polar axis OX directed along the initial radius OD_0) is

$$\varphi=\frac{\sqrt{\rho^2-k^2}}{k}-\arccos\frac{k}{\rho} \tag{5}$$

where k is the radius of the circle.

7. The **parametric equations** are

$$x=k(\cos\alpha+\alpha\sin\alpha);\quad y=k(\sin\alpha-\alpha\cos\alpha) \tag{6}$$

where $\alpha=\angle D_0OL$.

8. The **arc length** s of $\overset{\frown}{D_0M}$ is

$$s=\frac{1}{2}k\alpha^2=\frac{1}{2}\frac{(k\alpha)^2}{k}=\frac{1}{2}\frac{ML^2}{OL} \tag{7}$$

To obtain a segment of the same length, draw a straight line $MV \perp OM$ to intersection at the point V with the extension of the radius OL. Half the segment OV is equal in length to the arc $\overset{\frown}{D_0M}$:

$$s=\overset{\frown}{D_0M}=\frac{1}{2}OV \tag{8}$$

9. The **area** S **of the sector** D_0OM described by the radius vector, and also the area S_1 of the curvilinear triangle LMD_0 whose base is the segment LM and whose lateral sides are the arc D_0L of the circle and the arc D_0M of the involute, is one third the area of the triangle OMV (constructed in Item 8):

$$S=S_1=\frac{1}{3}\text{ area }OMV=\frac{1}{6}k^2a^3 \tag{9}$$

10. **Natural equation of the involute of a circle.** The *natural equation* of a curve is that equation which relates the length s of its arc $\overset{\frown}{M_0M}$, reckoned from some initial point M_0, to the radius R of curvature at the point M. The natural

equation of the involute of a circle is

$$R^2 = 2ks \tag{10}$$

It is obtained from (4) and (7) by eliminating α.

11. **Kinematic property.** In the language of kinematics, the natural equation (10) expresses the following property: if the arc of the involute of a circle rolls (without sliding) along a straight line, then the centre of curvature L corresponding to the point of tangency moves along a parabola with parameter k.

12. **Historical background.** The involutes of various curves were first studied by Huyghens in his celebrated study of the clock pendulum (1673) (cf. Sec. 514, Item 17). The basic properties of the involute of a circle were discovered by the French scholar La Hire (1640-1718) and described in a paper in 1706. Property 5 (Item 5) was found by Clairaut (1713-1765) in 1740. Property 9 and also the kinematic interpretation of the natural equation (of any curve) were pointed out by A. Mannheim in 1859.

513. Logarithmic Spiral

1. **Definition.** Let a straight line UV (Fig. 496) rotate uniformly about a fixed point O (the *pole*), and let a point M move along UV receding from O at a rate proportional to the distance OM. The curve described by M is called a *logarithmic spiral*.

2. **Basic geometrical property.** Rotation of the straight line UV from any position through a given angle $\omega\,(=\angle M_0OM_1)$ is associated with one and the same ratio $OM_1 : OM_0$ of the radius vectors. Put otherwise, if a pair of points M_0, M_1 of the logarithmic spiral is seen from the pole at the same angle as another pair of points N_0, N_1 of the same spiral, then the triangles OM_0M_1 and ON_0N_1 are similar.

The ratio q of a finite radius vector (OA_1) to the initial radius vector (OA_0) for a rotation of the straight line UV through an angle $+2\pi$ will be called the *coefficient of growth* of the logarithmic spiral.

3. **Right-handed and left-handed spirals.** If recession of point M from the pole O is accompanied by a counterclockwise rotation of UV, then the logarithmic spiral is called *right-handed*; otherwise, it is *left-handed*. For a right-handed spiral, the coefficient of growth $q > 1$; for a left-handed spiral, $q < 1$. For $q = 1$, the spiral degenerates into a circle.

If the product of the coefficients of growth of a left- and a right-handed spiral yields 1, then they may be brought to coincidence if one of them is turned over.

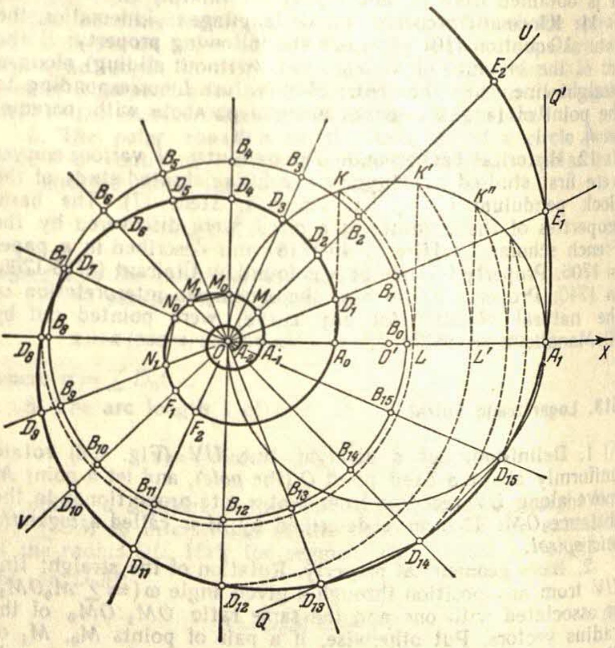

Fig. 496

4. Construction. To construct a right-handed logarithmic spiral with a given coefficient of growth q, [1] divide some circle with centre at O into $n=2^k$ equal parts by the points B_0, B_1, B_2, B_3, . . . in a counterclockwise sequence. [2] For

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[1] The left-handed spiral with coefficient of growth $\frac{1}{q}$ is built in the same way.

[2] When building a left-handed spiral, the points B_0, B_1, B_2, B_3, ... are in a clockwise sequence.

definiteness, put $n=2^4=16$. On ray OB_0 take an arbitrary point A_0 and lay off the segment $OA_1=qOA_0$. On the segment OA_1 as diameter, construct a circle O' and draw $A_0K \perp OA_1$ to intersection with this circle at the point K. The circle of radius OK will intersect ray OB_8 at point D_8 belonging to the sought-for spiral; the same circle will intersect ray OA_1 at a point L. Draw $LK' \perp OA_1$ to intersection with circle O' at point K'. The circle of radius OK' will intersect the ray OB_{12} at the point D_{12} belonging to the desired spiral, and it will intersect the ray OA_1 at some point L'. Through it we again draw $L'K'' \perp OA_1$, etc. We thus get the points D_{14} and D_{15}.

An infinity of other points of the spiral lying on the straight lines B_0B_8, B_1B_9, etc. may be constructed as follows. At point D_{14}, construct the angle $\angle OD_{14}Q$ equal to the angle $\angle OD_{15}D_{14}$; at the intersection with the ray OB_{13} we obtain point D_{13} of the required spiral. At point A_1 we construct $\angle OA_1Q'=\angle OD_{15}A_1$; at the intersection with the ray OB_1 we get the point E_1, etc.

5. The **polar equation** (the pole coincides with the pole of the spiral; the polar axis is drawn through an arbitrary point M_0 of the spiral) is

$$\rho=\rho_0 q^{\frac{\varphi}{2\pi}} \tag{1}$$

where $\rho_0=OM_0$ is the radius vector of the point M_0 and q is the coefficient of growth.

Example. The spiral constructed in Fig. 496 ($q=3$) is given by the equation

$$\rho=\rho_0 3^{\frac{\varphi}{2\pi}}$$

If for the polar axis we take the ray OB_0, then $\rho_0=OA_0$. In particular, putting $\varphi=\pi$, we get $\rho=\rho_0\sqrt{3}=OD_8$; for $\varphi=\frac{\pi}{2}$ we have $\rho=\rho_0\sqrt[4]{3}=OD_4$, etc.

Ordinarily, Eq. (1) is written as

$$\rho=\rho_0 e^{k\varphi} \tag{2}$$

where k is a parameter which is expressed in terms of the coefficient of growth q as

$$k=\frac{\ln q}{2\pi} \tag{3}$$

Conversely,

$$q = e^{2k\pi} \tag{4}$$

The geometrical significance of the parameter k is read from the relation

$$k = \cot\alpha \tag{5}$$

where $\alpha = \angle OMT$ is the angle between the straight line OM and the tangent MT (see Fig. 497).

The parameter k is positive for right-handed spirals and negative for left-handed spirals.

6. **Peculiarities of shape.** For an indefinite number of counterclockwise (clockwise) revolutions of the straight line UV, the point M, which describes a right-handed (left-handed) spiral, recedes without bound from the pole, sweeping out an infinity of circuits. In the case of an indefinite number of circuits in the opposite direction, point M approaches without bound the pole O, but does not coincide with O for any position of the straight line UV. Thus, the spiral performs an infinity of circuits about the pole. However, the length of the arc described by M in this operation and reckoned from some initial position A_0 of point M increases, but not without bound. It tends to a certain limit s which is called the *length of the arc* OA_0. The name is symbolic because, strictly speaking, the point O does not lie on the logarithmic spiral.

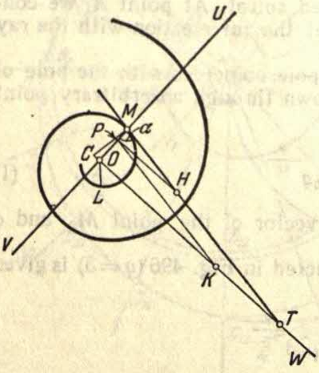

Fig. 497

7. **Tangent and arc length.** The angle $\alpha (= \angle OMT)$, through which the straight line UV must be rotated about point M of the logarithmic spiral (Fig. 497) so that UV coincides with MT, is the same for all points of the spiral. Segment MT of the tangent from the point of tangency to intersection with the straight line OW drawn through the pole O perpendicular to the radius vector OM has the same length

s as the arc of the spiral from point M to the pole O:

$$s=\overset{\frown}{OM}=MT=\frac{\rho}{\cos\alpha} \tag{6}$$

where ρ is the radius vector OM.

The length $\overline{s}$ of any arc LM of the logarithmic spiral

$$\overline{s}=\overset{\frown}{LM}=\overset{\frown}{OM}-\overset{\frown}{OL}=\frac{\rho_M-\rho_L}{\cos\alpha} \tag{7}$$

That is, the arc length $\overset{\frown}{LM}$ is proportional to the difference of the radius vectors at the end-points of the arc. In order to construct a line-segment of the same length, it is sufficient to lay off on the larger radius OM a segment OP equal to the smaller radius OL, and draw through P a straight line PH perpendicular to OM. It will intersect the tangent MT at some point H. MH is the desired segment.

The angle α is expressed in terms of the coefficient of growth q by the formula

$$\cot\alpha=\frac{\ln q}{2\pi} \tag{8}$$

For the spiral shown in Fig. 496, where $q=OA_1:OA_0=3$, we have

$$\cot\alpha=\frac{\ln 3}{2\pi}\approx 0.1748,$$

$$\alpha\approx 80°5'$$

8. **Characteristic triangle and the sectorial area.** The area described by the radius vector OL (Fig. 497) when point L, starting from some initial position M, unboundedly approaches the pole O along the logarithmic spiral, tends to the *finite* limit S (*sectorial area*). The sectorial area at the point M is one half the area of the *characteristic triangle* OMT formed by the radius vector OM, the straight line OW perpendicular to it, and the tangent MT:

$$S=\frac{1}{2}S_{OMT}=\frac{1}{4}\rho^2\tan\alpha \tag{9}$$

where ρ is the radius vector of point M.

The area $\overline{S}$ of any sector LOM (we assume that OM is the greater radius and that $\angle LOM$ does not exceed 2π in absolute value) is one half the area of the trapezoid $PMTK$ (Fig. 497) which will be cut out of the characteristic triangle OMT if on OM we lay off a segment $OP=OL$ and draw

$PK \parallel MT$:

$$\overline{S} = S_{PMTK} = \frac{1}{4}(\rho_2^2 - \rho_1^2)\tan\alpha \tag{10}$$

where ρ_1, ρ_2 are radius vectors of the points L and M.

9. **The radius and centre of curvature.** The centre of curvature C, which corresponds to point M of the logarithmic spiral (Fig. 497), lies at the intersection of the normal MC, drawn through M, and the straight line OW drawn through the pole perpendicular to the radius vector OM. The radius of curvature

$$R = \frac{\rho}{\sin\alpha} \tag{11}$$

This equality is evident from the triangle COM.

10. **Evolute.** The locus of the centres of curvature C (evolute) of a logarithmic spiral is a logarithmic spiral obtained from the original spiral by rotation about the pole through the angle

$$\omega = (2n+1)\frac{\pi}{2} - \tan\alpha\ln\tan\alpha \tag{12}$$

where n is any integer. Thus, if the original spiral intersects radius vectors at an angle $\alpha = 45°$, then it is coincident with its evolute in a rotation about the pole through an angle $\omega = \frac{\pi}{2}$, or $\omega = 5\frac{\pi}{2}$, or $\omega = -3\frac{\pi}{2}$, etc. In particular, there exists an infinity of logarithmic spirals which are their own evolutes. These are the spirals for which the angle α satisfies one of the equations

$$\tan\alpha\ln\tan\alpha = (2n+1)\frac{\pi}{2}$$

where n is an integer.

11. The **natural equation** (i. e. the equation relating the arc length and the radius of curvature; cf. Sec. 512, Item 10) is

$$R = ks\ (= s\cot\alpha) \tag{13}$$

It follows from (6) and (11) and is evident from the triangle CMT.

12. **Kinematic property.** In the language of kinematics, Eq. (13) expresses the following property: if an arc of the logarithmic spiral rolls (without sliding) along the straight line AB, then the centre of curvature corresponding to the point of tangency moves along a straight line inclined at an angle of $\frac{\pi}{2} - \alpha$ to AB.

13. **Cartographic property.** A spherical curve intersecting the meridians at a constant angle α (this curve is called a *loxodrome*[1]) is projected from the pole P of a sphere onto the plane of the equator as a logarithmic spiral; the pole of the spiral lies at the centre of the sphere. The meridians are projected as rays directed along the radius vectors of the spiral. These rays are intersected by the spiral at a constant angle α at which the loxodrome intersects the meridians.

14. **Historical background.** In 1638 Descartes found that a spiral whose arc grows in proportion to the radius vector has the property that its tangent forms a constant angle with the radius vector. At about that time Torricelli, independently of Descartes and in much more detail, made a study of the properties of the "geometrical spiral" (which was the name he gave the curve that he determined by means of the construction described in Item 3 above). Torricelli proved geometrically the properties given in Items 6 and 7. In 1692 James Bernoulli discovered Properties 8 to 11 and a number of other properties of this "spira mirabilis" (wonderful spiral). The term "logarithmic spiral" (the angle between the radius vectors is proportional to the logarithm of their ratios) was given by P. Varignon in 1704. Later, the logarithmic spiral was the subject of numerous investigations. Thus, its kinematic property (Item 12) was discovered by E. Catalan in 1856.

514. Cycloids

1. **Definition.** A *cycloid* is a curve described by a point (Fig. 498) fixed in the plane of a circle (the *generating circle*) when the circle rolls (without sliding) along some straight line KL (*directrix* or *base-line*).

If a point M describing the cycloid is taken inside the generating circle (i.e. at a distance $CM = d$ from the centre C, less than the radius r), then the cycloid is called *curtate* (Fig. 498*a*); if it is taken outside the circle (i.e. $d > r$), the cycloid is *prolate* (Fig. 498*b*); and if the point M lies on the circle (i.e. $d = r$), then the curve described by this point is termed a *common cycloid* (Fig. 498*c*) or, more often, simply a *cycloid* (cf. Sec. 253).

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[1] From the Greek meaning "oblique course". It is the path of a ship which cuts the meridians at a constant angle not equal to a right angle.

Example. When a railway car moves along rails, an interior point of a wheel traces out a curtate cycloid, a point on the outer rim, a prolate cycloid, and a point on the circumference of the wheel, a common cycloid.

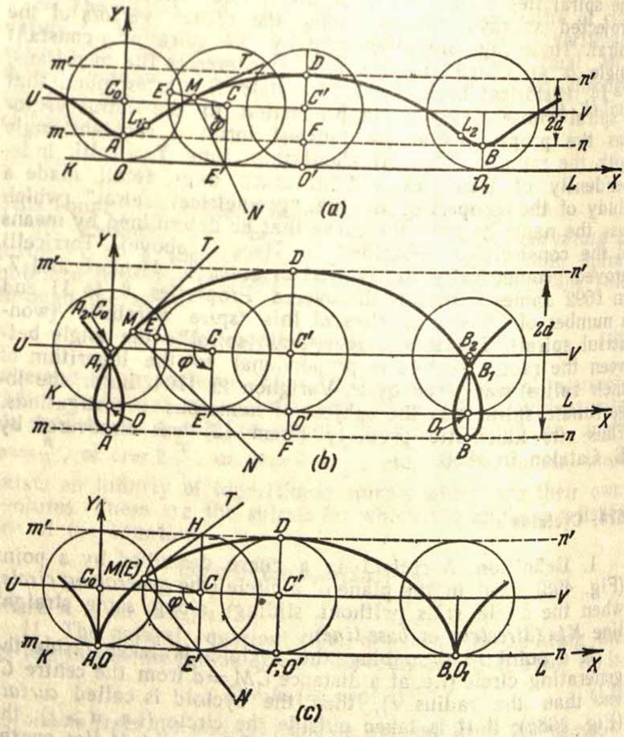

Fig. 498

The *starting point* of a cycloid (A in Fig. 498*a-c*) is a point lying on the straight line (C_0O) connecting the centre C_0 of the generating circle and the point of its support (O), and is located on the same side of the centre C_0 as the point of support O. Point B in Fig. 498*a-c* is also a starting point.

The initial points of a common cycloid (Fig. 498*c*) lie on the directrix and coincide with the corresponding points of support of the generating circle.

The *vertex* of a cycloid (D in Fig. 498*a-c*) is a point lying on the straight line $C'O'$ connecting the centre C' of the generating circle and the point of support O' but located on the extension of segment $C'O'$ beyond point C'.

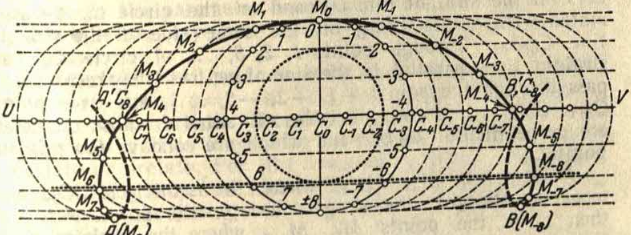

Fig. 499

Segment AB, which connects two adjacent starting points, is called the *base-line* of the cycloid; the perpendicular DF dropped from the vertex of the cycloid onto its base-line is the *altitude*. The arc traced out by the point M between two adjacent starting points is called a *cycloidal arch*; the straight line UV traced out by the centre C of the generating circle is the *line of centres* of the cycloid.

2. **Construction.** To construct a cycloid using the radius r of the generating circle and the distance d of the point M (which describes the cycloid) from the centre C of the generating circle, first draw (Fig. 499) the line of centres UV. From some point C_0 (on this line) as a centre, draw a circle of radius d.[1] Label one of the ends of its diameter, which is perpendicular to UV, as M_0. This will be the vertex of the desired curve.

Divide the circle C_0 into an even number, $2n$, of equal arcs (we have taken $2n=16$) making M_0 one of the division points; label the division points $0, \pm 1, \pm 2, \ldots, \pm n$ (the

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[1] In the case of the common cycloid, this is the circumference of the generating circle. Generally, however, neither the generating circle nor the directrix participate in this construction. In Fig. 499 they are shown as dotted lines for the purpose of pictorialness only.

points $+n$ and $-n$ coincide). On the line of centres, to the right and left of C_0, lay off segments C_0A', C_0B', equal to semicircles of the generating circle:

$$C_0A' = C_0B' = \pi r$$

and divide each of the segments into n equal parts. Denote the points of division by $C_{\pm 1}$, $C_{\pm 2}$, ..., $C_{\pm n}$ (the points C_n, C_{-n} coincide, respectively, with A', B'; positive numbers on the straight line UV and on the circle C_0 are associated with points lying on the same side of the straight line C_0M_0). Through points *1*, *2*, *3*, ... of circle C_0 draw straight lines parallel to the line of centres (they respectively pass through the points *—1*, *—2*, *—3*, ...), and from points $C_{\pm 1}$, $C_{\pm 2}$, ... draw semicircles of radius d whose diameters are perpendicular to UV and which are concave towards the point C_0.

Mark points M_1, M_{-1}, where the semicircles C_1, C_{-1} meet the straight line drawn through the points *+1*, *—1*; then mark the points M_2, M_{-2}, where the semicircles C_2, C_{-2} meet the straight line drawn through the points *+2*, *—2*, and so on. All the points M_1, M_{-1}, M_2, M_{-2} and so on lie on the desired cycloid. At points M_n, M_{-n} we find its starting points A, B.

That is how one arch of the cycloid is constructed from points. To construct adjacent arches, we have to continue the series of points C as shown in Fig. 499. These points have to be numbered from the beginning again. The circle C_0 need not be drawn anew since the straight lines parallel to the line of centres remain the same.

3. **The parametric equations** [where the axis of abscissas is the directrix KL; the coordinate origin O is a projection of one of the starting points (A in Fig. 498*a-c*) on the directrix KL] are

$$x = r\varphi - d \sin \varphi; \quad y = r - d \cos \varphi \tag{1}$$

where $\varphi = \angle MCE'$ is the angle of rotation of the generating circle reckoned from the position in which the point M coincides with the starting point A.

For the common cycloid ($d = r$)

$$x = r(\varphi - \sin \varphi); \quad y = r(1 - \cos \varphi) \tag{1a}$$

4. **Peculiarities of shape.** The cycloid extends along the straight line KL in both directions to infinity. Any arc reckoned from any starting point A is associated with a symmetrical arc reckoned from that same point in the opposite

direction; AC_0 is the axis of symmetry. The cycloid is also symmetric about the straight line DF drawn through any one of the vertices perpendicular to the directrix.

By displacement along the line of centres over a distance which is a multiple of $2\pi r$ (the length of the generating circle), the cycloid is brought to coincidence with itself. By successive displacements through the distance $\pm 2\pi r$ we can obtain the entire cycloid from any one of its arcs corresponding to a change in the parameter from some value $\varphi = \varphi_0$ to a value $\varphi = \varphi_0 + 2\pi$, say from $\varphi = -\pi$ to $\varphi = \pi$ or from $\varphi = 0$ to $\varphi = 2\pi$.

The cycloid lies inside a strip bounded by the straight lines $y = r + d$ and $y = r - d$. The first touches the cycloid at each one of its vertices. The second passes through all the starting points; it is tangent to the cycloid when this cycloid is curtate or prolate. For the common cycloid, the second straight line ($y = 0$) coincides with the directrix and is *perpendicular* to the (one-sided) tangents at the starting points of the cycloid.

5. **Nodes.** The prolate cycloid always has nodes. The number and location of the nodes depend on the ratio $d:r\ (=\lambda)$. So long as this ratio does not exceed the number $\lambda_0 = 4.60333\ldots$,[1] all the nodal points are located on the straight lines $x = 2k\pi r$ (k is an integer), and one node lies on each one of these lines: point A_1 (Fig. 498*b*) on the straight line $x = 0$, point B_1 on the straight line $x = 2\pi r$, etc.

These points may be found by solving the equation

$$\varphi - \lambda \sin \varphi = 0 \tag{2}$$

which, in the case at hand, $\lambda < \lambda_0$, has a unique positive root φ_1 located in the interval $(0, \pi)$. The values $\varphi = \varphi_1$ and $\varphi = -\varphi_1$ correspond to the point A_1 on the arch ADB $(0 < \varphi < 2\pi)$ and on the neighbouring arch $(-2\pi < \varphi < 0)$.[2]

Example 1. Let $d = 1.43r$, as in Fig. 498*b*. Solving

$$\varphi - 1.43 \sin \varphi = 0 \tag{2a}$$

(by the method indicated in Secs. 288-289), we find the value $\varphi_1 = 81°$, which corresponds to the point A_1 (on arch ADB). We find the ordinate y_1 of point A_1 from the second equation (1):

$$y_1 = OA_1 = r\,(1 - 1.43 \cos \varphi_1) \approx 0.78r$$

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[1] This irrational number is equal to sec α_0, where α_0 is the least positive root of the equation $\tan \alpha - \alpha = 0$.

[2] The zero root of Eq. (2) is associated with the starting point A, which is not a nodal point.

The nodal points of the given cycloid are

$$(2\pi kr,\ 0.78r)$$

Now let the ratio λ lie in the interval

$$\lambda_0 < \lambda \leqslant \lambda_1$$

where $\lambda_1 = 7.78968$;[1] then besides the above-considered nodes there appear nodal points on the straight lines $x = (2k+1)\pi r$, one pair of nodes on each of the straight lines: points P_1, P_2 (Fig. 500) on the straight line $x = \pi r$, points Q_1, Q_2 on the straight line $x = -\pi r$, points R_1, R_2 on the straight line $x = 3\pi r$, etc. These points may be found by solving the equation

$$\varphi - \lambda \sin \varphi = \pi \tag{3}$$

which in the case at hand has two positive roots: φ_1, φ_2. Both roots lie in the interval $(2\pi, 3\pi)$ and correspond to the points P_1, P_2 on the arch $BD''N$, which intersects the arch $LD'A$ here;[2] it is separated from $BD''N$ by one intermediate arch ADB.

When λ lies in the interval

$$\lambda_1 < \lambda \leqslant \lambda_2$$

where $\lambda_2 = 14.102\ldots$,[3] fresh nodes appear in the cycloid; this time again on the straight lines $x = 2k\pi r$, one pair on each of these straight lines: the points A_2, A_3 (see Fig. 500) on the straight line $x = 0$, the points B_2, B_3 on the straight line $x = 2\pi r$, the points L_2, L_3 on the straight line $x = -2\pi r$, etc. These points can be found by solving Eq. (2), which in the case at hand has three positive roots (not one, as in Example 1). The smallest root φ_1 lies in the interval $(0, \pi)$ and corresponds to the node A_1 (Fig. 500) lying at the intersection of the adjacent arches ADB and $LD'A$. The other two roots φ_2, φ_3 lie in the interval $(2\pi, 3\pi)$ and correspond to the points A_2, A_3 lying at the intersection of the arches $BD''N$ and $LD'''S$, which are separated by two intermediate arches ($LD'A$ and ADB).

As the ratio λ continues to increase, the cycloid acquires ever fresh pairs of nodes: first, one pair of points each on the straight lines $x=(2k+1)\pi r$ (these are the points of intersection of two arches separated by three intermediate arches), then one pair of points each on the straight lines $x=2k\pi r$ (here, two arches intersect which are separated by four intermediate arches) and so on alternately.

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[1] This irrational number is equal to sec α_1, where α_1 is the least positive root of the equation $\tan\alpha - \alpha = \pi$.

[2] If $\lambda = \lambda_0$, then points P_1 and P_2 coincide, so that the arches $BD''N$ and $LD'A$ contact one another.

[3] The number λ_2 is equal to sec α_2, where α_2 is the second (in order of increasing absolute value) positive root of the equation $\tan\alpha - \alpha = 0$.

Example 2. Let $\lambda=8$ as in Fig. 500. Since this value of λ lies in the interval (λ_1, λ_2), the given prolate cycloid has

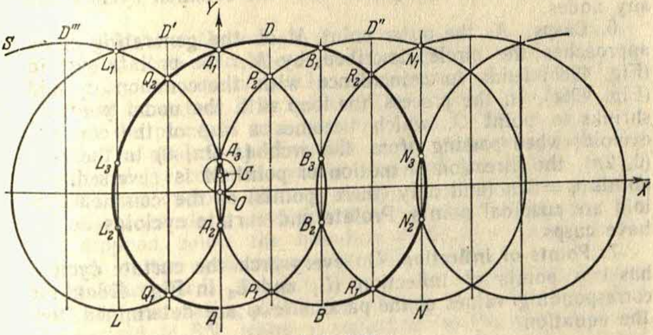

Fig. 500

three nodes on each of the straight lines $x=2k\pi r$ and two on each of the straight lines $x=(2k+1)\pi r$.

The nodes A_1, A_2, A_3 on the straight line $x=0$ are found from the equation

$$\varphi-8\sin\varphi=0 \tag{2b}$$

Its roots are

$$\varphi_1=159°40', \quad \varphi_2=360°+69°30', \quad \varphi_3=360°+95°54'$$

The ordinates of the points A_1, A_2, A_3 are found from the second equation (1):

$$y_1=OA_1=r(1-8\cos\varphi_1)\approx 8.50r$$

and, similarly,

$$y_2=OA_2\approx 1.80r, \quad y_3=OA_3\approx 1.83r$$

The nodes P_1, P_2 on the straight line $x=\pi$ are found from the equation

$$\varphi-8\sin\varphi=\pi \tag{3a}$$

Its roots are

$$\varphi_1'=360°+26°49', \quad \varphi_2'=360°+136°21'$$

The ordinates of the points P_1, P_2 will be

$$y_1' = OP_1 \approx -6.14r, \quad y_2' = OP_2 \approx 6.79r$$

Each arch of our cycloid has 10 nodal points (on arch ADB, the points Q_1, L_2, L_3, Q_2, A_1 and, symmetrical to them, the points R_1, N_2, N_3, R_2, B_1).

Neither the curtate cycloid nor the common cycloid has any nodes.

6. **Cusps.** As the outer point M of the generating circle approaches the circle described by M, the prolate cycloid (Fig. 498*b*) tends to coincidence with the common cycloid (Fig. 498*c*). In the process, the loop with the nodal point A_1 shrinks to point O, which becomes a *cusp* of the common cycloid; when passing from the arch $(-2\pi, 0)$ to the arch $(0, 2\pi)$, the direction of motion of point M is reversed. All points $\varphi = 2k\pi$ (and only these points) of the common cycloid are cuspidal points. Prolate and curtate cycloids do not have cusps.

7. **Points of inflection.** On every arch the curtate cycloid has two points of inflection (L_1 and L_2 in Fig. 498*a*); the corresponding values of the parameter φ are determined from the equation

$$\cos\varphi = \frac{d}{r}$$

For the cycloid depicted in Fig. 498*a*, where $d = 0.6r$, we have $\cos\varphi = 0.6$. The point L_1 is associated with the value $\varphi_1' \approx 52°25'$, the point L_2 with the value $\varphi_2' = 127°35'$. The coordinates x_1, y_1 of the point L_1 are

$$x_1 = r\varphi - d\sin\varphi = r(\varphi - 0.6\sin\varphi) \approx 0.43r,$$

$$y_1 = r - d\cos\varphi = r(1 - 0.6\cos\varphi) \approx 0.63r$$

The coordinates of the point L_2 are

$$x_2 = 2\pi - x_1 \approx 5.85r, \quad y_2 = y_1 \approx 0.63r$$

8. **Properties of normal and tangent.** The normal MN (Fig. 498*a-c*) of any cycloid passes through the point of support E' of the generating circle. The tangent line MT (Fig. 498*c*) of the common cycloid passes through point H, which is diametrically opposite to the point of support of the generating circle.

This makes it clear how to construct the tangent.

9. **Radius of curvature.** For any cycloid

$$R=\frac{(r^2+d^2-2dr\cos\varphi)^{3/2}}{d\,|d-r\cos\varphi|} \tag{4}$$

In particular, for the common cycloid

$$R=2r\sqrt{2}\sqrt{1-\cos\varphi}=4r\left|\sin\frac{\varphi}{2}\right|=2\sqrt{2ry}=2ME' \tag{4a}$$

(Fig. 498*c*); i.e. *the radius of curvature of the common cycloid is equal to twice the segment of the normal between the cycloid and the directrix*. In other words, to construct the centre of curvature it is sufficient to continue the chord ME' beyond the point E' to a distance equal to the chord.

10. **The evolute and involute of the common cycloid.** The evolute of the common cycloid (locus of the centres of curvature) is a cycloid which is congruent to the given cycloid, but displaced along the directrix one-half the base-line AB and dropped below the base-line a distance equal to the altitude of the cycloid (see Fig. 384).

In other words, the involute of the cycloid C_4BD (see Fig. 384) emanating from the vertex B of this cycloid is the cycloid M_2BN, which is congruent to the given one but is displaced along the directrix one-half the base C_4D and raised above the base a distance equal to the altitude of the cycloid.

11. **The cycloid and the sinusoid.** The locus of the feet of perpendiculars dropped from point M of a cycloid onto the generating-circle diameter passing through the point of support is a sinusoid (sine curve) with wavelength $2\pi r$ and amplitude d. The axis of the sine curve coincides with the line of centres of the cycloid.

12. **The cycloid as the projection of a helix.** *Notation:* h is the lead of the helix; a its radius, α the helix angle, β the angle between the axis of the helix and the projection plane; σ the angle of inclination of the projecting rays to the projection plane.

The *oblique projection* of a helix on a plane *perpendicular to the axis* is a cycloid. If $\sigma > \alpha$, the cycloid is prolate; if $\sigma < \alpha$, the cycloid is curtate; if $\sigma=\alpha$, the cycloid is common. The rectangular projection of a helix on the same plane is, obviously, a circle.

The *rectangular projection* of a helix on a plane *not perpendicular to the axis* but also not parallel to the axis, is a "compressed cycloid" (Fig. 501*a-c*), that is, a curve obtained from a cycloid by means of uniform compression (Sec. 40) towards some straight line perpendicular to the line of centres of the cycloid.

The coefficient of compression $k=\sin\beta$; the quantities r and d which characterize the cycloid (prior to compression) are expressed as

$$r=\frac{h}{2\pi}\cot\beta\,(=a\tan\alpha\cot\beta);\qquad d=a \tag{5}$$

Whence it is seen that for $\beta>\alpha$ the projection of the helix (Fig. 501*a*) is related to the prolate cycloid; for $\beta<\alpha$ (Fig. 501*b*) it is related to the curtate cycloid; for $\beta=\alpha$ (Fig. 501*c*) it is related to the common cycloid.

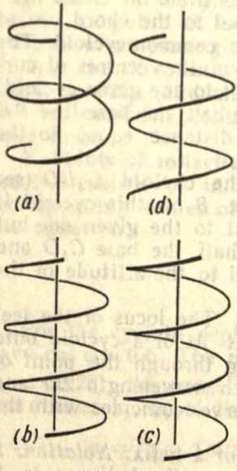

Fig. 501

The orthogonal projection of a helix on a plane parallel to the axis (Fig. 501*d*) is a sine curve whose amplitude is the radius a of the helix and whose wavelength is the projection $h\cos\beta$ of the lead h.

13. **The arc length** s of a cycloid between the points $\varphi=0$, $\varphi=\varphi_1$ is

$$s=\int_0^{\varphi_1}\sqrt{r^2+d^2-2rd\cos\varphi}\,d\varphi \tag{6}$$

This arc is equal, in length, to the arc of the ellipse

$$x=2\,(d+r)\cos\frac{\varphi}{2}$$
$$y=2\,(d-r)\sin\frac{\varphi}{2} \tag{7}$$

between points with the same values of the parameter φ.

In the general case, integral (6) cannot be expressed in terms of elementary functions of the argument φ_1. But *for the common cycloid* [ellipse (7) degenerates into a line segment of length $8r$] we have

$$s=2r\int_0^{\varphi_1}\sin\frac{\varphi}{2}\,d\varphi=4r\left(1-\cos\frac{\varphi_1}{2}\right)=8r\sin^2\frac{\varphi_1}{4}\qquad(\varphi_1\leqslant 2\pi) \tag{8}$$

In particular, *one arch of the common cycloid is equal in length to four times the diameter of the generating circle:*

$$s=4\cdot 2r \tag{8a}$$

14. **The natural equation** of the common cycloid (within the limits of a *single* arch) is

$$R^2+(s-4r)^2=(4r)^2 \quad (0 < s < 2\pi r) \tag{9}$$

This is obtained from (8) and (4a) by eliminating φ. The arcs are reckoned from the starting point. If the vertex is taken as the starting point of the arcs, then the natural equation will be

$$R^2+s^2=(4r)^2 \quad (-4r \leqslant s \leqslant 4r) \tag{9a}$$

15. **The kinematic property of the common cycloid.** Eq. (9) expresses in the language of kinematics the following property: if a common cycloid rolls (without sliding) along a straight line AB, then the centre of curvature of the point of tangency moves in a circle, whose radius is four times the radius of the generating circle and whose centre lies at the point of AB through which the vertex of the cycloid rolls.

16. **Areas and volumes.** The area S_1 swept out by the ordinate as φ varies from $\varphi=0$ to $\varphi=\varphi_1$ is

$$2S_1=(2r^2+d^2)\,\varphi-4dr\sin\varphi+\frac{d^2\sin 2\varphi}{2} \tag{10}$$

The "total area" S (for $\varphi_1=2\pi$) is

$$S=2\pi r^2+\pi d^2 \tag{11}$$

For the common and curtate cycloids, this is the area of the figure $OADBO_1$ (Fig. 498*a*, *c*); for the prolate cycloid, it is the area of the figure that remains after removing the rectangle $OABO_1$ from the figure AA_1DB_1B (Fig. 498*b*)

For the common cycloid ($d=r$)

$$S=3\pi r^2 \tag{12}$$

Thus, the figure bounded by an arch of the cycloid and the base is three times the area of the generating circle [Roberval (1634), Torricelli (1643)].

The area F_1 of a surface formed by rotation of a common cycloid about its base AB is

$$F_1=\frac{64}{3}\pi r^2=\frac{64}{9}S \tag{13}$$

where S is the area of a loop of the cycloid.

The volume V_1 of the corresponding solid of revolution is

$$V_1 = 5\pi^2 r^3 = \frac{5}{8} V \tag{14}$$

where V is the volume of the circumscribed cylinder.

The area F_2 of a surface formed by rotation of a common cycloid about the altitude DF is

$$F_2 = 8\pi\left(\pi - \frac{4}{3}\right) r^2 \tag{15}$$

The volume V_2 of the corresponding solid of revolution is

$$V_2 = \pi r^3\left(\frac{3}{2}\pi^2 - \frac{8}{3}\right) = \frac{3}{4} V' - 2V'' \tag{16}$$

where V' is the volume of the circumscribed cylinder and V'' is the volume of the inscribed sphere.

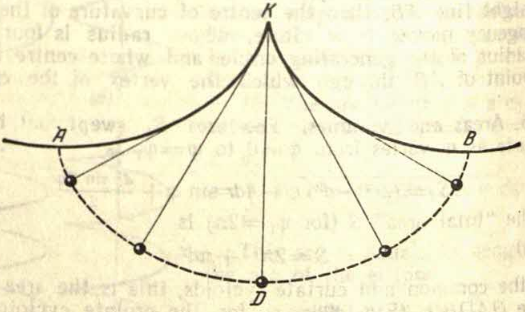

Fig. 502

17. **Tautochrone**[1] **property of the cycloid.** A mass point moving under gravity along a common cycloid ADB (Fig. 502) concave upward reaches its lowest position D in an interval of time

$$t = \pi\sqrt{\frac{r}{g}} \tag{17}$$

(r is the radius of the generating circle, g the acceleration of gravity). This interval is independent of the initial position of the point (Huyghens, 1673).

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[1] From the Greek meaning "equal time".

Therefore the period T of oscillation of a cycloidal pendulum ($T=4t$) is not dependent on its amplitude (a circular pendulum has this property only for small oscillations). The string of the cycloidal pendulum constructed by Huyghens is fixed at the starting point K of another cycloid AKB, which is the evolute of cycloid ADB (see Item 10).

18. **The cycloid as a brachistochrone.** [1] The brachistochrone of a mass point moving under gravity (in a medium whose resistance may be neglected) from a given point A to a lower-lying point B (not located on the same vertical as A) is a common cycloid. It is concave up, the point A is the starting point. The size of the generating circle is determined from the condition that the cycloid must pass through point B.

The time t of quickest descent is determined from the formula

$$t=\sqrt{\frac{r}{g}}\,\varphi_B \tag{18}$$

where φ_B is the angle of rotation (of the generating circle) corresponding to the point B.

Example. Point B is 0.83 m below point A, and 1.54 metres from A horizontally Find the time of quickest descent from A to B.

Solution. Take the coordinate origin at A, the x-axis pointing vertically down; for the xy-plane take the vertical plane passing through A and B. Take the y-axis so that point B has a positive abscissa. We take a scale unit of 1 metre. Then the coordinates of B will be

$$x_1=1.54, \quad y_1=0.83 \tag{19}$$

The cycloid ensuring quickest descent is given by the equations

$$x=r\,(\varphi-\sin\varphi), \quad y=r\,(1-\cos\varphi) \tag{20}$$

From the conditions of (19) it is possible to find the radius r of the generating circle and the value $\varphi=\varphi_B$ corresponding to the point B

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[1] The term "brachistochrone", derived from the Greek "*brachistos*", shortest and "*chronos*", time, is the curve of quickest descent from one point to another. The term was introduced by John Bernoulli (1667-1748) who posed the problem of finding the "curve of shortest descent". In 1696 he and James Bernoulli simultaneously published the solution.

To do this, eliminate r from (20) and solve the equation

$$1.54\,(1-\cos\varphi)=0.83\,(\varphi-\sin\varphi)$$

by the method given in Secs. 288-289. This yields

$$\varphi\approx 195^\circ\ (\approx 3.40 \text{ radians})$$

Now from the second equation of (20) we find

$$r\approx 0.42 \text{ (metre)}$$

Finally, from formula (18), putting $g=9.8\ \text{m/sec}^2$, we get

$$t=\sqrt{\frac{0.42}{9.8}}\cdot 3.40\approx 0.70 \text{ (second.}$$

The descent from A to B along an inclined plane would have lasted 0.87 sec, which is nearly 25% longer.

19. **Historical background.** The cycloid played an exceedingly important role in the history of higher mathematics. For over half a century it attracted the attention of outstanding mathematicians of the 17th century. A number of its properties found by geometrical methods confirmed the correctness of new analytical methods. Other properties were discovered by the new methods alone.

In 1590, Galileo, studying the path of a point on a rolling circle constructed a cycloid (Galileo gave it the name cycloid). He wanted to determine the area bounded by an arch of the cycloid and its base. Not having at his disposal the means for a theoretical solution of the problem, he attempted to find the ratio of the area of the cycloid to the area of the generating circle by means of weighing. He at first believed this ratio to be equal to 3, but then noticed that the experiment invariably yielded a number less than three. Since the difference was slight, the desired ratio seemed beyond the capability of small integers, and Galileo became convinced that the ratio was irrational.

After Galileo's death (1642), his pupils Torricelli and Viviani, who had shared with him the deprivations of imprisonment, undertook the mathematical investigation of the cycloid. Applying kinematic reasoning, Viviani found the property of the tangent (given in Item 5); Torricelli, using techniques that foreshadowed the integral calculus, determined the area of a cycloid (Item 14).

The area of a cycloid was found by Roberval too (independently of Torricelli and, probably, a few years earlier). Roberval's method is remarkably ingenious and simple (it is based on the property of Item 11). 1)

Using the same method, Roberval found the volumes of the solids of revolution of the cycloid about the base and about the altitude. Roberval considered the prolate and curtate cycloids in addition to the common cycloid and gave a method for constructing their tangents.

Remarkable as these discoveries were, they had to do with the same problems which long ago had been solved for a number of other figures. Yet all attempts to accomplish an *exact* rectification of curvilinear arcs had failed. The cycloid was the first curve to be rectified. This was first accomplished by the English astronomer, physicist, mathematician and architect Christopher Wren (1632-1723). Wren's work was published in 1658. The same problem was soon solved by a number of other scholars, and Fermat was first to rectify an *algebraic* curve (the semicubic parabola).

An exhaustive investigation into the geometrical properties of the cycloid was carried out by Blaise Pascal, whose work was published in 1659. .

During the next forty years, such first-class scientists as Huyghens, Newton, Leibniz and the Bernoulli brothers investigated the mechanical applications of the cycloid (see Items 15 and 16). In its generalized form, the problem of the brachistochrone (Item 16) was one of the basic sources for a new branch of mathematics called the *calculus of variations*, which was created in the 18th century in the works of Euler and Lagrange.

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1) The underlying idea of the method is this: reasoning from symmetry, it is clear that the sine curve *AQD* (Fig. 503) which is discussed in Property 11 partitions into equal parts the rectangle *AFDK* constructed on half the base of the cycloid and on its altitude. From elementary reasoning, it follows that the figure *AFDQ* formed by the altitude, half the base and by the sine curve is equal to the generating circle. To obtain the area of the semicycloid, we have to add the area of the "petal" *AQDP* between the semi-arch of the cycloid and the sine curve. Roberval proved this petal-shaped loop to be equal to half the generating circle. The proof consists in applying what is known as Cavalieri's principle (the semicircle is bounded by a vertical diameter, the sections are made parallel to the base).

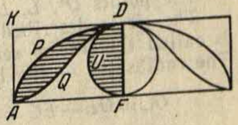


Fig. 503

515. Epicycloids and Hypocycloids

1. **Definition.** Like the epicycloid (Fig. 504*a*), the hypocycloid (Fig. 504*b*) is a curve L described by a point M fixed in the plane of some circle C (generating circle) of radius r when this circle rolls without sliding along a fixed circle (directrix) of radius R. The curve L is called an *epicycloid* when the circles C and O are externally tangent, and a *hypocycloid* when the tangency is internal.

The epicycloid of Fig. 504*a* and the hypocycloid of Fig. 504*b* are described by the motion of point M located on the circumference of the generating circle. Such epicycloids and hypocycloids are termed *common* in contrast to the curtate and prolate curves. The epicycloid (Fig. 505*a*) and the hypocycloid (Fig. 505*b*) are called *curtate* when point M is taken inside the generating circle, i. e. when $d < r$ ($d = CM$ is the distance of M from the centre C of the generating circle), and *prolate* (Fig. 506*a* and *b*) when M is exterior to the generating circle, i. e. when $d > r$.

The starting point of an epicycloid or hypocycloid (A in Figs. 504-506) is a point which lies on the straight line (C_1E_1) connecting the centre (C_1) of the generating circle and the point (E_1) of support, and is on the same side of the centre C_1 as the point E_1 of support. The points A', B, B' in Fig. 505*a* and *b* are also starting points.

The starting points of a common epicycloid and a common hypocycloid (A, B, K, in Fig. 504*a* and *b*) lie on the directing circle (directrix) and coincide with the respective points of support of the generating circle.

The vertex of an epicycloid or hypocycloid (D in Fig. 505*a* and *b*) is that point which lies on the straight line C_2E_2 joining the centre C_2 of the generating circle to the point E_2 of support but is located on the extension of the segment C_2E_2 beyond the point C_2.

The points D', L, L' in Fig. 505*a* and *b* are also vertices.

The circle described by the centre of the generating circle is called the *line of centres* of the epicycloid (hypocycloid). The radius OC of the line of centres is

$$OC = OE + EC = R + r \quad \text{for the epicycloid,}$$

$$OC = |OE - EC| = |R - r| \quad \text{for the hypocycloid}$$

2. **Construction.** To construct an epicycloid or hypocycloid from given R (radius of directrix), r (radius of generating circle) and d (distance of point M describing epicycloid or

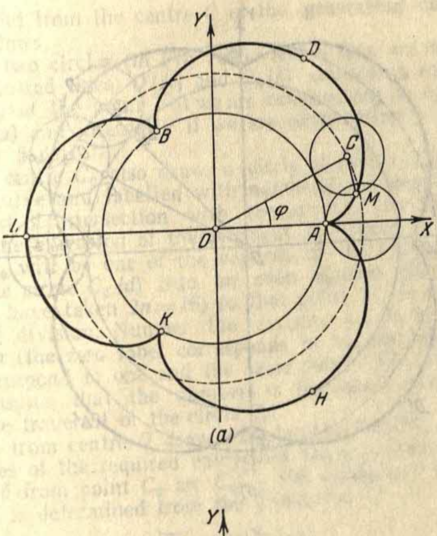
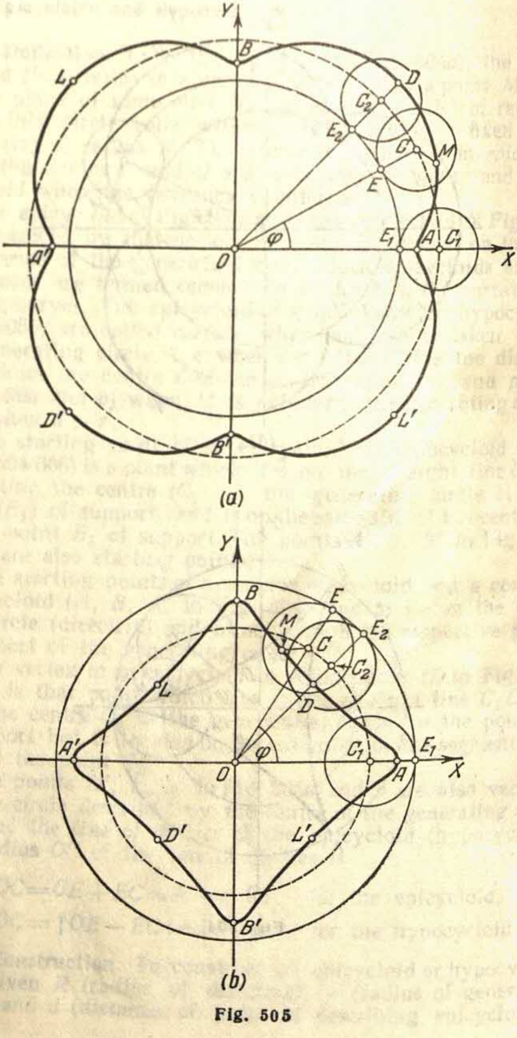


Fig. 504


Y
B
L
D
C_2
E_2
C
M
E
φ
O
A'
E_1
A
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E
M
C
E_2
C_2
L
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A'
O
φ
C_1
A
E_1
X
D'
L'
B'

(a)

(b)

Fig. 505

hypocycloid from the centre C of the generating circle) we do as follows.

Draw two circles (in Fig. 506*a* and *b* they are shown as heavily dotted lines) $O(R)$ and $C_0(r)$ contacting each other externally at the point V if we are constructing an epicycloid (Fig. 506*a*) and internally if we are constructing a hypocycloid (Fig. 506*b*).

From centre C_0 also draw a circle of radius d (shown as a solid curve and labelled with numbers) and denote by M_0 that point of intersection with the straight line OC_0 which lies on the extension of the segment C_0V beyond point C_0. Point M_0 will be one of the vertices of the required curve.

Divide circle $C_0(d)$ into an even number $(2n)$ of equal arcs (we have taken $2n=16$) so that point M_0 is one of the points of division. Number the division points 0, ± 1, ± 2, $\ldots$, $\pm n$ (the zero label corresponds to M_0, the labels n and $-n$ correspond to one and the same point). For definiteness let us assume that the numbers of the labels increase in a clockwise traversal of the circle C_0.

Then from centre O draw a circle of radius OC_0—the line of centres of the required epicycloid (hypocycloid)—and on it lay off from point C_0 arc C_0C_n, the measure of which (in degrees) is determined from the proportion

$$\overset{\frown}{C_0C_n} : 180^\circ = r : R \tag{1}$$

and which is clockwise if we are constructing an epicycloid, and counterclockwise if a hypocycloid. From the same point C_0 lay off arc C_0C_{-n} symmetric to C_0C_n.[1] In Fig. 506*a* and *b*, where $r:R=1:4$, the arcs C_0C_8 and C_0C_{-8} each contain 45° and, geometrically, are readily constructed exactly with straightedge and compass. In other cases, such a construction may prove complicated or impossible altogether. Then the construction is performed in approximate fashion to the required degree of accuracy.

Divide each of the arcs C_0C_n, C_0C_{-n} into n equal parts and label the division points $C_{\pm 1}$, $C_{\pm 2}$, $\ldots$, $C_{\pm n}$ starting with point C_0.

Now from point O draw a number of concentric circles passing successively through M_0, which has the label 0, through a pair of points labelled ± 1, through a pair of points labelled ± 2, and so on. On the first of these circles will lie all the vertices, on the last, all the starting points.

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[1] If $r > R$, then the arcs C_0C_n and C_0C_{-n} overlap, and if $r > 2R$, then, besides, each of the arcs C_0C_n, C_0C_{-n} overlaps itself.

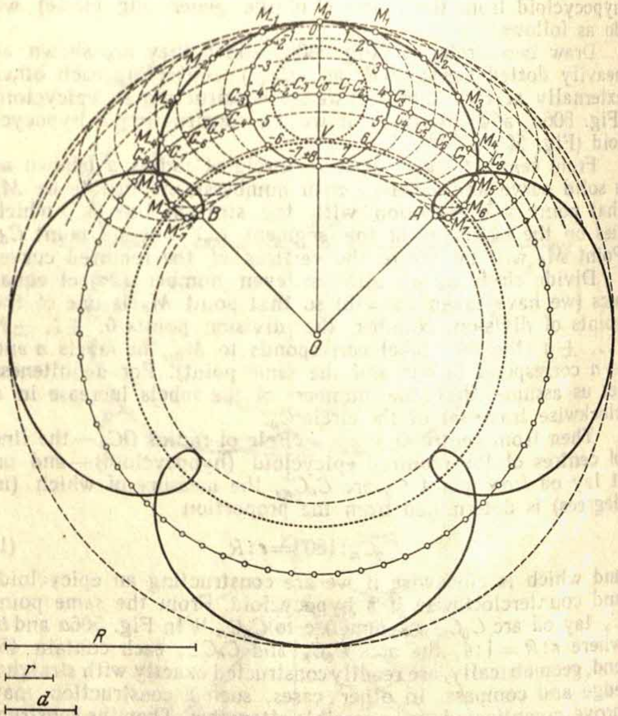

Fig. 506a

From points C_1, C_2, ..., C_n as centres, draw semicircles of radius d so that their extremities lie on the first and the last of the concentric circles and so that these semicircles can, by rotating about point O, be brought to coincidence with the semicircle bearing the labels *1*, *2*, *3*, ... Similarly, from centres C_{-1}, C_{-2}, C_{-3}, ... draw semicircles which, after rotation about point O, can be brought to coincidence with the semicircle labelled —*1*, —*2*, —*3*, ...

Mark points M_1, M_{-1}, where the semicircles $C_1(d)$, $C_{-1}(d)$ meet that one of the concentric circles which was drawn

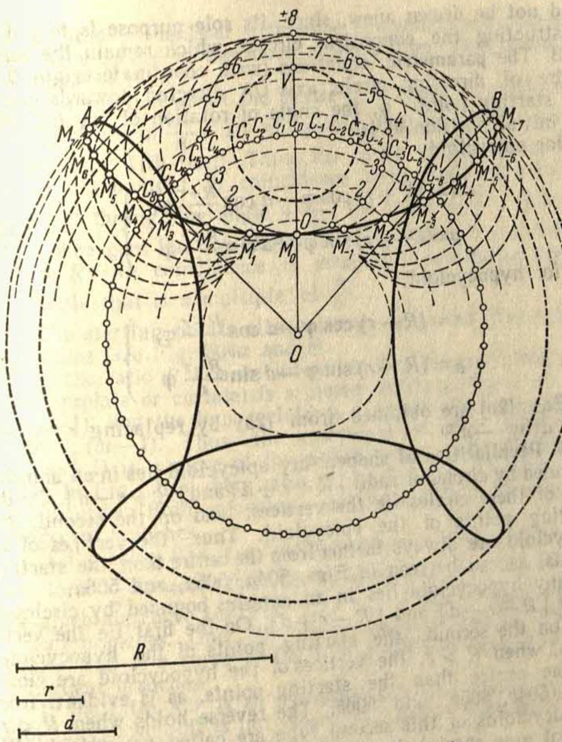
±8
7
-7
6
-6
V
5
-5
4
-4
3
-3
2
-2
1
0
-1
C_3 C_2 C_1 C_0 C_{-1} C_{-2} C_{-3} C_{-4}
C_4 C_5 C_6 C_7 C_8
C_{-5} C_{-6} C_{-7} C_{-8}
A
B
M_7 M_6 M_5 M_4 M_3 M_2 M_1 M_0 M_{-1} M_{-2} M_{-3} M_{-4} M_{-5} M_{-6} M_{-7}
O
R
r
d

Fig. 506b

through the points ±*1*; then mark points M_2, M_{-2} where the semicircles C_2, C_{-2} meet the circle drawn through the points ±*2*, etc. All the points $M_{\pm 1}$, $M_{\pm 2}$, $M_{\pm 3}$, ... lie on the desired curve, and points $M_8 M_{-8}$ coincide with the starting points *A*, *B* (these could have been obtained preliminarily by drawing the straight lines OC_n, OC_{-n}).

That is how one branch of the epicycloid (hypocycloid) is constructed from points. To construct adjacent branches, it suffices to continue the series of points *C* as shown in Fig. 506*a* and *b*. These points will have to be renumbered. The circle C_0

need not be drawn anew, since its sole purpose is to aid in constructing the concentric circles, which remain the same.

3. The **parametric equations** (with coordinate origin O at centre of directing circle; axis OX directed towards one of the starting points; φ, the angle of rotation of ray OC from its initial position[1]) is:

for epicycloid,

$$\left.\begin{aligned} x&=(R+r)\cos\varphi-d\cos\frac{R+r}{r}\varphi,\\ y&=(R+r)\sin\varphi-d\sin\frac{R+r}{r}\varphi \end{aligned}\right\}\tag{2a}$$

for hypocycloid,

$$\left.\begin{aligned} x&=(R-r)\cos\varphi+d\cos\frac{R-r}{r}\varphi,\\ y&=(R-r)\sin\varphi-d\sin\frac{R-r}{r}\varphi \end{aligned}\right\}\tag{2b}$$

Eqs. (2b) are obtained from (2a) by replacing r by $-r$ and d by $-d$.[2]

4. **Peculiarities of shape.** Any epicycloid lies in an annulus bounded by circles of radii $|R+r+d|$ and $|R+r-d|$. On the first of these circles lie the vertices, and on the second, the starting points of the epicycloid. Thus, the vertices of an epicycloid are always farther from the centre than the starting points, as can be seen in Figs. 504*a*, 505*a*, and 506*a*.

Any hypocycloid lies in an annulus bounded by circles of radii $|R-r-d|$ and $|R-r+d|$. On the first lie the vertices, on the second, the starting points of the hypocycloid. Thus, when $R > r$, the vertices of the hypocycloid are closer to the centre than the starting points, as is evident from Figs. 504*b*, 505*b*, and 506*b*. The reverse holds when $R < r$. Hypocycloids of this second type are called *pericycloids*. We do not give special drawings for the simple reason that every pericycloid is identical with some epicycloid and differs from the latter solely in the mode of generation. This is discussed in detail in Item 7.

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[1] This angle is equal to $\angle XOC$ for all epicycloids and for those hypocycloids the radius of the generating circle of which is less than the radius of the directing circle ($r < R$). But if $r > R$, then $\varphi = \angle XOC + \pi$. Note that there is no hypocycloid for which $r = R$ because then the generating circle could not roll without sliding along the directing circle and contact it internally.

[2] For the indicated choice of direction of x-axis and parameter φ, Eq. (2b) is also valid for those hypocycloids in which $r > R$ (such hypocycloids are called *pericycloids*). But if (as is often done) we take for the parameter φ the angle $\angle XOC$, then the parametric equations of the pericycloid will differ from the equations of the other hypocycloids.

In a rotation about the centre O through an angle which is a multiple of $\frac{2\pi r}{R}$, an epicycloid (hypocycloid) comes to coincidence with itself Thus, the curve in Fig. 504*a*, where $R=3r$, is brought to coincidence with itself in a rotation about O through the angle $\pm\frac{2\pi}{3}$, the angle $\pm\frac{4\pi}{3}$, $\pm 2\pi$, etc. The same goes for Fig. 504*b*. In Figs. 505*a*, *b* and 506*a*, *b*, where $R=4r$, coincidence is attained in a rotation through an angle that is a multiple of $\frac{\pi}{2}$.

The starting points of a common epicycloid (hypocycloid) are cusps (see Fig. 504*a* and *b*).

If the ratio $R:r$ is an integer m, then the epicycloid (common, prolate or curtate) is a closed algebraic curve of order $2(m+1)$, and the hypocycloid is a closed algebraic curve of order $2(m-1)$. Thus, the epicycloid of Fig. 504*a* (where $R:r=3:1$) is a curve of the eighth order, while the hypocycloid in Fig. 504*b* (here also $R:r=3:1$) is a curve of the fourth order. Both epicycloid and hypocycloid consist of m congruent branches.

If the ratio $R:r$ is a fraction which in lowest terms is of the form $\frac{p}{q}$ $(q \neq 1)$, then the epicycloid (hypocycloid) is also an algebraic curve [of order $2|p \pm q|$] and consists of p congruent branches. Thus, the common epicycloid in Fig. 507 $(R:r=3:2)$ is a curve of the 10th order and consists of three congruent branches.

If the ratio $R:r$ is an irrational number, then the epicycloid (hypocycloid) is not closed and has an infinity of branches which intersect one another

5. **Particular types.**

(1) For $R:r=2:1$ both the prolate and curtate hypocycloids constitute an ellipse with centre at O. The semiaxes of the ellipse are $a=r+d$; $b=|r-d|$; the end-points of the major axis are the starting points, the end-points of the minor axis are the vertices of the hypocycloid. This method of constructing ellipses is the underlying principle of an instrument for tracing ellipses

(1a) If for constant R and r connected by the relation $R:r=2:1$, the difference $r-d$ tends to zero, then the minor axis of the ellipse decreases indefinitely and the major axis tends to coincidence with the diameter of the directing circle.

The common hypocycloid obtained in the limiting case ($d=r$) is a segment of a straight line; namely, that diameter of the directing circle which joins the starting points. In one complete revolution of the generating circle, this diameter is traced out in one direction; in the next revolution, in the opposite direction. Thus, in this limiting case too the starting points of a common hypocycloid are cuspidal points.

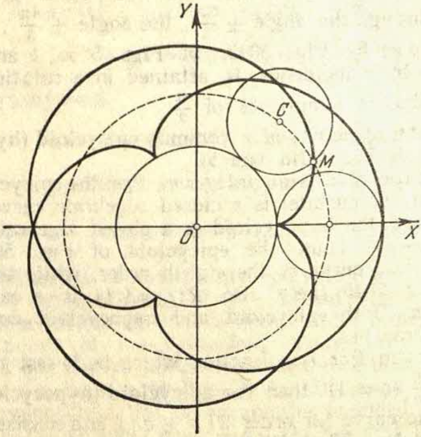

Fig. 507

(2) For $R=r$, each of the epicycloids represents a limaçon (Sec. 508); in particular, the common epicycloid of the type under consideration is nothing other than a cardioid.

(3) For $R:r=4:1$, the common hypocycloid is an *astroid* (Fig. 508); this curve is distinguished by the fact that the line-segment EF of its tangent that lies between two mutually perpendicular straight lines (passing through two pairs of opposite starting points) has the same length R. The equation of the astroid in rectangular coordinates (as in Fig. 508) is

$$x^{2/3}+y^{2/3}=R^{2/3}$$

or, in parametric form,

$$x=R\cos^3 u, \quad y=R\sin^3 u$$

6. Limiting cases.

Case 1. For an infinite radius of the directing circle and a given radius of the generating circle, an epicycloid (hypocycloid) turns into a cycloid (Sec. 514, Item 1) with the same radius of the generating circle.

Case 2. When the radius of the generating circle becomes infinite, it reduces to a straight line (*KL* in Fig. 509), which rolls without sliding along the directing circle *O*; then the epicycloid (hypocycloid) becomes a curve described by a point *M* fixed to the straight line *KL*.

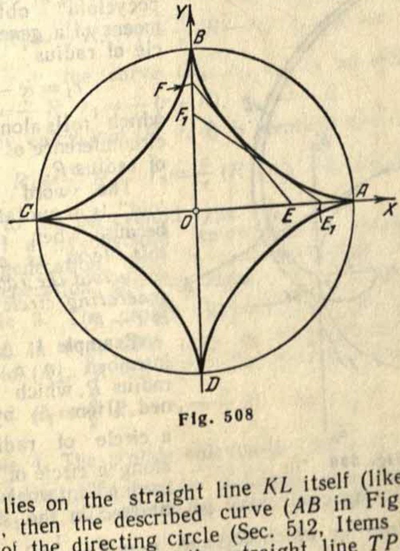

Fig. 508

If *M* lies on the straight line *KL* itself (like point *P* in Fig. 509), then the described curve (*AB* in Fig. 509) is the involute of the directing circle (Sec. 512, Items 1 and 2).

If the point, fixed to the straight line *TP*, lies on the same side as the directing circle (like point *M* in Fig. 509), then the projection *P* of this point traces out the involute *AB*, and point *M* itself traces out a *curtate involute of the circle*. This curve is the locus of the end-point of segment *PM* of a given length l laid off on the tangent *PT* to the involute *AB*; here, the direction of *PM* coincides with that of decreasing arc $\overset{\frown}{AP}$ of the involute.

But if the point attached to the straight line KL lies on the other side of this straight line, then it describes a *prolate involute*. This curve is constructed in similar fashion, with the sole difference that the segment of given length is laid off on tangent PT in the direction of increasing arc $\overset{\frown}{AP}$.

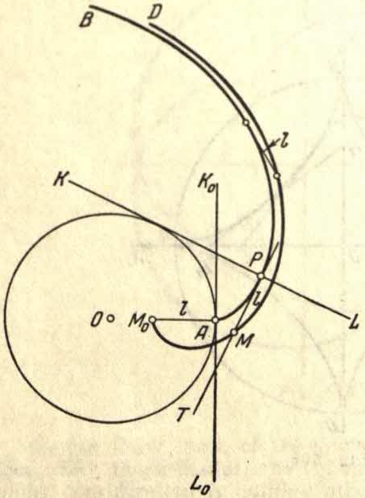

Fig. 509

7. **Double generation of hypocycloids and epicycloids.** The common hypocycloid obtained with the aid of a generating circle of radius r which rolls along the circumference of a circle of radius R is identical with a "hypocycloid" obtained by means of a generating circle of radius

$$r_1 = R - r \qquad (3)$$

which rolls along the same circumference of the circle of radius R.

The word "hypocycloid" is in quotation marks because when $r > R$, *by this term is meant an epicycloid the radius of the generating circle of which is* $r-R$.

Example 1. An astroid inscribed in a circle of radius R, which is obtained (Item 5) by rolling a circle of radius $\frac{1}{4}R$ along a circle of radius R having internal contact with the generating circle, may be obtained in the same way as a hypocycloid for which $R_1 = R$, $r_1 = R - \frac{1}{4}R = \frac{3}{4}R$.

Example 2. The common hypocycloid obtained by rolling a circle of radius $r = 4$ metres along the circumference of a circle of radius $R = 2$ m is identical with the "hypocycloid" obtained by rolling a circle of radius $r_1 = 2\text{ m} - 4\text{ m} = -2$ m along the circumference of a circle of radius $R_1 = 2$ m, that is, with an *epicycloid* for which $R_1 = 2$ m, $r_1 = 2$ m. *This epicycloid is a cardioid* (Item 5).

From the foregoing it follows that any common epicycloid (R, r) is identical with a hypocycloid[1] $(R, R+r)$. Thus, the common epicycloid of Fig. 504*a* $\left(r=\frac{1}{3}R\right)$ may be obtained as a hypocycloid corresponding to the values $R_1=R$, $r_1=\frac{4}{3}R$.

Double generation is also applicable to hypocycloids (epicycloids) of the general type, namely: a hypocycloid corresponding to given quantities R, r, d may be obtained in the same way as a "hypocycloid" (R_1, r_1, d_1), where R_1, r_1, d_1 are expressed in terms of R, r, d by the following formulas:

$$R_1=\frac{d}{r}R, \quad r_1=\frac{d}{r}(R-r), \quad d_1=R-r \tag{4}$$

When $R<r$,[2] the curve (R_1, r_1, d_1) is an epicycloid for which $R_1=\frac{d}{r}R$, $r_1=\frac{d}{r}(r-R)$, $d_1=r-R$.

Now any epicycloid (R, r, d) is identical to a hypocycloid

$$R_1=\frac{d}{r}R, \quad r_1=\frac{d}{r}(R+r), \quad d_1=R+r \tag{4a}$$

belonging to the pericycloid type.

Note. If a hypocycloid (epicycloid) is prolate in one mode of generation, then it becomes curtate in the other mode of generation, and vice versa.

Example 3. The prolate hypocycloid $\left(R, \frac{1}{4}R, \frac{3}{8}R\right)$ constructed in Fig. 506*b* may be obtained as a (curtate) hypocycloid (R_1, r_1, d_1), where [by formulas (4)]

$$R_1=\frac{3}{2}R, \quad r_1=\frac{9}{8}R, \quad d_1=\frac{3}{4}R$$

Example 4. The prolate epicycloid $\left(R, \frac{1}{4}R, \frac{3}{8}R\right)$ constructed in Fig. 506*a* may be obtained as a (curtate) hypocycloid (R_1, r_1, d_1), where [by formulas (4a)]

$$R_1=\frac{3}{2}R, \quad r_1=\frac{15}{8}R, \quad d_1=\frac{5}{4}R$$

8. **Property of normal and tangent.** The normal drawn through point M of any epicycloid (hypocycloid) passes through the corresponding point E of tangency of the generating circle to the directrix. The tangent to a common

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[1] Belonging to the type of pericycloids; see Item 4.

[2] That is, when the original hypocycloid is a pericycloid.

epicycloid (hypocycloid) passes through point E' of the generating circle diametrically opposite the point E (cf. Sec. 514, Item 8).

The mode of construction of the tangent is thus clear.

9. **The radius of curvature** $\bar{R}$ of any epicycloid is

$$\overline{R}=(R+r)\frac{\left(r^2+d^2-2dr\cos\frac{R\varphi}{r}\right)^{3/2}}{\left|r^3+d^2(R+r)-dr(R+2r)\cos\frac{R\varphi}{r}\right|} \tag{5}$$

The corresponding formula for the hypocycloid is obtained from (5) by replacing r by $-r$ and d by $-d$.

For the common epicycloid (hypocycloid) we get

$$\overline{R}=\frac{4r\,|R\pm r|}{|R\pm 2r|}\sin\frac{R\varphi}{2r} \tag{5a}$$

where the plus sign corresponds to the epicycloid and the minus sign to the hypocycloid.

Formula (5a) may be rewritten thus:

$$\overline{R}=2l\left|\frac{R\pm r}{R\pm 2r}\right| \tag{5b}$$

Here, l is the chord ME of the generating circle joining point M of the epicycloid (hypocycloid) to the corresponding support point E of the generating circle. Formula (5b) yields a simple method for constructing the centre of curvature

At the starting points of the common epicycloid (hypocycloid) $\overline{R}=0$.

At the vertices

$$\overline{R}=\frac{4r\,|R\pm r|}{|R\pm 2r|}$$

10. **Evolute.** The evolute of a common epicycloid or hypocycloid (i.e. the locus of its centres of curvature) is a curve similar to the original one. The similarity ratio for the epicycloid is $R:(R+2r)$ and for the hypocycloid, $R:(R-2r)$. The evolute has the same centre as the original epicycloid (hypocycloid). The vertices of the evolute coincide with the starting points of the original curve (cf. Sec. 514, Item 10) so that one of these curves may be obtained from the other by rotation through the angle $\pi\cdot\frac{r}{R}$ with subsequent proportional alteration of the distance from the centre.

Example. The evolute of the astroid $ABCD$ (Fig. 508), that is, of a hypocycloid for which $R=4r$, is also an astroid obtained from the given one by a rotation through an angle of 45° about the centre and a proportional alteration of the distances to the centre in the ratio $R:(R-2r)=4:2=2:1$. The starting points A, B, C, D will be vertices of the evolute.

11. The **arc length** s of an epicycloid between points $\varphi=0$, $\varphi=\varphi_1$ is

$$s=\frac{R+r}{r}\int_0^{\varphi_1}\sqrt{r^2+d^2-2rd\cos\frac{R\varphi}{2r}}\,d\varphi \tag{6}$$

The length of this arc is equal to that of the arc of the ellipse

$$x=2(d+r)\frac{R+r}{R}\cos\frac{R\varphi}{2r},\quad y=2(d-r)\frac{R+r}{R}\sin\frac{R\varphi}{2r} \tag{7}$$

between the points with the same values of the parameter φ.

Integral (6) cannot, in the general case, be expressed in terms of elementary functions of the argument φ_1. However, for the common epicycloid (the ellipse degenerates into a segment whose length is $8r$) we have

$$s=\frac{8r\,(R+r)}{R}\sin^2\frac{R\varphi_1}{4r} \tag{8}$$

In particular, the arc between two adjacent starting points is

$$8r\left(1+\frac{r}{R}\right) \tag{9}$$

The foregoing remains valid for the hypocycloid if we replace d, r by $-d$, $-r$ respectively.

12. The **natural equation** of the common epicycloid (hypocycloid) is

$$\frac{\bar{R}^2}{a^2}+\frac{(s-b)^2}{b^2}=1\quad (0\leqslant s\leqslant 2b) \tag{10}$$

where $a=\frac{4r\,(R\pm r)}{R\pm 2r}$, $b=\frac{4r\,(R\pm r)}{R}$, $\bar{R}$ is the radius of curvature, s, the arc length reckoned from one of the starting points. In the expressions for a, b, the upper signs refer to the case of an epicycloid, the lower signs, to the case of a hypocycloid. Eq. (10) is obtained by eliminating the parameter φ from (8) and (5a).

If we take one of the vertices for the origin of arc reckoning, then the natural equation will be

$$\frac{\overline{R}^2}{a^2}+\frac{s^2}{b^2}=1 \quad (-b \leqslant s \leqslant b) \tag{10a}$$

Cf. Sec 514, Item 13.

13. **Kinematic property.** Eq. (10) or (10a) expresses the following property in the language of kinematics: if an arc of a common epicycloid or a common hypocycloid rolls without sliding along a straight line AB, then the centre of curvature of the point of contact moves in an ellipse; the centre of the ellipse lies at the point of AB through which the vertex of the epicycloid (hypocycloid) rolls; one of the semiaxes coincides with AB and is equal in length to the half-branch of the epicycloid (hypocycloid) $\left|\frac{4r(R \pm r)}{R}\right|$, the other semiaxis is the radius of curvature at the vertex and is equal to $\left|\frac{4r(R \pm r)}{R \pm 2r}\right|$ Cf. Sec. 514, Item 14.

14. The **sectorial area** S described by a radius vector OM, which in the starting position leads to the starting point of the epicycloid, is expressed by the formula

$$S=\frac{R+r}{2}\left\{\left(R+r+\frac{d^2}{r}\right)\varphi-\frac{d\,(R+2r)}{R}\sin\frac{R\varphi}{r}\right\} \tag{11}$$

In particular, for the common epicycloid

$$S=\frac{(R+r)\,(R+2r)}{2}\left\{\varphi-\frac{r}{R}\sin\frac{R\varphi}{r}\right\} \tag{12}$$

(Newton).

In the case of a hypocycloid, we have to replace r by $-r$ in formulas (11) and (12).

In (11) and (12), the area is regarded as a directed quantity, that is, it is assumed that in those ranges of the parameter φ where the radius vector rotates in the negative direction, it sweeps out a negative area.

The area S_1 of the sector described by the radius vector OM of a common epicycloid (hypocycloid) when point M traverses one branch is expressed by the formula

$$S_1=\frac{\pi r\,(R \pm r)\,(R \pm 2r)}{R} \tag{13}$$

where the upper signs are taken for the epicycloid, the lower signs, for the hypocycloid.

The area S_2 of the corresponding sector of the directing circle is

$$S_2=\pi Rr \tag{14}$$

Therefore the area $\overline{S}$ of the figure bounded by one branch of a common epicycloid (hypocycloid) and the corresponding arc of the directing circle is expressed by the formula

$$S=|S_1-S_2|=\pi r^2\left|3\pm 2\frac{r}{R}\right| \tag{15}$$

Example. Consider a common hypocycloid for which $r:R=1:4$, that is, the astroid $ABCD$ (Fig. 508). By formula (15) we get

$$\overline{S}=\pi r^2\left|3-2\cdot\frac{1}{4}\right|=\frac{5}{2}\pi r^2=\frac{5}{32}\pi R^2 \tag{15a}$$

This is an area bounded by one of the branches of the astroid, for example, the branch AB, and the corresponding arc $\overset{\frown}{AB}$ of the directing circle O ($\overset{\frown}{AB}=90°$).

However, this same astroid may also be regarded as a hypocycloid for which $r:R=3:4$ (Item 7). Then, using (15), we get

$$\overline{S}=\pi r^2\left|3-2\cdot\frac{3}{4}\right|=\frac{3}{2}\pi r^2=\frac{27}{32}\pi R^2 \tag{15b}$$

This result may appear absurd, but one must take into account that now, for branch AB of the astroid, the corresponding arc is not $\overset{\frown}{AB}$ which contains 90°, but the second arc ($BCDA$) containing 270°, so that formula (15b) expresses that area which together with the area (15a) fills the entire circle O. Indeed, adding (15a) and (15b), we get

$$\frac{27}{32}\pi R^2+\frac{5}{32}\pi R^2=\pi R^2$$

15. **Historical background.** In order to explain the retrograde movements of the planets, the ancient Greek astronomers, following Hipparchus (second century B. C.), attributed to them a uniform motion along a circle *(epicycle)*, the centre of which was in uniform motion along another circle *(deferent)*. The curve traced out by a point under these conditions is an epicycloid. However, we do not know which of the geometrical properties were known to the ancients. In the middle of the 13th century, an outstanding Persian astronomer and mathematician Muhammed Nasr-ed-din (1201-1274) established that the point of the circumference of a circle rolling on a fixed circle of double radius and contacting it internally describes

the diameter of the fixed circle (Item 5). Independently of Nasr-ed-din this property was discovered by the Polish astronomer N. Copernicus (1473-1543). It is described in his celebrated "De Revolutionibus Orbium Coelestium" ("The Revolutions of the Heavenly Bodies") published in 1543. The theorem of Nasr-ed-din—Copernicus was widely used in applied mechanics.

The systematic study of epicycloids and hypocycloids was begun in 1525 by the noted German painter Albrecht Dürer (1471-1528), who employed geometrical methods in his painting. However, mathematicians were not acquainted with Dürer's investigations.

In the middle of the 17th century, Gérard Desargues (1593-1662) made a study of the properties of epicycloids in connection with the problem of designing gear wheels with minimum friction. Desargues combined profound mathematical ideas with engineering talents, but his results (like those in other fields) were not published, though they were known to his friends.

La Hire continued the studies of Desargues and published in 1675 a "Treatise on Epicycloids and their Employment in Mechanics", where he established a number of important properties, including the properties discussed in items 7, 8, 10, 11, 14 and 15.

Newton, in his immortal "Principia" ("The Mathematical Principles of Natural Philosophy") (1687) generalized the investigations of Huyghens on the cycloidal pendulum (Sec. 514, Item 17) and established that in a spherical field of gravitation the line of isochronous vibration of a pendulum is an epicycloid.

Being a natural generalization of cycloids, the epicycloids and hypocycloids repeatedly attracted the attention of investigators, particularly Leibniz, Euler and Daniel Bernoulli (1700-1782).

516. Tractrix

1. **Hictorical background.** In the year 1693, a French physician Claude Perrot posed the following problem: one end of an inextensible string is attached to a point M lying in a horizontal plane; the other end is in motion along a straight line $X'X$ lying in the same plane. What kind of curve will the point carried by the taut string trace out?

This problem was solved by Leibniz, who set up the differential equation of the curve, proceeding from the fact that the segment of its tangent line from the point of contact M to intersection with the straight line $X'X$ must be of constant length (equal to the length of the string). Independently of Leibniz and at the same time, the problem was solved by Huyghens, who gave it the name "tractory" (today it is called the *tractrix*) [1]

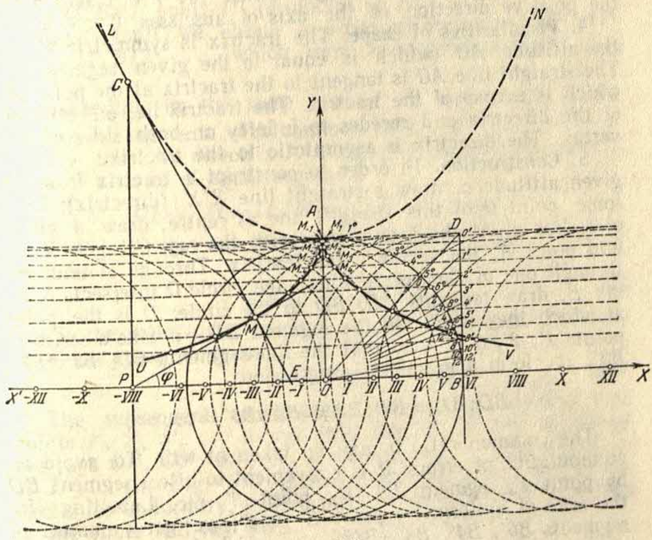

Fig. 510

2. Definition. The *tractrix* (Fig. 510) is the locus of points having the property that the segment MP of the tangent line from the point of tangency M to intersection with the given straight line $X'X$ *(directrix)* has a given magnitude a. Point A of the tractrix which is farthest from the directrix is called the *vertex*; the perpendicular AO dropped from the vertex to the directrix is the *altitude* of the tractrix.

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[1] More historical details are given in item 14.

3. The **parametric equations** (with x-axis the directrix of the tractrix and with the axis of ordinates directed along the altitude towards the vertex A) are

$$\left.\begin{aligned} x&=a\cos\varphi+a\ln\tan\frac{\varphi}{2}, \\ y&=a\sin\varphi \end{aligned}\right\} \tag{1}$$

where $\varphi=\angle XPM$ is the angle which ray PM makes with the positive direction of the axis of abscissas $(0<\varphi<\pi)$.

4. **Peculiarities of shape.** The tractrix is symmetric about the altitude AO (which is equal to the given segment a). The straight line AO is tangent to the tractrix at the point A, which is a cusp of the tractrix. The tractrix lies on one side of the directrix and recedes to infinity on both sides of the vertex. The directrix is asymptotic to the tractrix.

5. **Construction.** In order to construct a tractrix from its given altitude a, draw a straight line $X'X$ (directrix); from some point O of this straight line as centre, draw a circle of radius a. At the intersection with the ray $OY\perp X'X$ we find point A, the vertex of the tractrix. Through A and also through one of the points where the circle O intersects $X'X$, say B, draw tangents AD, BD to the circle; D is the point at which they meet. On the segment $BD=a$ take a series of points $1'$, $2'$, $3'$, ... so that the line-segments BD, $B1'$, $B2'$, $B3'$, ... form a geometric progression:

$$BD:B1'=B1':B2'=B2':B3'=\ldots=q$$

The common ratio q may be taken at will. To avoid an accumulation of errors, it is convenient to bisect segment BD by point $4'$, segment $B4'$ by point $8'$, etc. Labelling (for the sake of uniformity) point D as $0'$, we get a number of segments $B0'$, $B4'$, $B8'$, $B16'$, ... which form a progression with ratio $\frac{1}{2}$. Now, between points $0'$, $4'$ construct intermediate points $1'$, $2'$, $3'$ [1] in the following order: first find $2'$ so that $B2'$ is the mean proportional between $B0'$ and $B4'$. While we are at it, label point $6'$, which bisects $B2'$, and point $10'$, which bisects $B6'$. [2] We then get a series of segments $B0'$, $B2'$, $B4'$, $B6'$, $B8'$, $B10'$, ... which form a geometric progression with ratio $1:2^{1/2}$.

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[1] Having in view the division of segment $0'4'$ into four parts, we numbered its lower end $4'$; for greater accuracy we can divide this segment into 8, 16, etc. parts and then alter the numbering accordingly.

[2] The numbers 2, 6, 10 form an arithmetic progression with the same difference (4) as the numbers 0, 4, 8, 12, ...

Further, we construct point $1'$ so that segment $B1'$ is the mean proportional between $B0'$ and $B2'$, and then label point $5'$, the midpoint of segment $B1'$, and point $9'$, the midpoint of segment $B5'$, etc. Point $3'$ (extremity of segment $B3'$ the mean proportional between $B2'$ and $B4'$) is constructed in the same manner; then label point $7'$ (midpoint of $B3'$), point $11'$ (midpoint of $B7'$), etc.[1]

As a result we obtain a series of segments $B0'$, $B1'$, $B2'$, ..., $B12'$, ... which form a progression with ratio $1:2^{1/4}$

Proceeding in the same manner we could build a progression of segments with common ratio $1:2^{1/8}$, $1:2^{1/16}$, etc.

Now take the directrix $X'X$ and lay off on both sides of point O a series of equal segments

$$OI = I\,II = II\,III = III\,IV = \dots = d$$

Theoretically, the exact value of d is determined from the proportion

$$d:a = \ln(a:B1') \tag{2}$$

But if the ratio $a:B1'$ is close to 1, it is sufficient, for practical purposes, to take

$$d = 0'1'^{2)} \tag{2a}$$

The subsequent construction proceeds as follows: join points $1'$, $2'$, $3'$, ... to centre O and at the intersection of the rays $O1'$, $O2'$, $O3'$, ... with the circle, label points 1, 2, 3, ... (in Fig. 510 these labels are given inside the circle; for uniformity, label 0 denotes the point of intersection of the circle and ray OD).

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[1] Points $7'$, $9'$, $11'$ are not indicated in Fig. 510 to avoid confusion. Generally speaking, there is no need to indicate the points which fall inside earlier constructed segments that are too small, since such points will not increase the precision of the construction.

[2] In our case, when $a:B1' = 2^{1/4}$, the proportion (2) yields

$$d = a \ln\left(2^{1/4}\right) = \frac{1}{4} a \ln 2 \approx 0.173a$$

whereas when we assume $d = 0'1'$, we get

$$d = 0'1' = B0' - B1' = a\left(1 - \frac{1}{2^{1/4}}\right) = 0.160a$$

Thus, the error amounts to 7.5%.

Lay off on arc BA, from point B, arc $BI^{\circ}=2\ BI$, arc $B2^{\circ}=2 \cdot B2$, and so forth. Through the end-points I°, 2°, 3°,... of the doubled arcs (the appropriate numbers are given outside the circle in Fig. 510) draw straight lines parallel to the directrix $X'X$, [1] and from the points $\pm I$, $\pm II$, $\pm III$ as centres draw semicircles of radius a, as shown in Fig. 510 (the semicircles I II, III are concave towards increasing labels of the centres, and the semicircles $-I$, $-II$, $-III$ are concave towards decreasing labels).

Finally, we label the pair of points where the semicircles $\pm I$ intersect the straight line drawn through I°, the pair of points where the semicircles $\pm II$ intersect the straight line drawn through 2°, etc. All these pairwise symmetric points lie on the required curve.

6. **The tractrix as an orthogonal trajectory; approximate construction.** The orthogonal path of a family of circles of radius a with centres on a given straight line $X'X$ (i. e. a curve which intersects all these circles at right angles) is a tractrix. This family of circles has an infinite number of orthogonal paths: through every point of one of the circles there passes one tractrix orthogonal to this circle. One of the trajectories is shown in Fig. 510; the other is symmetric to it about $X'X$. The others are obtained by a parallel displacement of that pair of tractrices along $X'X$.

This property enables one to make a rather exact sketch of a tractrix in the following manner. Draw a number of semicircles of radius a with centres closely spaced on the straight line $X'X$ and, choosing on one of the circles an arbitrary point distant about $\frac{1}{3}a$ from $X'X$, draw through it by eye a curve which intersects a number of adjacent semicircles at right angles that is to say, which is directed along the corresponding radius each time. Or proceed as follows. Label the point of intersection of a radius of a circle (or the prolongation of that radius) with an adjacent semicircle; join the centre of the semicircle to the point thus found and label the point of intersection of the new radius with the following semicircle, etc. We obtain a polygonal curve which for all practical purposes takes the place of the required tractrix if the centres are sufficiently dense. The accuracy of the construction diminishes as we approach the vertex.

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[1] To do this, it is best to lay off on circle O, from point A, arcs symmetric to arcs AI°, $A2^{\circ}$, and join each of the points I°, 2° to the symmetric point.

7. **Construction of a tangent.** To construct a tangent line at a given point M of a tractrix with given vertex A and directrix $X'X$, it suffices on $X'X$ to strike an arc, at point P, from M as centre with radius $AO=a$. The straight line MP is the required tangent line.

8. **Radius of curvature:**

$$R=a\cot\varphi \tag{3}$$

Geometrically, this formula expresses (see Fig. 510) the fact that the *radius of curvature of the tractrix at point M is a segment MC of the normal from M to intersection with the straight line PC drawn perpendicular to the directrix $X'X$ through the point P of its intersection with the tangent at M.*

The thus constructed point C is the centre of curvature of the tractrix at the point M.

The radius of curvature at the vertex A is

$$R_A=a \tag{3a}$$

The radius of curvature MC and the segment of the normal ME (from M to the intersection E with the directrix) are connected by the relation

$$MC\cdot ME=a^2 \tag{4}$$

That is, the *radius of curvature MC and the segment of the normal ME are inversely proportional.*

9. **Evolute.** The evolute LAN of the tractrix (Fig. 510), that is, the locus of its centres of curvature C is a *catenary* (Sec. 517). In the OXY system of coordinates of Fig. 510, the equation of the evolute is of the form

$$\frac{y}{a}=\frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right) \tag{5}$$

or, what is the same,

$$\frac{y}{a}=\cosh\frac{x}{a} \tag{5a}$$

10. The **length s of the arc $\breve{AM}$** is given by the formula

$$s=a\ln\operatorname{cosec}\varphi=a\ln\frac{a}{y}$$

The difference $s-|x|$ between the length of the arc $\breve{AM}$ and the length of its projection on the directrix, as M recedes indefinitely from vertex A, tends to the limit $a(1-\ln 2)$:

$$\lim_{x\to\infty}(s-|x|)=a(1-\ln 2)\approx 0.307a \tag{6}$$

11. **Natural equation:**

$$s=a\ln\sqrt{\frac{R^2}{a^2}+1} \tag{7}$$

12. The **area** S of the infinite strip between the tractrix and its asymptote $X'X$ is one half the area of a circle whose radius is the altitude AO of the tractrix:

$$S=\frac{1}{2}\pi a^2 \tag{8}$$

13. The **solid of revolution** of the tractrix about the asymptote $X'X$ (which extends to infinity along $X'X$) has a finite surface S_1 equal to the surface of a sphere of radius R, and a finite volume V, equal to half the volume of the sphere:

$$S_1=4\pi a^2, \tag{9}$$

$$V=\frac{2}{3}\pi a^3 \tag{10}$$

14. **The tractrix and the pseudosphere.** The surface (Fig. 511) formed by the revolution of the tractrix about its asymptote is called a *pseudosphere*. The name is due to the profound analogy between it and the surface of a sphere. Thus, if three points B, C, D on the surface of a sphere are joined pairwise by shortest arcs, then in the resulting spherical triangle BCD the sum of the interior angles is always greater than π, and the excess of the sum $B+C+D$ over π is equal to the ratio of the surface S of the spherical triangle to the square of the radius a of the sphere:

$$(B+C+D)-\pi=\frac{S}{a^2} \tag{11}$$

Now if we take three points B, C, D (Fig. 511) on a pseudosphere (on one side of the parallel UV described by the vertex of the tractrix) and join them too by shortest arcs, then in the resulting pseudospherical triangle the sum of the interior angles will always be less than π, and the deficit in the sum of $B+C+D$ up to π will be equal to the ratio of the area S of the pseudospherical triangle to the square of the radius a of the parallel line UV:

$$\pi-(B+C+D)=\frac{S}{a^2} \tag{12}$$

It is a remarkable fact that *rectilinear* triangles in Lobachevskian geometry possess the property (12). And, generally, any patch of the pseudosphere which does not contain points of the parallel *UV* exhibits all (without exception) properties which any piece of a plane in Lobachevskian geometry possesses. This discovery was made in 1863 by the Italian geometer E. Beltrami (1835-1900) and did much to dispel the distrust of Lobachevsky's geometry on the part of nearly all mathematicians including some very outstanding ones.

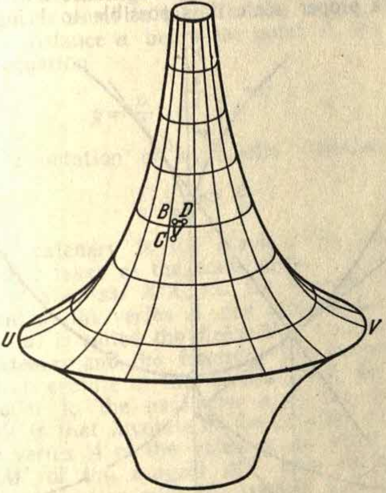

Fig. 511

517. Catenary

1. **Definition.** The *catenary* is a curve in which a homogeneous inextensible string hangs when suspended from two fixed end-points.

Note 1. In the original statement of the problem (see Item 9) we dealt with the line of suspension of a chain, whence the name (chain-line, or catenary). In replacing the chain by a string, we are able to disregard a number of circumstances (size of links, friction, etc.) which complicate the investigation. The intensity of gravity is assumed constant in magnitude and direction.

Note 2. The arc of suspension is shaped differently depending on the position of the points P, Q from which the string is suspended, and on the length l of the string ($l > PQ$). However, an investigation shows that by depicting arc PQ on a proper scale it is possible to bring it to coincidence with some arc P_0Q_0 (Fig. 512) of quite a definite infinite curve LAN. It is to this infinite curve as a whole (and not to the arc of suspension, which constitutes only a part of it) that the name "catenary" refers.

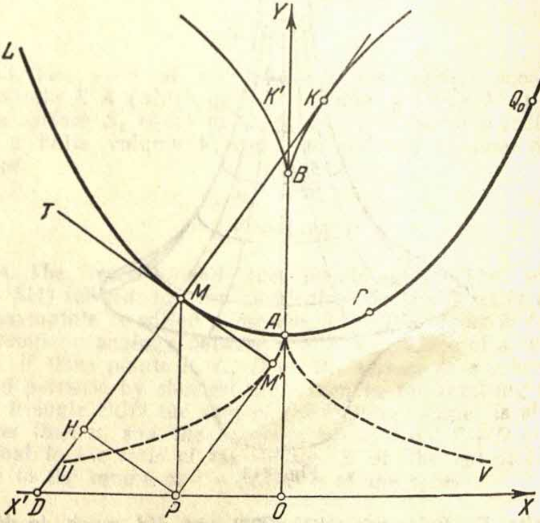

Fig. 512

The lowest point A of the catenary curve is called its *vertex*.

2. Equation. If for the coordinate origin we take the vertex of the catenary (this appears to be rather natural) and take the axis of ordinates vertically upwards, the catenary curve is given by the equation

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)-a \qquad (1)$$

where a (the *parameter* of the catenary) is the length of a segment of the string such that its weight is equal to the horizontal component of the tension of the string (this component is constant throughout the arc of suspension).

However, it is usual to take for the origin the point O, which is at a distance a below the point A. We then have the simpler equation

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right) \tag{2}$$

or, using the notation of hyperbolic functions (Sec. 403),

$$\frac{y}{a}=\cosh\frac{x}{a} \tag{2a}$$

Thus, the catenary is the graph of the function $\cosh x$ (if segment a is taken as the scale unit).

The axis of abscissas $X'X$, i.e. the straight line parallel to the tangent at the vertex A and lying below the vertex at a distance a, is called the *directrix* of the catenary curve.

3. **The catenary and the tractrix.** The catenary (LAN in Fig. 512) is an evolute of the tractrix UAV, the altitude of which is equal to the parameter a of the catenary. The tractrix UAV is that involute of the catenary whose starting point is the vertex A of the catenary. In other words, the segment MM' of the tangent MT from the point M of tangency to intersection with the tractrix UAV at point M' is equal in length to the arc MA of the catenary.

4. **Construction.** To construct a catenary with a given parameter a, find a number of points of a tractrix of altitude a (Sec. 516, Item 5). Then join each such point M' (Fig. 512) to the centre P of the corresponding semicircle. The straight line $M'P$ is tangent to the tractrix. Now draw the normal $M'M$ of the tractrix ($MM' \perp M'P$) to intersection, at point M, with the perpendicular PM erected from point P to the directrix $X'X$. The point M (the centre of curvature of the tractrix) lies on the desired catenary curve LAN.

Note. The normal $M'M$ of the tractrix is the tangent of its evolute LAN (Sec. 346, Item 1). This property simplifies drawing a smooth curve through the series of constructed points M. At the same time it permits checking the accuracy of the construction.

5. **Arc length.** The length s of the arc $\overset{\frown}{AM}$ of the catenary reckoned from the vertex A is equal to the projection MM' of the ordinate PM on the tangent MT and is given

by the formula

$$s = \overset{\frown}{AM} = MM' = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \tag{3}$$

or

$$s = a \sinh \frac{x}{a} \tag{3a}$$

The arc s is connected with the ordinate $PM = y$ by the relation

$$s^2 + a^2 = y^2 \tag{4}$$

This relation follows from (2) and (3) and is evident from the triangle $PM'M$ where $PM = y$, $MM' = s$ and $PM' = a$ (by the basic property of the tractrix).

6. **Projection of an ordinate on the normal.** The projection MH of the ordinate MP of the catenary on the normal MD is of constant length a:

$$HM = OA = a \tag{5}$$

This relation is read from the rectangle $MM'PH$, where $MH = M'P = a$ (by the basic property of the tractrix).

7. **Radius of curvature.** The radius of curvature $MK = R$ of the catenary is equal to the segment MD of the normal from point M to the directrix $X'X$ and is given by the formula

$$R = MD = \frac{a}{4}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^2 \tag{6}$$

or

$$R = a \cosh^2 \frac{x}{a} \tag{6a}$$

8. **Constructing the centre of curvature; the evolute of the catenary.** To construct the centre of curvature of the catenary at a given point M of it, continue the normal MD beyond M and lay off segment $MK = MD$. Point K is the desired centre of curvature. That is how we construct, by points, the curve $K'BK$ traced out by the centre of curvature (that is, the evolute of the catenary). Its parametric equations are

$$\left.\begin{aligned} x_K &= a\left[\cosh \frac{x}{a} \sinh \frac{x}{a} + \ln\left(\cosh \frac{x}{a} - \sinh \frac{x}{a}\right)\right], \\ y_K &= 2a \cosh \frac{x}{a} \end{aligned}\right\} \tag{7}$$

The point B (centre of curvature for vertex A) is a cusp of the evolute (7).

9. **Natural equation of the catenary:**

$$R=\frac{s^2}{a}+a \tag{8}$$

It is obtained from (3a) and (6a) by eliminating x. In the language of kinematics, Eq. (8) signifies the following: if a catenary curve rolls without sliding along a straight line, then the centre of curvature of the point of tangency describes a parabola, the axis of which is vertical; the vertex lies at point B; the parameter of the parabola is equal to the semiparameter $\frac{a}{2}$ of the catenary curve.

10. The **area** S of the "curvilinear trapezoid" $OAMP$ ($OA=a$ is the ordinate of the vertex; PM, the ordinate of the end-point M of the arc $\overset{\frown}{AM}=s$) is equal to the area of a rectangle with sides a,s so that

$$S=as=a^2\sinh^2\frac{x}{a} \tag{9}$$

11. **Historical background.** When the suspension points of a chain are at the same height and the chain is somewhat longer than the distance between the suspension points, the arc of suspension appears to be identical with the arc of a parabola. That was believed to be the case for a long time. The studies of Galileo in the field of mechanics cast doubt on the correctness of this view, but Galileo was not able either to corroborate or refute the idea. In 1669 Jungius established, both theoretically and experimentally, that the line of suspension of the chain is not a parabola. But the mathematics of that day was not sufficiently equipped to find the true shape of the curve. Soon after Newton and Leibniz worked out the methods of infinitesimal analysis it was possible to solve the problem of the curve of suspension of a chain. The problem was formulated in 1690 by James Bernoulli, and was immediately solved by his brother John Bernoulli, Huyghens and Leibniz.

James Bernoulli posed another problem as well: disregarding the weight of a sail filled with wind, find the shape (line of profile) of the sail. James himself was only able to set up the differential equation. John Bernoulli solved it. It turned out that the sought-for profile was the catenary curve.

In 1744, Euler posed and solved the following problem: given a straight line AB and two points C, D (not on AB) in a plane; draw through A and B a curve such that the surface formed by its rotation about the axis AB is a minimum area. This too proved to be a catenary curve (the straight line AB is its directrix).

The surface of revolution of a catenary about its directrix (*catenoid*)[1] has a still more general property, namely: *any* piece of it has a smaller area than any other surface bounded by the same contour. This property of the catenoid was discovered in 1776 by the celebrated French mathematician, engineer and officer Jean B. M. C. Meusnier. A whole class of surfaces (so-called *minimal* surfaces)[2] have this property. However, the catenoid is the only surface of this class from among the surfaces of revolution.

The value of the catenary curve in engineering practice stems, incidentally, from the fact that the weight proper of an arch having the shape of a catenary does not affect the depression of the arch

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[1] From the Latin *catena* meaning "chain".

[2] Meusnier indicated yet another minimal surface: the *helicoid* (it is formed by helical motion of a horizontal straight line intersecting a vertical axis).

TABLES

1. Natural Logarithms [1]

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0100	0.0198	0.0296	0.0392	0.0488	0.0583	0.0677	0.0770	0.0862
1.1	0.0953	0.1044	0.1133	0.1222	0.1310	0.1398	0.1484	0.1570	0.1655	0.1740
1.2	0.1823	0.1906	0.1989	0.2070	0.2151	0.2231	0.2311	0.2390	0.2469	0.2546
1.3	0.2624	0.2700	0.2776	0.2852	0.2927	0.3001	0.3075	0.3148	0.3221	0.3293
1.4	0.3365	0.3436	0.3507	0.3577	0.3646	0.3716	0.3784	0.3853	0.3920	0.3988
1.5	0.4055	0.4121	0.4187	0.4253	0.4318	0.4383	0.4447	0.4511	0.4574	0.4637
1.6	0.4700	0.4762	0.4824	0.4886	0.4947	0.5008	0.5068	0.5128	0.5188	0.5247
1.7	0.5306	0.5365	0.5423	0.5481	0.5539	0.5596	0.5653	0.5710	0.5766	0.5822
1.8	0.5878	0.5933	0.5988	0.6043	0.6098	0.6152	0.6206	0.6259	0.6313	0.6366
1.9	0.6419	0.6471	0.6523	0.6575	0.6627	0.6678	0.6729	0.6780	0.6831	0.6881
2.0	0.6931	0.6981	0.7031	0.7080	0.7129	0.7178	0.7227	0.7275	0.7324	0.7372
2.1	0.7419	0.7467	0.7514	0.7561	0.7608	0.7655	0.7701	0.7747	0.7793	0.7839
2.2	0.7885	0.7930	0.7975	0.8020	0.8065	0.8109	0.8154	0.8198	0.8242	0.8286
2.3	0.8329	0.8372	0.8416	0.8459	0.8502	0.8544	0.8587	0.8629	0.8671	0.8713
2.4	0.8755	0.8796	0.8838	0.8879	0.8920	0.8961	0.9002	0.9042	0.9083	0.9123
2.5	0.9163	0.9203	0.9243	0.9282	0.9322	0.9361	0.9400	0.9439	0.9478	0.9517
2.6	0.9555	0.9594	0.9632	0.9670	0.9708	0.9746	0.9783	0.9821	0.9858	0.9895
2.7	0.9933	0.9969	1.0006	1.0043	1.0080	1.0116	1.0152	1.0188	1.0225	1.0260
2.8	1.0296	1.0332	1.0367	1.0403	1.0438	1.0473	1.0508	1.0543	1.0578	1.0613
2.9	1.0647	1.0682	1.0716	1.0750	1.0784	1.0818	1.0852	1.0886	1.0919	1.0953

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[1] The natural logarithm of a number not found among the arguments of this table is determined as follows. Let it be required to find ln 753. We have $\ln 753 = \ln(7.53 \cdot 10^2) = \ln 7.53 + 2 \ln 10$. We find the first term from the table of natural logarithms, the second from Table III. This yields: $\ln 753 = 2.0189 + 4.6052 = 6.6241$.

We thus obtain $\ln 0.00753 = \ln(7.53 \cdot 10^{-3}) = 2.0189 - 6.9078 = -4.8889$.

Continued

N	0	1	2	3	4	5	6	7	8	9
3.0	1.0986	1.1019	1.1053	1.1086	1.1119	1.1151	1.1184	1.1217	1.1249	1.1282
3.1	1.1314	1.1346	1.1378	1.1410	1.1442	1.1474	1.1506	1.1537	1.1569	1.1600
3.2	1.1632	1.1663	1.1694	1.1725	1.1756	1.1787	1.1817	1.1848	1.1878	1.1909
3.3	1.1939	1.1969	1.2000	1.2030	1.2060	1.2090	1.2119	1.2149	1.2179	1.2208
3.4	1.2238	1.2267	1.2296	1.2326	1.2355	1.2384	1.2413	1.2442	1.2470	1.2499
3.5	1.2528	1.2556	1.2585	1.2613	1.2641	1.2669	1.2698	1.2726	1.2754	1.2782
3.6	1.2809	1.2837	1.2865	1.2892	1.2920	1.2947	1.2975	1.3002	1.3029	1.3056
3.7	1.3083	1.3110	1.3137	1.3164	1.3191	1.3218	1.3244	1.3271	1.3297	1.3324
3.8	1.3350	1.3376	1.3403	1.3429	1.3455	1.3481	1.3507	1.3533	1.3558	1.3584
3.9	1.3610	1.3635	1.3661	1.3686	1.3712	1.3737	1.3762	1.3788	1.3813	1.3838
4.0	1.3863	1.3888	1.3913	1.3938	1.3962	1.3987	1.4012	1.4036	1.4061	1.4085
4.1	1.4110	1.4134	1.4159	1.4183	1.4207	1.4231	1.4255	1.4279	1.4303	1.4327
4.2	1.4351	1.4375	1.4398	1.4422	1.4446	1.4469	1.4493	1.4516	1.4540	1.4563
4.3	1.4586	1.4609	1.4633	1.4656	1.4679	1.4702	1.4725	1.4748	1.4770	1.4793
4.4	1.4816	1.4839	1.4861	1.4884	1.4907	1.4929	1.4951	1.4974	1.4996	1.5019
4.5	1.5041	1.5063	1.5085	1.5107	1.5129	1.5151	1.5173	1.5195	1.5217	1.5239
4.6	1.5261	1.5282	1.5304	1.5326	1.5347	1.5369	1.5390	1.5412	1.5433	1.5454
4.7	1.5476	1.5497	1.5518	1.5539	1.5560	1.5581	1.5602	1.5623	1.5644	1.5665
4.8	1.5686	1.5707	1.5728	1.5748	1.5769	1.5790	1.5810	1.5831	1.5851	1.5872
4.9	1.5892	1.5913	1.5933	1.5953	1.5974	1.5994	1.6014	1.6034	1.6054	1.6074
5.0	1.6094	1.6114	1.6134	1.6154	1.6174	1.6194	1.6214	1.6233	1.6253	1.6273
5.1	1.6292	1.6312	1.6332	1.6351	1.6371	1.6390	1.6409	1.6429	1.6448	1.6467
5.2	1.6487	1.6506	1.6525	1.6544	1.6563	1.6582	1.6601	1.6620	1.6639	1.6658
5.3	1.6677	1.6696	1.6715	1.6734	1.6752	1.6771	1.6790	1.6808	1.6827	1.6845
5.4	1.6864	1.6882	1.6901	1.6919	1.6938	1.6956	1.6974	1.6993	1.7011	1.7029

Continued

N	0	1	2	3	4	5	6	7	8	9
5.5	1.7047	1.7066	1.7084	1.7102	1.7120	1.7138	1.7156	1.7174	1.7192	1.7210
5.6	1.7228	1.7246	1.7263	1.7281	1.7299	1.7317	1.7334	1.7352	1.7370	1.7387
5.7	1.7405	1.7422	1.7440	1.7457	1.7475	1.7492	1.7509	1.7527	1.7544	1.7561
5.8	1.7579	1.7596	1.7613	1.7630	1.7647	1.7664	1.7681	1.7699	1.7716	1.7733
5.9	1.7750	1.7766	1.7783	1.7800	1.7817	1.7834	1.7851	1.7867	1.7884	1.7901
6.0	1.7918	1.7934	1.7951	1.7967	1.7984	1.8001	1.8017	1.8034	1.8050	1.8066
6.1	1.8083	1.8099	1.8116	1.8132	1.8148	1.8165	1.8181	1.8197	1.8213	1.8229
6.2	1.8245	1.8262	1.8278	1.8294	1.8310	1.8326	1.8342	1.8358	1.8374	1.8390
6.3	1.8405	1.8421	1.8437	1.8453	1.8469	1.8485	1.8500	1.8516	1.8532	1.8547
6.4	1.8563	1.8579	1.8594	1.8610	1.8625	1.8641	1.8656	1.8672	1.8687	1.8703
6.5	1.8718	1.8733	1.8749	1.8764	1.8779	1.8795	1.8810	1.8825	1.8840	1.8856
6.6	1.8871	1.8886	1.8901	1.8916	1.8931	1.8946	1.8961	1.8976	1.8991	1.9006
6.7	1.9021	1.9036	1.9051	1.9066	1.9081	1.9095	1.9110	1.9125	1.9140	1.9155
6.8	1.9169	1.9184	1.9199	1.9213	1.9228	1.9242	1.9257	1.9272	1.9286	1.9301
6.9	1.9315	1.9330	1.9344	1.9359	1.9373	1.9387	1.9402	1.9416	1.9430	1.9445
7.0	1.9459	1.9473	1.9488	1.9502	1.9516	1.9530	1.9544	1.9559	1.9573	1.9587
7.1	1.9601	1.9615	1.9629	1.9643	1.9657	1.9671	1.9685	1.9699	1.9713	1.9727
7.2	1.9741	1.9755	1.9769	1.9782	1.9796	1.9810	1.9824	1.9838	1.9851	1.9865
7.3	1.9879	1.9892	1.9906	1.9920	1.9933	1.9947	1.9961	1.9974	1.9988	2.0001
7.4	2.0015	2.0028	2.0042	2.0055	2.0069	2.0082	2.0096	2.0109	2.0122	2.0136
7.5	2.0149	2.0162	2.0176	2.0189	2.0202	2.0215	2.0229	2.0242	2.0255	2.0268
7.6	2.0281	2.0295	2.0308	2.0321	2.0334	2.0347	2.0360	2.0373	2.0386	2.0399
7.7	2.0412	2.0425	2.0438	2.0451	2.0464	2.0477	2.0490	2.0503	2.0516	2.0528
7.8	2.0541	2.0554	2.0567	2.0580	2.0592	2.0605	2.0618	2.0631	2.0643	2.0656
7.9	2.0669	2.0681	2.0694	2.0707	2.0719	2.0732	2.0744	2.0757	2.0769	2.0782

IV. The Exponential Function e^x (natural antilogarithms)

	0	1	2	3	4	5	6	7	8	9
0.0	1.0000	1.0101	1.0202	1.0305	1.0408	1.0513	1.0618	1.0725	1.0833	1.0942
0.1	1.1052	1.1163	1.1275	1.1388	1.1503	1.1618	1.1735	1.1853	1.1972	1.2092
0.2	1.2214	1.2337	1.2461	1.2586	1.2712	1.2840	1.2969	1.3100	1.3231	1.3364
0.3	1.3499	1.3634	1.3771	1.3910	1.4049	1.4191	1.4333	1.4477	1.4623	1.4770
0.4	1.4918	1.5068	1.5220	1.5373	1.5527	1.5683	1.5841	1.6000	1.6161	1.6323
0.5	1.6487	1.6653	1.6820	1.6989	1.7160	1.7333	1.7507	1.7683	1.7860	1.8040
0.6	1.8221	1.8404	1.8589	1.8776	1.8965	1.9155	1.9348	1.9542	1.9739	1.9937
0.7	2.0138	2.0340	2.0544	2.0751	2.0959	2.1170	2.1383	2.1598	2.1815	2.2034
0.8	2.2255	2.2479	2.2705	2.2933	2.3164	2.3396	2.3632	2.3869	2.4109	2.4351
0.9	2.4596	2.4843	2.5093	2.5345	2.5600	2.5857	2.6117	2.6379	2.6645	2.6912
1.0	2.7183	2.7456	2.7732	2.8011	2.8292	2.8577	2.8864	2.9154	2.9447	2.9743
1.1	3.0042	3.0344	3.0649	3.0957	3.1268	3.1582	3.1899	3.2220	3.2544	3.2871
1.2	3.3201	3.3535	3.3872	3.4212	3.4556	3.4903	3.5254	3.5609	3.5966	3.6328
1.3	3.6693	3.7062	3.7434	3.7810	3.8190	3.8574	3.8962	3.9354	3.9749	4.0149
1.4	4.0552	4.0960	4.1371	4.1787	4.2207	4.2631	4.3060	4.3492	4.3929	4.4371
1.5	4.4817	4.5267	4.5722	4.6182	4.6646	4.7115	4.7588	4.8066	4.8550	4.9037
1.6	4.9530	5.0028	5.0531	5.1039	5.1552	5.2070	5.2593	5.3122	5.3656	5.4195
1.7	5.4739	5.5290	5.5845	5.6407	5.6973	5.7546	5.8124	5.8709	5.9299	5.9895
1.8	6.0496	6.1104	6.1719	6.2339	6.2965	6.3598	6.4237	6.4883	6.5535	6.6194
1.9	6.6859	6.7531	6.8210	6.8895	6.9588	7.0287	7.0993	7.1707	7.2427	7.3155

Continued

	0	1	2	3	4	5	6	7	8	9
2.0	7.3891	7.4633	7.5383	7.6141	7.6906	7.7679	7.8460	7.9248	8.0045	8.0849
2.1	8.1662	8.2482	8.3311	8.4149	8.4994	8.5849	8.6711	8.7583	8.8463	8.9352
2.2	9.0250	9.1157	9.2073	9.2999	9.3933	9.4877	9.5831	9.6794	9.7767	9.8749
2.3	9.9742	10.074	10.176	10.278	10.381	10.486	10.591	10.697	10.805	10.913
2.4	11.023	11.134	11.246	11.359	11.473	11.588	11.705	11.822	11.941	12.061
2.5	12.182	12.305	12.429	12.554	12.680	12.807	12.936	13.066	13.197	13.330
2.6	13.464	13.599	13.736	13.874	14.013	14.154	14.296	14.440	14.585	14.732
2.7	14.880	15.029	15.180	15.333	15.487	15.643	15.800	15.959	16.119	16.281
2.8	16.445	16.610	16.777	16.945	17.116	17.288	17.462	17.637	17.814	17.993
2.9	18.174	18.357	18.541	18.728	18.916	19.106	19.298	19.492	19.688	19.886
3.0	20.086	20.287	20.491	20.697	20.905	21.115	21.328	21.542	21.758	21.977
3.1	22.198	22.421	22.646	22.874	23.104	23.336	23.571	23.807	24.047	24.288
3.2	24.533	24.779	25.028	25.280	25.534	25.790	26.050	26.311	26.576	26.843
3.3	27.113	27.385	27.660	27.938	28.219	28.503	28.789	29.079	29.371	29.666
3.4	29.964	30.265	30.569	30.877	31.187	31.500	31.817	32.137	32.460	32.786
3.5	33.115	33.448	33.784	34.124	34.467	34.813	35.163	35.517	35.874	36.234
3.6	36.598	36.966	37.338	37.713	38.092	38.475	38.861	39.252	39.646	40.045
3.7	40.447	40.854	41.264	41.679	42.098	42.521	42.948	43.380	43.816	44.256
3.8	44.701	45.150	45.604	46.063	46.525	46.993	47.465	47.942	48.424	48.911
3.9	49.402	49.899	50.400	50.907	51.419	51.935	52.457	52.985	53.517	54.055

V. Table of Indefinite Integrals

1. Functions Containing $a+bx$ to Integral Powers

(1) $\int \frac{dx}{a+bx}=\frac{1}{b}\ln(a+bx)+C$

(2) $\int (a+bx)^n\,dx=\frac{(a+bx)^{n+1}}{b(n+1)}+C,\ n\neq -1$

(3) $\int \frac{x\,dx}{1+bx}=\frac{1}{b^2}[a+bx-a\ln(a+bx)]+C$

(4) $\int \frac{x^2\,dx}{a+bx}=\frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2-2a(a+bx)+\right.$
$\left.+a^2\ln(a+bx)\right]+C$

(5) $\int \frac{dx}{x(a+bx)}=-\frac{1}{a}\ln\frac{a+bx}{x}+C$

(6) $\int \frac{dx}{x^2(a+bx)}=-\frac{1}{ax}+\frac{b}{a^2}\ln\frac{a+bx}{x}+C$

(7) $\int \frac{x\,dx}{(a+bx)^2}=\frac{1}{b^2}\left[\ln(a+bx)+\frac{a}{a+bx}\right]+C$

(8) $\int \frac{x^2\,dx}{(a+bx)^2}=\frac{1}{b^3}\left[a+bx-2a\ln(a+bx)-\frac{a^2}{a+bx}\right]+C$

(9) $\int \frac{dx}{x(a+bx)^2}=\frac{1}{a(a+bx)}-\frac{1}{a^2}\ln\frac{a+bx}{x}+C$

(10) $\int \frac{x\,dx}{(a+bx)^3}=\frac{1}{b^2}\left[-\frac{1}{a+bx}+\frac{a}{2(a+bx)^2}\right]+C$

2. Functions Containing a^2+x^2, a^2+x^2, $a+bx^2$

(11) $\int \frac{dx}{1+x^2}=\arctan x+C$

(12) $\int \frac{dx}{a^2+x^2}=\frac{1}{a}\arctan\frac{x}{a}+C$

(13) $\int \frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\frac{a+x}{a-x}+C$

or

(14) $\int \frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\frac{x+a}{x-a}+C$

(15) $\int \frac{dx}{a+bx^2}=\frac{1}{\sqrt{ab}}\arctan x\sqrt{\frac{b}{a}}+C$ for $a>0$ and $b>0$

If a and b are negative, the minus sign is taken outside the integral, but if a and b are of different sign, then use No. 16.

(16) $$\int \frac{dx}{a-bx^2}=\frac{1}{2\sqrt{ab}}\ln\frac{\sqrt{a}+x\sqrt{b}}{\sqrt{a}-x\sqrt{b}}+C$$

(17) $$\int \frac{x\,dx}{a+bx^2}=\frac{1}{2b}\ln\left(x^2+\frac{a}{b}\right)+C$$

(18) $$\int \frac{x^2\,dx}{a+bx^2}=\frac{x}{b}-\frac{a}{b}\int\frac{dx}{a+bx^2}$$

see No. 15 or No. 16.

(19) $$\int \frac{dx}{x\,(a+bx^2)}=\frac{1}{2a}\ln\frac{x^2}{a+bx^2}+C$$

(20) $$\int \frac{dx}{x^2\,(a+bx^2)}=-\frac{1}{ax}-\frac{b}{a}\int\frac{dx}{a+bx^2}$$

see No. 15 or No. 16.

(21) $$\int \frac{dx}{(a+bx^2)^2}=\frac{x}{2a\,(a+bx^2)}+\frac{1}{2a}\int\frac{dx}{a+bx^2}$$

see No. 15 or No. 16.

3. Functions Containing $\sqrt{a+bx}$

(22) $$\int \sqrt{a+bx}\,dx=\frac{2}{3b}\sqrt{(a+bx)^3}+C$$

(23) $$\int x\sqrt{a+bx}\,dx=-\frac{2\,(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}+C$$

(24) $$\int x^2\sqrt{a+bx}\,dx=\frac{2\,(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}+C$$

(25) $$\int \frac{x\,dx}{\sqrt{a+bx}}=-\frac{2\,(2a-bx)}{3b^2}\sqrt{a+bx}+C$$

(26) $$\int \frac{x^2\,dx}{\sqrt{a+bx}}=\frac{2\,(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}+C$$

(27) $$\int \frac{dx}{x\sqrt{a+bx}}=\frac{1}{\sqrt{a}}\ln\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}+C \text{ for } a>0$$

(28) $$\int \frac{dx}{x\sqrt{a+bx}}=\frac{2}{\sqrt{-a}}\arctan\sqrt{\frac{a+bx}{-a}}+C \text{ for } a<0$$

(29) $$\int \frac{dx}{x^2\sqrt{a+bx}}=\frac{-\sqrt{a+bx}}{ax}-\frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}}$$

see No. 27 or No. 28.

(30) $\int \frac{\sqrt{a+bx}\,dx}{x} = 2\sqrt{a+bx} + a\int \frac{dx}{x\sqrt{a+bx}}$

see No. 27 or No. 28.

4. Functions Containing $\sqrt{x^2+a^2}$

(31) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + C$

(32) $\int \sqrt{(x^2+a^2)^3}\,dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} +$

$+ \frac{3a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right) + C$

(33) $\int x\sqrt{x^2+a^2}\,dx = \frac{\sqrt{(x^2+a^2)^3}}{3} + C$

(34) $\int x^2\sqrt{x^2+a^2}\,dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} -$

$- \frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right) + C$

(35) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) + C$

(36) $\int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C$

(37) $\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2} + C$

(38) $\int \frac{x^2\,dx}{\sqrt{x^2+a^2}} = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + C$

(39) $\int \frac{x^2\,dx}{\sqrt{(x^2+a^2)^3}} = -\frac{x}{\sqrt{x^2+a^2}} + \ln\left(x+\sqrt{x^2+a^2}\right) + C$

(40) $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\frac{x}{a+\sqrt{x^2+a^2}} + C$

(41) $\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x} + C$

(42) $\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\ln\frac{a+\sqrt{x^2+a^2}}{x} + C$

(43) $\int \frac{\sqrt{x^2+a^2}\,dx}{x} = \sqrt{x^2+a^2} - a\ln\frac{a+\sqrt{x^2+a^2}}{x} + C$

(44) $\int \frac{\sqrt{x^2+a^2}\,dx}{x^2} = -\frac{\sqrt{x^2+a^2}}{x} + \ln\left(x+\sqrt{x^2+a^2}\right) + C$

5. Functions Containing $\sqrt{a^2-x^2}$

(45) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

(46) $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$

(47) $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C$

(48) $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2} + C$

(49) $\int \frac{x\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{1}{\sqrt{a^2-x^2}} + C$

(50) $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin \frac{x}{a} + C$

(51) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin \frac{x}{a} + C$

(52) $\int \sqrt{(a^2-x^2)^3}\,dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} +$
$+ \frac{3a^4}{8}\arcsin \frac{x}{a} + C$

(53) $\int x\sqrt{a^2-x^2}\,dx = -\frac{\sqrt{(a^2-x^2)^3}}{3} + C$

(54) $\int x\sqrt{(a^2-x^2)^3}\,dx = -\frac{\sqrt{(a^2-x^2)^5}}{5} + C$

(55) $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} +$
$+ \frac{a^4}{8}\arcsin \frac{x}{a} + C$

(56) $\int \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin \frac{x}{a} + C$

(57) $\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln \frac{x}{a+\sqrt{a^2-x^2}} + C$

(58) $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

(59) $\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} + \frac{1}{2a^3}\ln \frac{x}{a+\sqrt{a^2-x^2}} + C$

(60) $\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a \ln \frac{a+\sqrt{a^2-x^2}}{x} + C$

(61) $\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \arcsin \frac{x}{a} + C$

6. Functions Containing $\sqrt{x^2-a^2}$

(62) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln (x + \sqrt{x^2-a^2}) + C$

(63) $\int \frac{dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2 \sqrt{x^2-a^2}} + C$

(64) $\int \frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2} + C$

(65) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \ln (x + \sqrt{x^2-a^2}) + C$

(66) $\int \sqrt{(x^2-a^2)^3}\,dx = \frac{x}{8} (2x^2-5a^2) \sqrt{x^2-a^2} +$

$+ \frac{3a^4}{8} \ln (x + \sqrt{x^2-a^2}) + C$

(67) $\int x \sqrt{x^2-a^2}\,dx = \frac{\sqrt{(x^2-a^2)^3}}{3} + C$

(68) $\int x \sqrt{(x^2-a^2)^3}\,dx = \frac{\sqrt{(x^2-a^2)^5}}{5} + C$

(69) $\int x^2 \sqrt{x^2-a^2}\,dx = \frac{x}{8} (2x^2-a^2) \sqrt{x^2-a^2} -$

$- \frac{a^4}{8} \ln (x + \sqrt{x^2-a^2}) + C$

(70) $\int \frac{x^2\,dx}{\sqrt{x^2-a^2}} = \frac{x}{2} \sqrt{x^2-a^2} + \frac{a^2}{2} \ln (x + \sqrt{x^2-a^2}) + C$

(71) $\int \frac{x^2\,dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{\sqrt{x^2-a^2}} + \ln (x + \sqrt{x^2-a^2}) + C$

(72) $\int \frac{dx}{x \sqrt{x^2-1}} = \operatorname{arcsec} x + C$

(73) $\int \frac{dx}{x \sqrt{x^2-a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{x}{a} + C$

(74) $\int \frac{dx}{x^2 \sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x} + C$

$$(75)\quad \int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\operatorname{arcsec}\frac{x}{a} + C$$

$$(76)\quad \int \frac{\sqrt{x^2-a^2}\,dx}{x} = \sqrt{x^2-a^2} - a\arccos\frac{a}{x} + C$$

$$(77)\quad \int \frac{\sqrt{x^2-a^2}\,dx}{x^2} = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left(x+\sqrt{x^2-a^2}\right) + C$$

7. Functions Containing $\sqrt{2ax-x^2}$, $\sqrt{2ax+x^2}$

A function containing $\sqrt{2ax-x^2}$ is integrated by the substitution $t=x-a$. Then $\sqrt{2ax-x^2}$ becomes $\sqrt{a^2-t^2}$ and the integral is found in Group 5 of this table. If it is not in the table, an attempt is made to reduce it to a form which is.

The same may be said of a function containing the expression $\sqrt{2ax+x^2}$. In this case the substitution $t=x+a$ reduces the radical to the form $\sqrt{t^2-a^2}$ (Group 6 of this table).

8. Functions Containing $a+bx+cx^2$ $(c>0)$

$$(78)\quad \int \frac{dx}{a+bx+cx^2} = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2cx+b}{\sqrt{4ac-b^2}} + C, & \text{if } b^2<4ac \\[2ex] \dfrac{1}{\sqrt{b^2-4ac}}\ln\dfrac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}} + C, & \text{if } b^2>4ac \end{cases}$$

$$(79)\quad \int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\ln\left(2cx+b+2\sqrt{c}\sqrt{a+bx+cx^2}\right) + C$$

$$(80)\quad \int \sqrt{a+bx+cx^2}\,dx = \frac{2cx+b}{4c}\sqrt{a+bx+cx^2} - \frac{b^2-4ac}{8\sqrt{c^3}}\ln\left(2cx+b+2\sqrt{c}\sqrt{a+bx+cx^2}\right) + C$$

$$(81)\quad \int \frac{x\,dx}{\sqrt{a+bx+cx^2}} = \frac{\sqrt{a+bx+cx^2}}{c} - \frac{b}{2\sqrt{c^3}}\ln\left(2cx+b+2\sqrt{c}\sqrt{a+bx+cx^2}\right) + C$$

9. Functions Containing $a+bx-cx^2$ $(c>0)$

$$(82)\quad \int \frac{dx}{a+bx-cx^2} = \frac{1}{\sqrt{b^2+4ac}}\ln\frac{\sqrt{b^2+4ac}+2cx-b}{\sqrt{b^2+4ac}-2cx+b} + C$$

(83) $$\int \frac{dx}{\sqrt{a+bx-cx^2}} = \frac{1}{\sqrt{c}} \arcsin \frac{2cx-b}{\sqrt{b^2+4ac}} + C$$

(84) $$\int \sqrt{a+bx-cx^2}\, dx = \frac{2cx-b}{4c} \sqrt{a+bx-cx^2} + \\ + \frac{b^2+4ac}{8\sqrt{c^3}} \arcsin \frac{2cx-b}{\sqrt{b^2+4ac}} + C$$

(85) $$\int \frac{x\, dx}{\sqrt{a+bx-cx^2}} = -\frac{\sqrt{a+bx-cx^2}}{c} + \frac{b}{2\sqrt{c^3}} \arcsin \frac{2cx-b}{\sqrt{b^2+4ac}} + C$$

10. Other Algebraic Functions

(86) $$\int \sqrt{\frac{a+x}{b+x}}\, dx = \sqrt{(a+x)(b+x)} + (a-b) \ln\left(\sqrt{a+x} + \\ + \sqrt{b+x}\right) + C$$

(87) $$\int \sqrt{\frac{a-x}{b+x}}\, dx = \sqrt{(a-x)(b+x)} + \\ + (a+b) \arcsin \sqrt{\frac{x+b}{a+b}} + C$$

(88) $$\int \sqrt{\frac{a+x}{b-x}}\, dx = -\sqrt{(a+x)(b-x)} - \\ -(a+b) \arcsin \sqrt{\frac{b-x}{a+b}} + C$$

(89) $$\int \sqrt{\frac{1+x}{1-x}}\, dx = -\sqrt{1-x^2} + \arcsin x + C$$

(90) $$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2 \arcsin \sqrt{\frac{x-a}{b-a}} + C$$

11. Exponential and Trigonometric Functions

(91) $\int a^x\, dx = \frac{a^x}{\ln a} + C$ (92) $\int e^x\, dx = e^x + C$

(93) $\int e^{ax}\, dx = \frac{e^{ax}}{a} + C$ (94) $\int \sin x\, dx = -\cos x + C$

(95) $\int \cos x\, dx = \sin x + C$ (96) $\int \tan x\, dx = -\ln \cos x + C$

(97) $\int \cot x\, dx = \ln \sin x + C$

(98) $\int \sec x\,dx = \ln(\sec x + \tan x) + C = \ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C$

(99) $\int \operatorname{cosec} x\,dx = \ln(\operatorname{cosec} x - \cot x) + C = \ln\tan\frac{x}{2} + C$

(100) $\int \sec^2 x\,dx = \tan x + C$

(101) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

(102) $\int \sec x \tan x\,dx = \sec x + C$

(103) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$

(104) $\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

(105) $\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

(106) $\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

This formula is used several times until it brings us to the integral $\int \sin x\,dx$ or $\int \sin^2 x\,dx$ (depending on whether n is even or odd); or these integrals see No. 94 and No. 104.

(107) $\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

(see note of No. 106 and also see No. 95 and No. 105).

(108) $\int \frac{dx}{\sin^n x} = -\frac{1}{n-1} \cdot \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}$

Use several times until it leads to the integral $\int dx$ if n is even, or to the integral $\int \frac{dx}{\sin x}$ if n is odd (the latter integral is given in No. 99).

(109) $\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\,\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}$

(see note of No. 108, also see No. 98).

(110) $\int \sin x \cos^n x\,dx = -\frac{\cos^{n+1} x}{n+1} + C$

(111) $$\int \sin^n x \cos x \, dx = \frac{\sin^{n+1} x}{n+1} + C$$

(112) $$\int \cos^m x \sin^n x \, dx = \frac{\cos^{m-1} x \sin^{m+1} x}{m+n} + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x \, dx$$

Use several times until the power of the cosine is zero (if m is even) or unity (if m is odd). In the former case, see No. 106, in the latter, No. 111. This formula should be used when $m < n$. If $m > n$, then it is better to use the following formula:

(113) $$\int \cos^m x \sin^n x \, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx$$

(see note of No. 112, and also see No. 107 and No. 110).

$$\left.\begin{array}{ll}
(114) & \displaystyle\int \sin mx \sin nx \, dx = \\
& \displaystyle = -\frac{\sin (m+n) x}{2 (m+n)} + \frac{\sin (m-n) x}{2 (m-n)} + C \\
(115) & \displaystyle\int \cos mx \cos nx \, dx = \\
& \displaystyle = \frac{\sin (m+n) x}{2 (m+n)} + \frac{\sin (m-n) x}{2 (m-n)} + C \\
(116) & \displaystyle\int \sin mx \cos nx \, dx = \\
& \displaystyle = -\frac{\cos (m+n) x}{2 (m+n)} - \frac{\cos (m-n) x}{2 (m-n)} + C
\end{array}\right\} (m \neq n)$$

(117) $$\int \frac{dx}{a+b \cos x} = \frac{2}{\sqrt{a^2-b^2}} \arctan \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right) + C, \quad \text{if } a > b$$

(118) $$\int \frac{dx}{a+b \cos x} = \frac{1}{\sqrt{b^2-a^2}} \ln \frac{\sqrt{b-a} \tan \frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a} \tan \frac{x}{2} - \sqrt{b+a}} + C, \quad \text{if } a < b$$

(119) $$\int \frac{dx}{a+b \sin x} = \frac{2}{\sqrt{a^2-b^2}} \arctan \frac{a \tan \frac{x}{2} + b}{\sqrt{a^2-b^2}} + C, \quad \text{if } a > b$$

$$(120)\quad \int \frac{dx}{a+b\sin x}=\frac{1}{\sqrt{b^2-a^2}}\ln\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}+C,$$

$$\text{if } a<b$$

$$(121)\quad \int \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{1}{ab}\arctan\left(\frac{b\tan x}{a}\right)+C$$

$$(122)\quad \int e^x\sin x\,dx=\frac{e^x(\sin x-\cos x)}{2}+C$$

$$(123)\quad \int e^{ax}\sin nx\,dx=\frac{e^{ax}(a\sin nx-n\cos nx)}{a^2+n^2}+C$$

$$(124)\quad \int e^x\cos x\,dx=\frac{e^x(\sin x+\cos x)}{2}+C$$

$$(125)\quad \int e^{ax}\cos nx\,dx=\frac{e^{ax}(n\sin nx+a\cos nx)}{a^2+n^2}+C$$

$$(126)\quad \int xe^{ax}\,dx=\frac{e^{ax}}{a^2}(ax-1)+C$$

$$(127)\quad \int x^n e^{ax}\,dx=\frac{x^n e^{ax}}{a}-\frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

The formula is used several times until the power of x becomes unity; then the integral is found in No. 126.

$$(128)\quad \int xa^{mx}\,dx=\frac{xa^{mx}}{m\ln x}-\frac{a^{mx}}{m(\ln a)^2}+C$$

$$(129)\quad \int x^n a^{mx}\,dx=\frac{a^{mx}x^n}{n\ln a}-\frac{n}{m\ln a}\int a^{mx}x^{n-1}\,dx$$

The formula is used until the power of x is unity; then the integral is found in No. 128.

$$(130)\quad \int e^{ax}\cos^n x\,dx=\frac{e^{ax}\cos^{n-1}x\,(a\cos x+n\sin x)}{a^2+n^2}+$$

$$+\frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2}x\,dx$$

The formula is used until the cosine disappears (in the case of even n) or until its power is unity (in the case of odd n). In the latter case, see No. 122.

$$(131)\quad \int \sinh x\,dx=\cosh x+C$$

(132) $\int \cosh x\,dx = \sinh x + C$

(133) $\int \tanh x\,dx = \ln \cosh x + C$

(134) $\int \coth x\,dx = \ln \sinh x + C$

(135) $\int \operatorname{sech} x\,dx = 2 \arctan e^x + C$

(136) $\int \operatorname{csch} x\,dx = \ln \tanh \frac{x}{2} + C$

(137) $\int \operatorname{sech}^2 x\,dx = \tanh x + C$

(138) $\int \operatorname{csch}^2 x\,dx = -\coth x + C$

(139) $\int \operatorname{sech} x \tanh x\,dx = \operatorname{sech} x + C$

(140) $\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C$

(141) $\int \sinh^2 x\,dx = -\frac{x}{2} + \frac{1}{4} \sinh 2x + C$

(142) $\int \cosh^2 x\,dx = \frac{x}{2} + \frac{1}{4} \sinh^2 2x + C$

12. Logarithmic Functions

Only functions containing the natural logarithm are given. If it is required to find the integral of a function containing the logarithm to a different base, first change to the natural logarithm by the formula $\log_a x = \frac{\ln x}{\ln a}$ and then use the table

(143) $\int \ln x\,dx = x \ln x - x + C$

(144) $\int \frac{dx}{x \ln x} = \ln(\ln x) + C$

(145) $\int x^n \ln x\,dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right] + C$

(146) $\int \ln^n x\,dx = x \ln^n x - n \int \ln^{n-1} x\,dx$

The formula is used until the integral $\int \ln x\,dx$ is obtained; this integral is taken by formula No. 143.

$$(147)\quad \int x^m \ln^n x\,dx = \frac{x^{m+1}}{m+1}\ln^n x - \frac{n}{m+1}\int x^m \ln^{n-1} x\,dx$$

The formula is used until it leads to integral No. 145.

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